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An empirical Bayesian analysis applied to the globular cluster pulsar population

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This version implements a number of small corrections to the text which were recently submitted as an erratum

ABSTRACT

We describe an empirical Bayesian approach to determine the most likely size of an astronomical population of sources of which only a small subset are observed above some limiting flux density threshold. The method is most naturally applied to astronomical source populations at a common distance (e.g., stellar populations in globular clusters), and can be applied even to populations where a survey detects no objects. The model allows for the inclusion of physical parameters of the stellar population and the detection process. As an example, we apply this method to the current sample of radio pulsars in Galactic globular clusters. Using the sample of flux density limits on pulsar surveys in 94 globular clusters published by Boyles et al., we examine a large number of population models with different dependencies. We find that models which include the globular cluster two-body encounter rate, Γ, are strongly favoured over models in which this is not a factor. The optimal model is one in which the mean number of pulsars is proportional to $\exp(1.5 \log \Gamma)$. This model agrees well with earlier work by Hui et al. and provides strong support to the idea that the two-body encounter rate directly impacts the number of neutron stars in a cluster. Our model predicts that the total number of potentially observable globular cluster pulsars in the Boyles et al. sample is 1070^{+1280}_{-700}, where the uncertainties signify the 95% confidence interval. Scaling this result to all Galactic globular clusters, and to account for radio pulsar beaming, we estimate the total population to be 2280^{+2720}_{-1490}.

Key words: pulsars: general — methods: statistical

1 INTRODUCTION

Virtually all observational samples of astronomical sources are subject to selection biases. As a result, the distributions of observed physical parameters (e.g. luminosity) are often significantly different to the underlying population. For example, as a result of the so-called "inverse square law" where the observed flux density of an object scales as its luminosity divided by the square of its distance from Earth, populations of objects are biased in favour of bright and/or relatively nearby sources whose flux densities are above the threshold of a given survey. Correcting these observationally biased samples to infer the size and properties of the underlying population has been carried out by numerous authors over the years for a variety of different astronomical sources. A number of techniques have been carried out in these studies, for example the "$V/V_{\rm max}$" approach pioneered for samples of quasars (Schmidt 1968), Monte Carlo population syntheses (Emmering & Chevalier 1987; Browne & Marcha 1993), as well as Bayesian statistical inference (Boyles et al. 2011). A common theme among samples of objects is that the number of sources observed may be low, yet difficulties in detection mean that the underlying population size can be significantly larger. Appropriate accounting of these small-number statistics and assigning confidence intervals for the underlying source populations have been the subject of significant research over the past decade. For example, in the population of double neutron star binaries in the Milky Way (Kim et al. 2003; Kalogera et al. 2004) and their implications for the neutron star inspiral rates observed by gravitational wave interferometers (Kalogera et al. 2001).

The population of radio pulsars in globular clusters (GCs) is an excellent example of the selection bias problem discussed above. This observable radio pulsar population currently amounts to 144 pulsars detected in 28 GCs[1].

[1] An online list detailing currently known pulsars in GCs is maintained at http://www.naic.edu/∼pfreire/GCpsr.html

These pulsars were found as a result of extensive pulsar surveys carried out with large radio telescopes over the past 25 years (see, e.g., Lyne et al. 1987; Anderson et al. 1990; Manchester et al. 1991; Biggs et al. 1994; Camilo et al. 2000; Ransom et al. 2004, 2005). Excellent reviews of this field can be found in Kulkarni & Anderson (1996) and Camilo & Rasio (2005).

Determining the properties of the underlying population of pulsars across all GCs, which themselves have significantly different physical properties (e.g. cluster mass, metallicity, central density, etc.), remains a challenging problem. Of particular interest in this case is whether underlying relationships between pulsar abundance and GC properties might exist. One such example is the proposed relationship between pulsar abundance and stellar encounter rate, Γ, and metallicity proposed recently by Hui et al. (2010). Their analysis was straightforward, being based on the luminosity function of GC pulsars. However, a similar analysis by Bagchi et al. (2011) concluded that the evidence in favour of such a correlation was tentative. While the existence of such a correlation is expected from a well-established relationship found for low-mass X-ray binaries (Pooley et al. 2003), it is important to confirm or refute the findings of Hui et al. (2010) based on the sample of radio pulsars in GCs.

