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On Lagrangian Solutions for the Semi-geostrophic System with Singular Initial Data

Mikhail Feldman

Adrian Tudorascu

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Mikhail Feldman
Department of Mathematics
University of Wisconsin-Madison
Madison, WI 53706, USA
feldman@math.wisc.edu

Adrian Tudorascu
Department of Mathematics
West Virginia University
Morgantown, WV 26506, USA
adriant@math.wvu.edu

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Abstract

We show that weak (Eulerian) solutions for the Semi-Geostrophic system in physical space exhibiting some mild regularity in time cannot yield point masses in dual space. However, such solutions are physically relevant to the model. Thus, we discuss a natural generalization of weak Lagrangian solutions in the physical space to include the possibility of singular measures in dual space. We prove existence of such solutions in the case of discrete measures in dual space. We also prove that weak Lagrangian solutions in physical space determine solutions in the dual space. This implies conservation of geostrophic energy along the Lagrangian trajectories in the physical space.

1 Introduction

The Semi-Geostrophic (abbreviated SG in this work) equations have been proposed as simplifications of the primitive equations (Boussinesq) when the rate of change of momentum is much smaller than the Coriolis "force" (small Rossby number) [16]. The advected quantity momentum is approximated by its geostrophic value, but the trajectories are not.

Throughout the entire paper, $\Omega \subset I\!R^3$ is a given open, bounded set, and $T \in (0, \infty)$ is fixed. A

*Key words: SG system, flows of maps, optimal mass transport, Wasserstein metric, optimal maps, absolutely continuous curves.

version of the 3D Semi-Geostrophic system [6, 9] is

$$\begin{aligned}
&D_t X = J\big[X - x\big],\\
&X = \nabla P,\\
&\nabla \cdot u = 0,\\
&u \cdot \nu = 0, \quad \text{on } [0,T) \times \partial\Omega,\\
&P(0,\cdot) = P_0 \quad \text{in } \Omega,
\end{aligned} \tag{1}$$

where P_0 is convex defined on Ω. One looks for solutions (P, u) satisfying the Cullen-Purser stability condition (see, e.g., [10]), which amounts to imposing that $P_t(\cdot) := P(t,\cdot)$ be convex for all $t \in [0,T)$. Here,

$$J = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Henceforth, we shall assume without loss of generality that $\mathcal{L}^3(\Omega) = 1$ (otherwise all the measures considered will have total mass equal to the volume of Ω) and denote

$$\chi := \mathcal{L}^3|_\Omega.$$

If $\nabla P_{t\#}\chi =: \alpha_t$ are all absolutely continuous with respect to $\mathcal{L}^3$, then one can use the Legendre-Fenchel transforms P_t^* of P_t to formally rewrite (1) as the so-called SG in dual variables

$$\partial_t \alpha + \nabla \cdot (U\alpha) = 0 \quad \text{in } [0,T) \times I\!R^3, \tag{2}$$

$$\nabla P(t,\cdot)\#\chi = \alpha(t,\cdot) \quad \text{for any } t \in [0,T); \tag{3}$$

$$U(t,X) = J[X - \nabla P^*(t,X)], \tag{4}$$

$$\alpha(0,X) = \alpha_0(X) \quad \text{for a.e. } X \in I\!R^3. \tag{5}$$

Existence of solutions of the problem recast in dual variables was obtained by J. D. Benamou, Y. Brenier [6], and for some related models by M. Cullen and W. Gangbo [10], and M. Cullen and H. Maroofi [12]. They considered case when $\alpha_0 \in L^q(\Omega)$ for some $q > 1$, and the solution satisfies $\alpha(t,\cdot) \in L^q$ for all t. These results were extended to the case $q = 1$ in [18].

These solutions are not known to be regular enough to be translated into Eulerian solutions of the problem in physical space. Recently, existence of Eulerian solutions for a class of initial data, where the conditions include the requirement that the support of $\alpha_0 = \nabla P_{0\#}\chi$ in the dual space is the whole space, was obtained by L. Ambrosio, M. Colombo, G. De Philippis, A. Figalli [2, 3] based on the results of G. De Philippis and M. Figalli [13] on regularity of solutions for the Monge-Ampere equation. Existence of Eulerian solutions in physical space for more general initial data, when the support of α_0 in the dual space may have a non-empty boundary, is presently not known.