In this paper, we present a new approach using empirical Bayesian methods which attempt to address this problem. This work is synergistic with our other recent studies of the pulsar content of GCs (Boyles et al. 2011; Bagchi et al. 2011; Lynch et al. 2012; Chennamangalam et al. 2013) which also rely on Bayesian and/or Monte Carlo techniques to infer the parent population. While we apply our model to the pulsar content of GCs, the method could equally well be used elsewhere, where one is interested in searching for dependencies of source abundance with environment. The methods we describe in this paper have been known within the statistics community for some time (Royle 2004; Kéry et al. 2005), but this study represents (to our knowledge) the first application to astrophysical sources. The advantage of this approach is that it makes use of the fact that in many GCs, no pulsars are currently known despite being extensively surveyed. This information was not taken account of by the analyses of Hui et al. (2010) or Bagchi et al. (2011).

The outline for the rest of this paper is as follows. In Section 2, we briefly describe the sample of pulsars in GCs whose underlying properties we wish to constrain. In Section 3, we describe our approach to the problem which makes use of existing flux-density limits for GCs compiled by Boyles et al. (2011). Our results are presented in Section 4. In Section 5 we discuss the implications of our results. Finally, in Section 6, we summarize our main conclusions and give suggestions for future extensions to this work.

2 THE PULSAR SAMPLE

The methods described below require a detection limit for each GC for the number of pulsars detected, as well as several physical parameters pertaining to the GC. We made use of the 95 GCs presented by Boyles et al. (2011) in which minimum detectable flux density limits, scaled to 1.4 GHz observing frequency, were collated for their study of young pulsars in GCs. We chose not to include the faint GC E3 in this analysis since it does not appear in the list of stellar encounter rates computed by Bahramian et al. (2013). Our final list therefore applies to 94 GCs. In order to investigate the relationship of the total number of pulsars in each GCs with physical parameters of that cluster, we collated V-band luminosities, stellar encounter rates, escape velocities and metallicities for each GC. The V-band luminosities and metallicities were taken from the catalog (December 2010 edition) of Harris (1996). The stellar encounter rates were taken from Bahramian et al. (2013). Since the input data used for this study are tabulated in these papers, we do not list them explicitly here. However, an ASCII file containing all of the data, is available at http://astro.phys.wvu.edu/gcpsrs/empbayes. This URL also contains a copy of the R code version 2.14.1, and the package *unmarked*, version 0.9-8 which was used to carry out the statistical analysis presented in this paper.

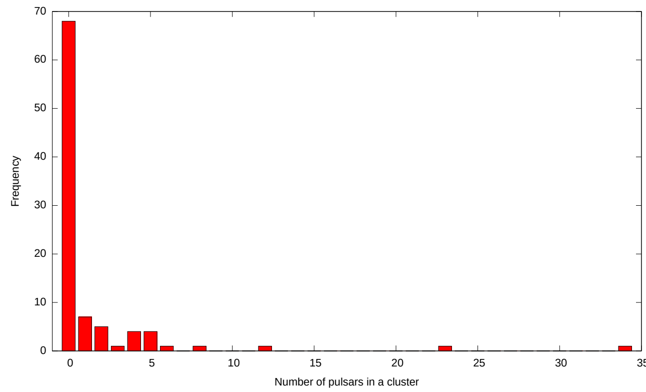

Figure 1. Histogram showing the distribution of the number of detected pulsars across our sample of GCs.

3 METHODS

As can be seen in Fig. 1, the number of clusters as a function of the number of currently detectable pulsars in each of the 94 GCs, the sample has many clusters with few or no detectable pulsars. The data analysis method described in this section is specifically designed to extract estimates of the true, unknown number of pulsars, or abundance, for each GC from this limited number of detections, accounting for imperfect detection. Briefly, the overall modeling strategy proceeds in the following manner. We first construct a predictor variable that will be used to model the detection probability for each GC. Next, we fit a variety of empirical Bayesian models to the currently detectable pulsars for the 94 GCs and use a model selection criterion to pick the best model. Once this has been done, we use the fitted model to conduct inference using standard hypothesis tests and confidence intervals. Next, we generate specific plots, and conduct goodness-of-fit and likelihood ratio tests. Lastly, we use bootstrapping and Bayesian inference to estimate the actual population size of the total number of pulsars in all of the the 94 GCs.

We evaluated so-called N-mixture models (Royle 2004) for obtaining dectability-corrected estimates of pulsar abundance for each GC, and estimating covariate effects on abundances and detectability. Specifically, let the number of pulsars counted at the ith GC, where $i = 1, 2, \ldots, 94$, n_i follow a binomial model in which N_i is the unknown abundance of pulsars in the ith GC, and the parameter p is the true, unknown detection probability for any pulsar. In this case, the likelihood, $\mathcal{L}$, for the number of pulsars from the ith GC is:

$$\mathcal{L}(N_i, p \mid n_i) = \binom{N_i}{n_i} p^{n_i}(1-p)^{N_i - n_i}. \quad (1)$$

Viewing the number of pulsars counted at the different GCs as independent samples, we obtain 94 likelihoods conditioned on $\{N_1, N_2, \ldots, N_{94}\}$ and p, which gives a joint likelihood:

$$\mathcal{L}(\{N_i\}, p \mid \{n_i\}) = \prod_{i=1}^{94} \binom{N_i}{n_i} p^{n_i}(1-p)^{N_i - n_i}. \quad (2)$$

Next, to simplify Equation (2), we construe the abundance N_is as independent latent random variables with some probability mass function, and then integrate Equation (1) over this prior distribution. Several easily implemented models have been proposed for a prior distribution on abundance; specifically, Poisson, zero-inflated Poisson, and negative binomial (Royle 2004; Fiske & Chandler 2011). For example, the Poisson probability mass function

$$f(N; \lambda) = \frac{e^{-\lambda}\lambda^N}{N!}, \quad (3)$$

where λ is the mean, or expected value, of N. Accordingly, the approximate integrated likelihood is now a function of only two parameters:

$$\mathcal{L}(\lambda, p \mid \{n_i\}) \cong \prod_{i=1}^{94} \left(\sum_{N_i = n_i}^{K} \binom{N_i}{n_i} p^{n_i}(1-p)^{N_i - n_i} \times \frac{e^{-\lambda}\lambda^{N_i}}{N_i!} \right), \quad (4)$$

where K is a finite large bound (e.g., 500, say) chosen in order to fit the model and achieve stable estimates of the parameters. By substituting in maximum likelihood estimates, obtained numerically, into Equation (4), we adopt an empirical Bayesian approach.

We modeled covariate effects on abundance using a linear sub-model where the covariate effects were physical properties of the GCs previously mentioned (e.g., metallicity). For example, using the Poisson model, we could use a log-linear model on the prior mean as:

$$\ln(\lambda_i) = \beta_0 + \beta_1 X_{i1} + \beta_2 X_{i2} + \cdots + \beta_r X_{ir}, \quad (5)$$

where λ_i is the mean, or expected, number of pulsars in the ith GC and X_{ij} is the jth covariate, $j = 1, 2, \ldots, r$. The β_j's are the usual regression coefficients that characterize the effect of the jth covariate.

Similarly, we may model the detection probability p as a function of covariates using a linear sub-model:

$$\mathrm{logit}(p_i) = \alpha_0 + \alpha_1 Z_{i1} + \alpha_2 Z_{i2} + \cdots + \alpha_r Z_{is}, \quad (6)$$

where Z_{ik} is the kth covariate, $k = 1, 2, \ldots, s$ and the logit function:

$$\mathrm{logit}(p) = \ln\left(\frac{p}{1-p}\right). \quad (7)$$

Note that our subsequent use of the subscripted notation "p_i" simply reflects the idea that the detection probability is free to vary for each GC as a function of the covariates. A covariate may appear in both the abundance model and the detection probability model (Kéry 2008).

Let $\widehat{p}_i$ be an estimate of the detection probability p_i for the ith GC, $i = 1, 2, \ldots, 94$, as determined in the following fashion. Let L be the luminosity of a pulsar in a GC and $L_{\min}$ be the minimum allowable luminosity of a pulsar in a GC. For each GC, we used the $L_{\min}$ values derived from the compilation of 1400 MHz survey flux density limits $S_{\min}$ and GC distances D tabulated by Boyles et al. (2011). Following standard practice, we computed $L_{\min}$ for the ith cluster as $L_{\min,i} = S_{\min,i} D_i^2$. Assuming that L follows a log-normal distribution with mean μ and standard deviation σ (Faucher-Giguère & Kaspi 2006; Bagchi et al. 2011), then the estimated detection probability

$$\widehat{p}_i = P[\log(L) \geqslant \log(L_{\min,i})], \quad (8)$$

where $\log(L)$ follows a normal distribution with mean μ and standard deviation σ. Note that, throughout this paper, we shall refer to base-10 logarithms as "log". Previous research suggests that values of $\mu = -1.1$ and $\sigma = 0.9$ are consistent with the observed GC pulsar population (see, e.g., Bagchi et al. 2011).