Another approach is to consider Lagrangian solutions in the physical space. Such solutions were introduced by M. Cullen and M. Feldman [9], and existence of Lagrangian solutions of (1) was shown in [9] for the case $\alpha_0 = \nabla P_{0\#}\chi \in L^q(\Omega)$ for $q > 1$, on the basis of Ambrosio's theory on transport equations and flows of BV vector fields [1]. These results were extended to the case $q = 1$ in [14]. The definition of weak Lagrangian solutions in the physical space is following (where we slightly modify the definition given in [9] by relaxing assumptions on P).

Definition 1.1. *Let $P_0 \in H^1(\Omega)$ be convex, $F : [0,T) \times \Omega \to \Omega$ be a Borel map such that $F \in C([0,T); L^2(\Omega; I\!R^3))$, and let $P \in C([0,T); H^1(\Omega))$. Assume $P(t,\cdot)$ is convex in Ω for each $t \in [0,T)$. Then the pair (P, F) is called a weak Lagrangian solution of (1) in $[0,T) \times \Omega$ if* *i.* $F(0,x) = x$, $P(0,x) = P_0(x)$ *for a.e.* $x \in \Omega$,

ii. *for any* $t > 0$ *the mapping* $F_t = F(t,\cdot) : \Omega \to \Omega$ *is Lebesgue measure preserving, in the sense that* $F_{t\#}\chi = \chi$;

iii. *There exists a Borel map* $F^* : [0,T) \times \Omega \to \Omega$ *such that for every* $t \in (0,T)$ *the map* $F_t^* = F^*(t,\cdot) : \Omega \to \Omega$ *is Lebesgue measure preserving:* $F_{t\#}^*\chi = \chi$, *and satisfies* $F_t^* \circ F_t(x) = x$ *and* $F_t \circ F_t^*(x) = x$ *for a.e.* $x \in \Omega$;

iv. *The function*

$$Z(t,x) = \nabla P(t, F_t(x)) \tag{6}$$

is a distributional solution of

$$\begin{array}{ll} \partial_t Z(t,x) = J\big[Z(t,x) - F(t,x)\big] & \text{in } [0,T) \times \Omega, \\ Z(0,x) = \nabla P_0(x) & \text{in } \Omega. \end{array} \tag{7}$$

Note that the sense in which (7) must be satisfied is

$$\int_0^T \int_\Omega [Z(t,x) \cdot \partial_t \varphi(t,x) + J\big[Z(t,x) - F(t,x)\big] \cdot \varphi(t,x)] dx dt + \int_\Omega \nabla P_0(x) \cdot \varphi(0,x) dx = 0 \tag{8}$$

for any $\varphi \in C_c^1([0,T) \times \Omega; I\!R^3)$.

In this paper we consider the case of singular initial data, i.e. when $\alpha_0 = \nabla P_{0\#}\chi$ is a singular measure. The dual problem in this case was studied by G. Loeper [17], and L. Ambrosio and W. Gangbo [4].

Note that equation (2) represents the fact that the dual flow $t \to \alpha_t$ is weakly (in the sense of distributions) transported by the dual velocity U defined by (4). The change of variable $X = \nabla P_t(x)$ is reversible (although there may not be enough smoothness to transport the dual space solutions back to physical space even in this case) if and only if ∇P_t^* pushes α_t forward to χ, one such situation being provided if $\alpha_t \ll \mathcal{L}^3$. In general, if ∇P_t^* is not necessarily the a.e. inverse of ∇P_t, then ∇P_t^* may not necessarily push α_t forward to χ, α_t may not necessarily be absolutely continuous with respect to $\mathcal{L}^3$ and the equation (4) in the dual-variable system *must* be generalized to (see [4])

$$U(t,X) = J\big[X - \bar{\gamma}_t(X)\big], \tag{9}$$

where $\bar{\gamma}_t$ is the barycentric projection onto α_t of the (unique) optimal Kantorovich plan [20] $\gamma_t := \big(\nabla P_t \times \mathrm{Id}\big)_{\#}\chi$ having α_t and χ as first and second marginals, respectively. It is defined by (see [4], [5])

$$\int_{I\!R^3} \xi(X) \cdot \bar{\gamma}_t(X) d\alpha_t(X) = \iint_{I\!R^3 \times \Omega} \xi(X) \cdot y d\gamma_t(X,y) \tag{10}$$

for all continuous $\xi : I\!R^3 \to I\!R^3$ of at most quadratic growth. Since χ is absolutely continuous with respect to $\mathcal{L}^3$, we deduce

$$\int_{I\!R^3} \xi(X) \cdot \bar{\gamma}_t(X) d\alpha_t(X) = \int_\Omega \xi(\nabla P_t(x)) \cdot x dx. \tag{11}$$

To justify generalizing (4) to (9), let us assume (1) has a smooth, stable solution (P,u). Set $Y := \nabla P$ and compute, for any $\zeta \in C_c^\infty(I\!R^3)$,

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} \int_{I\!R^3} \zeta(X) d\alpha_t(X) &= \int_\Omega \partial_t Y_t(x) \cdot \nabla\zeta(Y_t(x)) dx \\ &= -\int_\Omega \big[(u(t,x) \cdot \nabla) Y_t(x)\big] \cdot \nabla\zeta(Y_t(x)) dx + \int_\Omega J\big[Y_t(x) - x\big] \cdot \nabla\zeta(Y_t(x)) dx. \end{aligned}$$

Since u is divergence-free and $\nabla Y_t = \nabla^2 P_t$ is symmetric, we deduce that the first integral in the right hand side of the second equation of the above display is

$$\begin{aligned}\int_\Omega \big[\big(\nabla Y_t(x)\big)^t u(t,x)\big]\cdot\nabla\zeta(Y_t(x))dx &= -\int_\Omega \big[\big(\nabla Y_t(x)\big)^t \nabla\zeta(Y_t(x))\big]\cdot u(t,x)dx \\ &= -\int_\Omega \nabla\big[\zeta(Y_t(x))\big]\cdot u(t,x)dx = 0.\end{aligned}$$

As for the second integral, we have, due to $J^t = -J$, $Y_{t\#}\chi = \alpha_t$ and (11),

$$\begin{aligned}\int_\Omega J\big[Y_t(x)-x\big]\cdot\nabla\zeta(Y_t(x))dx &= -\int_\Omega Y_t(x)\cdot(J\nabla\zeta)(Y_t(x))dx + \int_\Omega x\cdot(J\nabla\zeta)(Y_t(x))dx \\ &= \int_{I\!R^3}(JX)\cdot\nabla\zeta(X)d\alpha_t(X) - \int_{I\!R^3}[J\bar\gamma_t(X)]\cdot\nabla\zeta(X)d\alpha_t(X).\end{aligned}$$

Thus,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{I\!R^3}\zeta(X)d\alpha_t(X) = \int_{I\!R^3} J\big[X-\bar\gamma_t(X)\big]\cdot\nabla\zeta(X)d\alpha_t(X), \tag{12}$$

which implies

$$\partial_t\alpha(t,X) + \nabla_X\cdot\big(J\big[X-\bar\gamma(t,X)\big]\alpha(t,X)\big) = 0 \tag{13}$$

is satisfied in the sense of distributions.

Existence of a solution for the dual problem (2), (3), (9), (5) for any probability measure α_0 with finite second moment was proved in [4].

In this paper we define weak Lagrangian solutions in physical space for any convex initial data P_0 (i.e. any initial probability measure $\alpha_0 = \nabla P_{0\#}\chi$), and prove existence of such solutions in the case of discrete measures. Moreover, we show conservation of geostrophic energy along the Lagrangian trajectories in the physical space.