The following candidate covariates were considered in the initial model selection. For the detection model, we used $\mathrm{logit}(\widehat{p}_i)$ without an intercept term. Note this is the only sensible detection model for this application as this forces the estimated detection probability from the fitted model to be 1/2 when $\mathrm{logit}(\widehat{p}_i) = 0$; that is, $\widehat{p}_i = 1/2$. For the abundance model, we separately considered five scenarios, each with an admissible intercept term: no estimated detection probability, and then including either $\widehat{p}_i, \log(\widehat{p}_i)$ and either $\widehat{p}_i$ and $\log(\widehat{p}_i)$ where their parameters β_j were constrained to equal 1 (so-called "offset terms"). For each of these scenarios, we chose 4-choose-l predictors where $l = 1, \ldots, 4$, from the set of predictors GC V-band luminosity, the log base 10 of the stellar encounter rate, the GC escape velocity, and metallicity. Accordingly, this leaves us with 80 models to be considered.

In addition to the Poisson model previously described, we considered two additional models as priors for abundance. We considered the zero-inflated Poisson (ZIP) model in which abundance is modeled as a mixture of two distributions. The ZIP model is characterized by λ as before and, ψ, the "excess zero" parameter. Hence, abundance N is modeled as:

$$N = \begin{cases} 0, & \text{with probability } \psi \\ \text{Poisson}, & \text{with probability } 1-\psi \end{cases}, \quad (9)$$

which gives rise to the following piecewise ZIP probability mass function:

$$f(N; \lambda, \psi) = \begin{cases} \psi + (1-\psi)e^{-\lambda}, & N = 0 \\ (1-\psi)\frac{e^{-\lambda}\lambda^N}{N!}, & N > 0. \end{cases}. \quad (10)$$

A negative binomial prior could also be considered as an alternative model for abundance when there is greater variation in the data than can be explained by the Poisson model, i.e., when the data are "overdispersed". One cause for overdispersion occurs when there are an excess number of zeros. The negative binomial probability mass function:

$$f(N;\alpha,r)=\frac{(N+\alpha-1)!}{(\alpha-1)!N!}r^{\alpha}(1-r)^{N}, \tag{11}$$

where the mean λ, say, is equal to $\alpha(1-r)/r$. In this context, α is a non-negative integer, or so-called overdispersion parameter, and r is a nuisance parameter on $[0,1]$.

Akaike's Information Criterion (AIC; Burnham & Anderson 2002) is often used for model selection in the sort of analysis we describe here. In general, AIC is defined as:

$$\mathrm{AIC}=-2\ln\mathcal{L}(\widehat{\theta})+2k, \tag{12}$$

where k is the number of estimable parameters in the model, $\widehat{\theta}$ are estimates of the model parameters obtained using maximum likelihood estimation, and $\ln\mathcal{L}(\widehat{\theta})$ is the natural logarithm of the maximized value of the likelihood function (the reader is referred to Casella & Berger (2001) for a detailed discussion of these concepts). Models with a smaller AIC value (by at least two AIC units) are considered to provide a better fit to the data than models with a larger AIC value (Burnham & Anderson 2002). Notice that AIC penalizes models containing more parameters to discourage overfitting of the data.

We used AIC to select a plausible model from the a-priori candidate set of 240 models, 80 for each of the three priors for abundance, containing the parameters previously described as well as null models without covariates for baseline comparison. Once the final model was selected, we conducted hypothesis tests on the parameters in both the abundance and detection components and obtained "P-values". Here a P-value is the probability of getting a value of a test statistic as extreme or more extreme than what is observed given the null hypothesis is true. As an example, which as we shall see will be relavant later on, suppose $\log(\Gamma)$ was a sole covariate in the model given in Equation (5). Much as would be done in simple linear regression, a hypothesis test that would be of interest would be to see if there is an association or not between the number of pulsars in a GC and $\log(\Gamma)$. Using parameters, the null hypothesis would be stated as $\beta_1=0$ and the alternative hypothesis would be $\beta_1\neq 0$. To achieve this, we use the so-called "Wald test" (Wald 1943) where we obtain the maximum likelihood estimate of β_1 ($\hat{\beta}_1$) and divide this by an estimate of the variation, or standard error, for $\hat{\beta}_1$. This gives us a test statistic z which we use to conduct the hypothesis test. The test statistic z follows a standard normal distribution with mean 0 and standard deviation 1. Hence, if z is large in absolute value, then the probability of getting a value of z as extreme or more extreme would be unlikely if the null hypothesis were true and we would therefore reject the null hypothesis in favor of the alternative hypothesis. In a nutshell, this probability, or P-value, can be thought of as a measure of evidence against the null hypothesis in favor of the alternative hypothesis; the smaller the P-value, the stronger the evidence.

Via maximum likelihood estimation, we obtained estimates of the mean number of pulsars at the ith GC, $\widehat{\lambda}_i$, possibly based on the values of known covariates. A derived estimator of the total number of pulsars in the sample GCs, N, is the sum of the estimates $\widehat{\lambda}_i$ obtained from the model (Royle 2004), henceforth referred to as $\widehat{N}_d$. Alternatively, for each GC, we used empirical Bayes methods to obtain the estimated posterior conditional probability distribution of N_i given the observed counts of pulsars and other parameters in the model (see Royle 2004, page 110 for details). We then obtained the estimated posterior mean to obtain predictions of N_i along with a 95% percentile-based credible interval for N_i. An estimate of N, $\widehat{N}_b$ say, was obtained as the sum of the estimated posterior mean pulsars in the ith GC. The GC-specific credible interval endpoints were then summed to yield a 95% percentile-based credible interval for N.

Table 1. Models within two units of the smallest AIC value.

Model Structure		AIC	ΔAIC
$\mathrm{logit}(p_i)$	$\ln\lambda$		
$\alpha_1\mathrm{logit}(\widehat{p}_i)$	$\beta_0+1\log(\widehat{p}_i)+\beta_2\log(\Gamma)$	208.10	0
$\alpha_1\mathrm{logit}(\widehat{p}_i)$	$\beta_0+1\widehat{p}_i+\beta_2\log(\Gamma)$	209.31	1.21
$\alpha_1\mathrm{logit}(\widehat{p}_i)$	$\beta_0+\beta_1\log(\Gamma)$	209.34	1.24

The goodness of fit of the final model was evaluated using a parametric bootstrapping procedure (Dixon 2002) where the sum-of-the-squared errors (SSE) was the bootstrap goodness of fit criterion. Briefly, the parameters of the final model were set to the maximum likelihood estimates and a large number of "bootstrap" replicate data sets were randomly generated. For each bootstrap sample, the parameters were estimated again and the SSE was generated. This gives us a bootstrap distribution for SSE from which a P-value for the original observed SSE can be obtained. For the sake of comparison, we did the same for the null model. We also generated $\widehat{N}_d$ for each bootstrap sample, and constructed the bootstrap distribution for $\widehat{N}_d$ to estimate bias and obtain a percentile-based confidence interval for N. Here, bias refers to the difference between N and the expected value of $\widehat{N}_d$. The number of bootstrap samples was set equal to 999, an amount deemed large enough to characterize the sampling distributions for SSE and $\widehat{N}_d$.

We conclude this section with a remark concerning multicollinearity, which refers to the situation where there is high correlation among the predictor variables. Consequently, there can be several problems that occur in modeling (e.g., estimated regression coefficients will have large standard errors). There was no evidence of multicollinearity among the covariates used in the abundance model as determined by condition indices (Belsley et al. 1980).

4 RESULTS

For the sake of brevity, Table 1 summarizes the most parsimonious models (i.e., those with the fewest parameters) and AIC values for those models within 2 AIC units of the model with the smallest AIC value. The model structure describes the terms in the detection and abundance sub-models, respectively, and Δ AIC is the change in AIC relative to the smallest AIC value. In the three cases shown, the prior distribution on N was the negative binomial, and the number of parameters was four in each case (including α).

Results from AIC suggested a negative binomial model provides the best fit. Based on the parsimonious exclusion of the offset term, the model in the last row of Table 1 (with AIC of 209.34) was selected. This final model is now described with results rounded and reported to one decimal

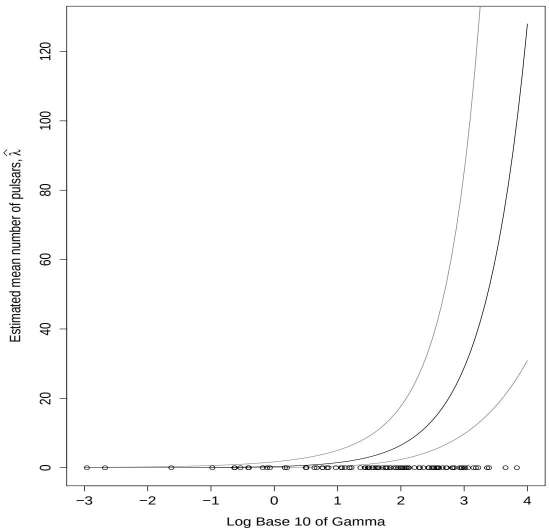

Figure 2. Estimated mean number of pulsars as a function of the base-10 logarithm of the two-body encounter rate, Γ. The observed GC $\log(\Gamma)$ values are displayed along the x-axis.