One stand-out feature of Lagrangian solutions in physical space in the case of singular initial data is the absence of the Lebesgue measure preserving property for the flow map F_t, given in (ii) of Definition 1.1. Heuristically, since the solution of the dual problem (2), (3), (9), (5) does not provide enough information to separate the particle paths (in physical space) within the subgradient set $\partial P_t^*(X)\subset\Omega$ for $X\in\operatorname{supp}\alpha_t$, we define the flow map F_t which collapses the set $\partial P_t^*(X)$ to one point, its "α_t-barycenter" $\bar\gamma_t(X)\in\partial P_t^*(X)$, and thus the χ-measure carried by the set $\partial P_t^*(X)$ should now be concentrated at the point $\bar\gamma_t(X)$. Then the measure preserving property $F_{t\#}\chi=\chi$ is replaced by

$$F_{t\#}\chi = \mu_t \ \text{ for } \ t>0, \quad \text{where} \quad \mu_t := (\bar\gamma_t\circ\nabla P_t)_\#\chi. \tag{14}$$

Note that in the case when α_t is absolutely continuous with respect to the Lebesgue measure, we have $(\bar\gamma_t(X)\circ\nabla P_t)(x) = (\nabla P_t^*\circ\nabla P_t)(x) = x$ for a.e. $x\in\Omega$, and thus we recover the Lebesgue measure preserving property of F_t given in (ii) of Definition 1.1. We justify this definition of F_t in the case of singular data, and its generalized measure-preserving property (14), by giving, in Section 4.1 below, an explicit example of a sequence of Lagrangian solutions $(P^{(i)},F^{(i)})$ in the sense of Definition 1.1 with absolutely continuous initial measures $\alpha_0^i := \nabla P^{(i)}_{0\#}\chi$ in the dual space, such that the sequence $(P^{(i)},F^{(i)})$ converges weakly to a Lagrangian solution (P,F) in the sense of Definition 4.12 with a singular initial measure $\nabla P_{0\#}\chi = \delta_{z_0}$. We show that this limiting solution (P,F) satisfies the (generalized) measure-preserving property (14), yet does not satisfy (ii) of Definition 1.1. Thus, Definition 4.12 appears to be a natural extension of Definition 1.1

to the case of singular initial data if we expect to have stability of weak Lagrangian solutions in physical space. We note that in the case of Lagrangian solutions with absolutely continuous initial measure $\alpha_0 = \nabla P_{0\#}\chi$, a stability property was proved in [14]. For solutions with singular initial data α_0, stability is not proved at present, although the example described above suggests that some stability properties may be expected in this case too.

The rest of the paper is organized as following. In the next section we collect some results on the existence and properties of solutions in dual space. In Section 3 we prove that, under a mild time-regularity assumption, Eulerian solutions do not exist if the dual-space solutions have point masses at non-negligibly many times. This will advocate finding a suitable notion of weak Lagrangian solutions in physical case that makes sense in the general case of measure-valued solutions α in the dual space. This will be achieved in Section 4. There we will also prove that such solutions exist in the case where α_t are convex combinations of point masses, or equivalently, when P_0 is the maximum of finitely many affine functions. In Section 5 we will show that weak Lagrangian solutions in physical space can be translated into weak solutions of the problem in the dual space. This will lead to conservation of energy for weak Lagrangian solutions for the Semi-Geostrophic system.

2 Solutions in dual space

In this section we collect a number of results on the existence and some properties of solutions in dual space. Our main source is [4]. Before that, let us recall the definitions of some important objects. In the spirit of [5], one defines $AC^p(0,T;\mathcal{P}_2(I\!R^3))$ (for $1 \le p \le \infty$) as the set of all paths $\mu : [0,T] \ni t \to \mu_t \in \mathcal{P}_2(I\!R^3)$ for which there exists $\beta \in L^p(0,T)$ such that

$$W_2(\mu_s, \mu_t) \le \int_s^t \beta(\tau)d\tau \text{ for all } 0 \le s \le t \le T,$$

where W_2 is the quadratic Wasserstein distance [20]. The smallest of the functions β satisfying the inequality above is called the *metric derivative* of the curve μ, it is denoted by $|\mu'|$ and it satisfies [5] that

$$|\mu'|(t) = \lim_{s\to t} \frac{W_2(\mu_s, \mu_t)}{|s-t|} \quad \text{for a.e. } t \in (0,T).$$

There exists a Borel velocity $\mathbf{v} : (0,T) \times I\!R^3 \to I\!R^3$ transporting μ in the sense of distributions, i.e.

$$\partial_t \mu + \nabla_x \cdot (\mathbf{v}\mu) = 0 \text{ in } \mathcal{D}'((0,T) \times I\!R^3), \tag{15}$$

such that $\mathbf{v}(t,\cdot) \in L^2(\mu_t; I\!R^3)$ and $\|\mathbf{v}_t\|_{L^2(\mu_t;I\!R^3)} = |\mu'|(t)$ for a.e. $t \in (0,T)$. It turns out that this velocity (called "of minimal norm") minimizes $\|\mathbf{v}_t\|_{L^2(\mu_t;I\!R^3)}$ among all possible Borel velocities (i.e. satisfying (15)), and it can be uniquely selected (μ_t–a.e.) for a.e. $t \in (0,T)$ by requiring that it belong to the closure of $\nabla C_c^\infty(I\!R^3)$ in $L^2(\mu_t; I\!R^3)$, which is denoted [5] by $\mathcal{T}_{\mu_t}\mathcal{P}_2(I\!R^3)$ and called the *tangent space* to $\mathcal{P}_2(I\!R^3)$ at μ_t. Finally, if $F : \mathcal{P}_2(I\!R^3) \to I\!R$ is lower semicontinuos with respect to the topology induced by the Wasserstein distance W_2, then we can define the *subdifferential* of F at some $\mu \in \mathcal{P}_2(I\!R^3)$ as the set of all $\xi \in L^2(\mu; I\!R^3)$ such that

$$F(\nu) \ge F(\mu) + \iint_{I\!R^3\times I\!R^3} \xi(x)\cdot(y-x)\gamma(dx,dy) + o\big(W_2(\mu,\nu)\big)$$

for all $\nu \in \mathcal{P}_2(I\!R^3)$ and all the optimal transport plans [20] γ between μ and ν. This set is denoted by $\partial F(\mu)$ and its element of minimal $L^2(\mu; I\!R^3)$–norm is denoted by $\nabla_w F(\mu)$ and called [4] the *Wasserstein gradient* of F at μ.

Adapting to our context (by, for example, replacing J by $J^t = -J$ which still satisfies the required orthogonality property in [4]), a Hamiltonian ODE solution/trajectory for the lower semicontinuous "Hamiltonian" $H : \mathcal{P}_2(\mathbb{R}^3) \to \mathbb{R}$ is defined as follows.

Definition 2.1. *A curve $\mu \in AC^1(0,T;\mathcal{P}_2(\mathbb{R}^3))$ with the property that there exists a vector field $\mathbf{v} : (0,T) \times \mathbb{R}^3 \to \mathbb{R}^3$ such that*

$$(0,T) \ni t \to \|\mathbf{v}_t\|_{L^2(\mu_t;\mathbb{R}^3)} \text{ belongs to } L^1(0,T) \tag{16}$$

and

$$\partial_t \mu - \nabla \cdot (J\mathbf{v}\mu) = 0, \;\; \mu_0 = \tilde{\mu} \text{ as distributions,} \tag{17}$$

$$\mathbf{v}_t \in \mathcal{T}_{\mu_t}\mathcal{P}_2(\mathbb{R}^3) \cap \partial H(\mu_t) \text{ for a.e. } t \in (0,T), \tag{18}$$

is called a solution of the Hamiltonian ODE associated to the Hamiltonian H, starting at $\tilde{\mu}$.

Fix $\nu \in \mathcal{P}_2(\mathbb{R}^3)$ and define

$$H_\nu : \mathcal{P}_2(\mathbb{R}^3) \to \mathbb{R}, \quad H_\nu(\mu) = -\frac{1}{2}W_2^2(\mu,\nu), \tag{19}$$

which is, obviously, continuous. Then $\partial H_\nu(\mu)$ consists of all functions of the type $\bar{\gamma} - \mathrm{Id}$, where γ is any optimal plan between μ and ν [20], and $\bar{\gamma}$ is its barycentric projection onto μ, i.e. given by (see, e.g. [5])

$$\iint_{\mathbb{R}^3\times\mathbb{R}^3} y \cdot \xi(x)\gamma(dx,dy) = \int_{\mathbb{R}^3} \bar{\gamma}(x) \cdot \xi(x)\mu(dx)$$

for all $\xi \in C(\mathbb{R}^3;\mathbb{R}^3)$ of at most quadratic growth. In light of the fact that the y–marginal of γ is ν, the above definition implies $\|\bar{\gamma}\|_{L^2(\mu;\mathbb{R}^3)} \leq \|\mathrm{Id}\|_{L^2(\nu;\mathbb{R}^3)}$. Thus,