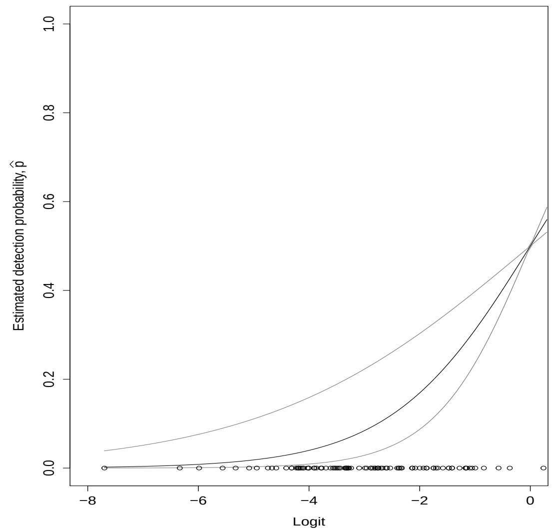

Figure 3. Estimated detection probability as a function of $\mathrm{logit}(\widehat{p}_i)$. The observed pulsar $\mathrm{logit}(\widehat{p}_i)$ values are displayed along the x-axis.

place. Within the framework of this model, the number of pulsars in a GC (on the natural log-scale) was modeled by an intercept and slope in which $\hat{\beta}_0 = -1.1$ and $\hat{\beta}_1 = 1.5$ as follows:

$$\ln \hat{\lambda} = -1.1 + 1.5 \log \Gamma. \tag{13}$$

The association between the number of pulsars in a GC and $\log(\Gamma)$ was highly significant (P-value $= 4.73 \times 10^{-7}$). Interpreting this expression in words, for a one-unit increase $\log(\Gamma)$, we estimate the mean number of pulsars in a GC increases by a factor of $\exp(\hat{\beta}_1) = 4.5$. To approximate the standard error in the above expression and obtain a confidence interval, we used a first-order Taylor series expansion of $\exp(\widehat{\beta}_1)$ about β_1. This is the so-called "delta method" (see, e.g., Casella & Berger 2001). The approximate 95% confidence interval for the factor is (2.5, 7.9). Fig. 2 shows the estimated mean number of pulsars versus $\log(\Gamma)$ with 95% confidence limits.

The detection probability for a GC (logit-scale) was modeled by $\mathrm{logit}(\widehat{p}_i)$, with the association being significant (P-value $= 3.87 \times 10^{-5}$). Fig. 3 shows the estimated detection probability versus $\mathrm{logit}(\widehat{p}_i)$ with 95% confidence limits.

Based on the parametric bootstrapping with 999 bootstrap samples, the final negative binomial model fit adequately (P-value $= 0.464$). By comparison, the negative binomial null model also provided a satisfactory fit (P-value $= 0.310$). However, the AIC value for the negative binomial null model was 246.68, much larger than the AIC value for the final fitted model.

An estimate of $\widehat{N}_d = 1073$ total pulsars was obtained by summing GC-specific estimates of each λ_i for a given value of $\log(\Gamma)$. Fig. 4 shows a somewhat right-skewed bootstrap distribution for the aggregate estimate of N. The 95% confidence interval was [370, 2352] and there was no evidence of non-zero bias (P-value $= 0.433$). $\widehat{N}_b$ was approximately 1084 with a 95% percentile-based credible interval of [465, 2599].

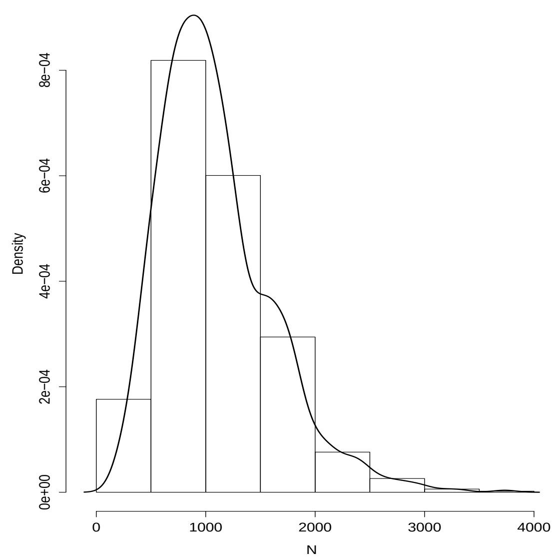

Figure 4. Parametric bootstrap distribution for $\widehat{N}_d$ obtained from 1000 bootstrap samples. The kernel density estimate is overlaid.