$$W_2^2(\mu,\nu) = \int_{\mathbb{R}^3} |x|^2\mu(dx) + \int_{\mathbb{R}^3} |y|^2\nu(dy) - 2\int_{\mathbb{R}^3} x \cdot \bar{\gamma}(x)\mu(dx) \geq \|\bar{\gamma} - \mathrm{Id}\|^2_{L^2(\mu;\mathbb{R}^3)}$$

for all such optimal plans, which implies

$$\|\nabla_w H_\nu(\mu)\|_{L^2(\mu;\mathbb{R}^3)} \leq 1 + \sqrt{-2H_\nu(\tilde{\mu})} =: a(\tilde{\mu}) \text{ whenever } W_2(\mu,\tilde{\mu}) \leq 1. \tag{20}$$

Then one combines Lemma 7.6 [4] and Theorem 7.4 [4] to obtain on $[0, 1/a(\tilde{\mu}) =: T]$ a solution $\mu \in AC^\infty(0,T;\mathcal{P}_2(\mathbb{R}^3))$ (Lipschitz curve) of the Hamiltonian ODE starting at $\tilde{\mu}$ and satisfying that it is $a(\tilde{\mu})$–Lipschitz and conservative, i.e. $[0,T] \ni t \to H_\nu(\mu_t)$ is constant. Thus, $a(\mu_T) = a(\tilde{\mu})$, which means that (20) is also satisfied with the same bounds if we replace $\tilde{\mu}$ by μ_T. Thus, in light of the same results from [4], we infer that we can extend the solution to $[T, 2T]$ (whereas it preserves the Lipschitz bound and conserves the Hamiltonian). By induction, we obtain:

Theorem 2.2. *For any $\tilde{\mu} \in \mathcal{P}_2(\mathbb{R}^3)$ there exists a solution $\mu \in AC^\infty(0,\infty;\mathcal{P}_2(\mathbb{R}^3))$ of (17) and (18), which starts at $\tilde{\mu}$, conserves $[0,\infty) \ni t \to H_\nu(\mu_t)$, and is globally $a(\tilde{\mu})$–Lipschitz.*

Let us now specialize to the case $\nu = \chi$ (defined in the introduction) and denote

$$H : \mathcal{P}_2(\mathbb{R}^3) \to \mathbb{R}, \quad H(\mu) = -\frac{1}{2}W_2^2(\mu,\chi). \tag{21}$$

The absolute continuity of χ with respect to $\mathcal{L}^3$ adds the benefit of the fact that there is a unique optimal plan now between μ and χ for every $\mu \in \mathcal{P}_2(\mathbb{R}^3)$, namely $\gamma_\mu = (\nabla\Phi \times \mathrm{Id})_\#\chi$, where $\nabla\Phi$ is the optimal map pushing χ forward to μ. Thus, we have

$$\partial H(\mu) \cap \mathcal{T}_\mu\mathcal{P}_2(\mathbb{R}^3) = \{\bar{\gamma}_\mu - \mathrm{Id}\} = \{\nabla_w H(\mu)\} \text{ for all } \mu \in \mathcal{P}_2(\mathbb{R}^3).$$

This means that Theorem 2.2 has the following:

Corollary 2.3. *Let $\alpha_0 \in \mathcal{P}_2(I\!R^3)$ be given. Then there exists a distributional solution $\alpha \in AC^\infty(0,\infty;\mathcal{P}_2(I\!R^3))$ for* (13) *with $\alpha(0) = \alpha_0$, where $\bar{\gamma}_t$ is the barycentric projection of the optimal plan between α_t and χ. In other words, the curve α is globally Lipschitz and satisfies the system* (2), (3), (9), (5) *in the sense of distributions in $(0,\infty) \times I\!R^3$. Also, the Hamiltonian energy $t \to H(\alpha_t)$ is conserved.*

Remark 2.4. *Since $[0,\infty) \ni t \to \alpha_t \in \mathcal{P}_2(I\!R^3)$ is continuous, we use Proposition 3.2 [19] to conclude that there is a family $P \in C([0,\infty); H^1(\Omega))$ of convex functions $P(t,\cdot)$ such that $\nabla_x P(t,\cdot) =: \nabla P_t$ pushes χ forward to α_t optimally for all $t \geq 0$.*

3 Lagrangian vs Eulerian

As of this date, the only weak Lagrangian solutions in the physical space have been shown to exist in the case $\nabla P_{0\#}\chi =: \alpha_0 \in L^1(\Omega)$. This is achieved in [14] by improving the L^q $(q > 1)$ result in [9]. The solution (P, F) constructed in these references satisfies $\nabla P_{t\#}\chi =: \alpha_t \in L^q(\Omega)$ for almost all $t \in (0,T)$. We work with solutions that may not satisfy any of these conditions, i.e. the measures α_t (for $t \in [0,T)$) may be singular. In particular, α_t can be Dirac measures $\delta_{z(t)}$ as in the example given in [14] (see also Section 4.1 below).

An interesting question is existence of Eulerian solutions in physical space. Recently, existence of such solutions for a class of initial data, where the conditions include the requirement that the support of the function $\alpha_0 = \nabla P_{0\#}\chi$ in the dual space be the whole space, was obtained by L. Ambrosio, M. Colombo, G. De Philippis, A. Figalli in two-dimensional periodic case [2] and in three-dimensional case with $\Omega = I\!R^3$ [3], based on the results of G. De Philippis and A. Figalli [13] on the regularity of solutions for the Monge-Ampere equation. Existence of Eulerian solutions in physical space for more general initial data $\alpha_0 \in L^q(\Omega)$, when the support of α_0 in the dual space may have a non-empty boundary, or if α_0 is a singular measure, is presently not known.

In this section, we show that Eulerian solutions in the physical space exhibiting some mild regularity in time cannot give rise to "very irregular" solutions in dual space. Let us begin by recalling the definition of a weak solution of (1).

Definition 3.1. *Let $u : [0,T) \times \Omega \to I\!R^3$ and $P : [0,T) \times \Omega \to I\!R$ satisfy $u \in L^1([0,T) \times \Omega; I\!R^3)$, $\nabla P \in L^\infty([0,T) \times \Omega) \cap C([0,T); L^1(\Omega))$, and $P(t,\cdot)$ is convex in Ω for every $t \in [0,\infty)$. The pair (P,u) is a weak Eulerian solution of (1) if*

$$\int_0^T \int_\Omega \{\nabla P(t,x) \cdot [\partial_t \phi(t,x) + \nabla\phi(t,x) u(t,x)] \tag{22}$$

$$+ J[\nabla P(t,x) - x] \cdot \phi(t,x)\} dxdt + \int_\Omega \nabla P_0(x) \cdot \phi(0,x) dx = 0$$

for any $\phi \in C_c^1([0,T) \times \Omega; I\!R^3)$, and

$$\int_0^T \int_\Omega u(t,x) \cdot \nabla\psi(t,x) dxdt = 0 \tag{23}$$

for any $\psi \in C_c^1([0,T) \times \overline{\Omega})$.

In what follows, we show that one cannot have weak Eulerian solutions exhibiting mild time regularity and spatial "flat parts" except, possibly, at negligibly many times. We begin with

Proposition 3.3 below, whose proof is a relatively straight-forward adaptation of Corollary 2.3 and Proposition 2.4 in [19]. The only difference is that in the said reference the set $\mathcal{O}$ is a subset of $(0,T)\times\Omega$ of full measure.

Before stating the result, we need some preliminary observations. If $X \in H^1(0,T;L^2(\Omega;I\!R^3))$, we denote by $\dot X \in L^2(0,T;L^2(\Omega;I\!R^3))$ its functional derivative, defined by

$$\lim_{h\to 0}\left\|\frac{X_{t+h}-X_t}{h}-\dot X_t\right\|_{L^2(\Omega;I\!R^3)} = 0 \quad \text{for} \quad \mathcal{L}^1-\text{a.e.} \quad t\in(0,T).$$

In the next lemma, we extend X to a map in $AC^2(I\!R;L^2(\Omega;I\!R^3))$ by setting $X_t = X_{0+}$ for $t\le 0$ and $X_t = X_{T-}$ for $t \ge T$.