Table 2. Sensitivity analysis results for various assumed μ and σ

μ	σ	$\widehat{N}_d$	Confidence interval
–1.1	0.9	1073	[370,2352]
–1.2	0.8	1118	[324, 2339]
–1.2	1.0	1156	[346, 2389]
–1.0	0.8	983	[354,2130]
–1.0	1.0	1010	[340, 2167]

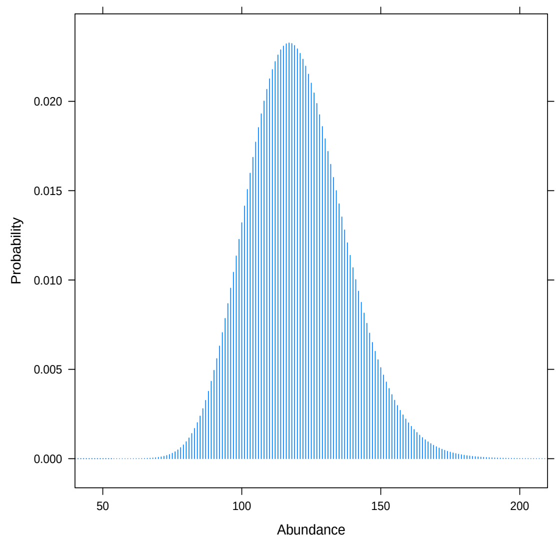

Figure 5. Estimated posterior conditional probability distribution for N_i for Ter 5.

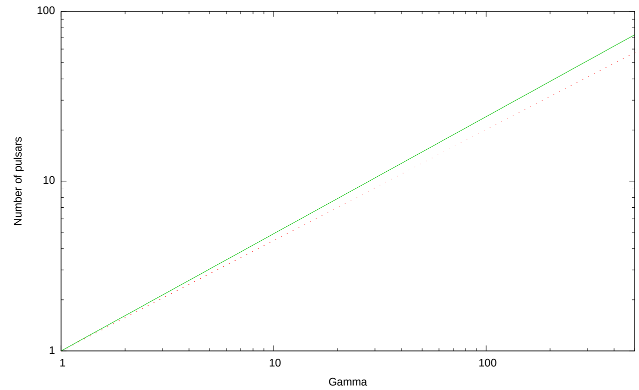

Figure 6. A comparison between the power-law relationship found by Hui et al. (red curve) and the functional form presented in Eq. 13 (green curve).

One implicit assumption in this work is the log-normal form of the luminosity function. As shown by Bagchi et al. (2011), the GC pulsar population is entirely consistent with a log-normal distribution albeit with a range of possible values of μ and σ. A sensitivity analysis was done to assess the sensitivity of $\widehat{N}_d$ to the parameterization of $\log(L) \sim \mathrm{Normal}(\mu, \sigma)$ by inputing several different values of the mean μ and the standard deviation σ and rerunning the procedure described in the previous section. As shown in Table 2, the results of the sensitivity analysis for the specification of the distribution for $\log(L)$ show a reasonable robustness with respect to estimates of N and the associated 95% confidence intervals.

It is also possible using the methodology described here to examine specific clusters. For example, Fig. 5 displays the estimated posterior conditional probability distribution for N_i for Ter 5. There were 34 observed pulsars and the observed $\widehat{p} = 0.2367$. The estimated mean of the distribution was 120 with a 95 per cent credible interval of [89, 157]. The model-based estimate of p was 0.2826. These results are in very good agreement with those found by Bagchi et al. (2011) and Chennamangalam et al. (2013) which are computed using a different technique. A complete set of distribution functions for all 94 GCs is available online.

5 DISCUSSION

The results presented here represent an estimate of the population of radio pulsars across 94 GCs whose emission beams intersect the line of sight to the Earth. A simple accounting for the fraction of 150 GCs not included in our analysis, and assuming a 75% beaming fraction for recycled pulsars in GCs (see, e.g., Kramer et al. 1998), means that our results should be scaled by a factor of $150/94/0.75 = 2.1$. The nominal estimate is then revised to 2280 with a 95% confidence interval of [790,5000]. This estimate is smaller than, and somewhat more well constrained than that found by Bagchi et al. (2011) which was based on scaling an analysis of only 10 GCs.

We caution the direct use of the above numbers, since further work is required to refine this population estimate. The input flux density limits from Boyles et al. (2011) are, strictly speaking, only applicable to long-period pulsars. Using the scheme developed here, where the detection probability can now be cast in terms of a linear model of other parameters, it should be possible to account for the reduced detectability of binary pulsars in a future analysis. Not only can orbital detectability be modeled (Johnston & Kulkarni 1992; Bagchi et al. 2013), but also the reduced detectability due to the presence of an eclipsing companion could also be taken into account.

The most important conclusion from the current work, however, is the independent verification of a trend seen earlier by Hui et al. (2010) in which the number of pulsars in a cluster directly scales with Γ. This correlation is well established for low-mass X-ray binaries (Pooley et al. 2003), but was not apparent in the recent analysis of 10 GCs by Bagchi et al. (2011). The trend we see in our analysis is based on 94 GCs in which we considered a wide variety of models with and without any explicit dependence on Γ. It should also be noted that the analysis was carried out by one of us (PJT) who was unaware of the proposed functional form found by Hui et al. (2010) As can be seen in Fig. 6, where we compare the functional form found here ($N \propto \exp(1.5 \log \Gamma)$) with that found by Hui et al. (2010; $N \propto \Gamma^{0.69}$), our results are in very good agreement.

To motivate future observations of GCs, in Table 3 we present the twenty clusters ranked in terms of Γ along with their estimated distances D, $L_{\rm min}$ values from searches so far (taken from Boyles et al. 2011), estimated values of $\widehat{\lambda}$ from Eq. 13 and currently observed number of pulsars, $N_{\rm obs}$. While most of the GCs with the highest Γ values have been searched repeatedly, resulting in significant numbers of detected pulsars, there are exceptions. The most notable of these is Terzan 6 which has so far no detected pulsars despite a nominal $L_{\rm min}$ of only 0.4 mJy kpc^2. There are number of promising GC search candidates with intermediate Γ values, e.g., NGC 2808, 6388 and 6293. These clusters would benefit from deeper observations than currently possible. More sensitive searches of GCs using exist-

Table 3. The top twenty GCs ranked in descending order of Γ.

Cluster	D (kpc)	$L_{\rm min}$ (mJy kpc^2)	Γ	$\widehat{\lambda}$	$N_{\rm obs}$
Terzan 5	6.9	0.6	6800	100	34
NGC 7078	10.4	2.3	4510	76	8
Terzan 6	6.8	0.4	2470	52	0
NGC 6441	11.6	1.7	2300	49	4
NGC 6266	6.8	1.0	1670	40	6
NGC 1851	12.1	4.4	1530	38	1
NGC 6440	8.5	0.7	1400	36	6
NGC 6624	7.9	1.0	1150	32	6
NGC 6681	9.0	6.3	1040	30	0
47 Tucanae	4.5	3.4	1000	29	23
Pal 2	27.2	25.2	929	27	0
NGC 2808	9.6	5.1	923	27	0
NGC 6388	9.9	5.4	899	27	0
NGC 6293	9.5	4.9	847	26	0
NGC 6652	10.0	7.8	700	23	1
NGC 6284	15.3	12.8	666	22	0
M28	5.5	0.1	648	22	12
M80	10.0	0.6	532	19	0
NGC 7089	11.5	0.8	518	19	0
NGC 5286	11.7	7.5	458	17	0

ing instrumentation (e.g., the state-of-the art systems at the Green Bank Telescope) as well as planned facilities (e.g., the MeerKAT array[2]) will undoubtably result in further discoveries in these and other GCs.

6 CONCLUSIONS

In summary, we have presented a new method to model populations of astronomical objects which is particularly applicable to stellar clusters. One of the benefits of our approach is that it allows use to be made of cases in which no sources are detected in a particular cluster, and it allows one to statistically study physical models which might affect the abundance of sources in a cluster. We have applied the method to the observed sample of pulsars in globular clusters and find very strong evidence in favour of the correlation between pulsar abundance and stellar encounter rate found previously by Hui et al. (2010). Our estimate of the total pulsar content in Galactic pulsars using this approach is 2280 with a 95% confidence interval of [790,5000]. Further refinements of the current model, which are beyond the scope of the current work, are planned to account for the detectability of binary pulsars will allow more realistic determinations of the pulsar content in globular clusters.

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[2] http://www.ska.ac.za/meerkat

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