Lemma 3.2. *Let $X \in H^1(0,T;L^2(\Omega;I\!R^3))$ and $\dot X$ be its functional derivative. Then*

$$\lim_{h\to 0}\int_0^T\int_\Omega\left|\frac{X_{t+h}x-X_tx}{h}-\dot X_tx\right|^2 dxdt = 0. \tag{24}$$

As a consequence, there exist sequences $h_k^+\to 0^+$, $h_k^-\to 0^-$ and a measurable subset $\mathcal{A}\subset I\!R\times\Omega$ such that $\mathcal{L}^4((I\!R\times\Omega)\setminus\mathcal{A})=0$ and

$$\lim_{k\to\infty}\frac{X_{t+h_k^+}x-X_tx}{h_k^+} = \lim_{k\to\infty}\frac{X_{t+h_k^-}x-X_tx}{h_k^-} = \dot X_tx \tag{25}$$

for all $(t,x)\in\mathcal{A}$.

The proof in [15] needs no modification. The philosophy behind this result is that, in some specified sense, $\dot X$ can be viewed as almost a classical pointwise time-derivative of X. Also, since $X \in H^1(0,T;L^2(\Omega;I\!R^3))$, we have that it admits a Borel representative. Equation (24) shows that $\dot X$ itself has that property. Throughout the paper we identify both X and $\dot X$ with their Borel representatives.

Proposition 3.3. *Let $X\in H^1(0,T;L^2(\Omega;I\!R^3))$ be such that $X_t=\nabla P_t$, where P_t is convex for all $t\in(0,T)$. Let $\mathcal{A}\subset(0,T)\times\Omega$ as in Lemma 3.2. Furthermore, let $\mathcal{O}\subset\mathcal{A}$ be a Borel set with $\mathcal{L}^4(\mathcal{O})>0$ and such that $\mathcal{L}^3([X_tx])>0$ for all $(t,x)\in\mathcal{O}$, where $[X_tx]:=\{y\in\Omega\ :\ X_ty=X_tx\}$. Then, there exists a Borel map $\mathbf{w}:(0,T)\times I\!R^3\to I\!R^3$ such that*

$$\dot X(t,x)=\mathbf{w}(t,X(t,x)) \text{ for } \mathcal{L}^4-\ \text{a.e. } (t,x)\in\mathcal{O}. \tag{26}$$

Proof: Let λ denote the $\mathcal{L}^4$–measure restricted to $\mathcal{O}$, $\Psi:\mathcal{O}\to(0,T)\times I\!R^3$ given by $\Psi(t,x)=\big(t,X(t,x)\big)$, and set $\vartheta:=\Psi_\#\lambda$. Denote by η the vector-measure whose density with respect to λ is $\dot X$, then set $\sigma:=\Psi_\#\eta$. Clearly, $\sigma\ll\vartheta$, which means there exists a Borel vector field $\mathbf{w}:(0,T)\times I\!R^3\to I\!R^3$ such that $d\sigma=\mathbf{w}d\vartheta$.
The disintegration theorem (see, for example, Theorem 5.3.1 [5]) applies to the Borel vector field Ψ and the measure λ. Thus, for ϑ–a.e. $(t,y)\in(0,T)\times I\!R^3$, there exists a unique Borel probability measure $\lambda_{t,y}$ on $\mathcal{O}$ such that the map $(t,y)\to\lambda_{t,y}(B)$ is Borel measurable for each Borel set $B\subset\mathcal{O}$. Furthermore, $\lambda_{t,y}\big(\Psi^{-1}(t,y)\big)=1$ for ϑ–a.e. $(t,y)\in(0,T)\times I\!R^3$ and

$$\iint_{\mathcal{O}} f(t,x)\,d\lambda(t,x)=\int_0^T\int_{I\!R^3}\Big(\int_{\Psi^{-1}(t,y)} f(t,x)\,\lambda_{t,y}(dt,dx)\Big)\vartheta(dt,dy)$$

for every Borel measurable $f : (0,T) \times \Omega \to [0,\infty]$. We showed in [19], Theorem 2.2, that

$$\mathbf{w}(t, X(t,x)) = \int_{\Psi^{-1}(t,X(t,x))} \dot{X}(t,z)\, d\lambda_{t,X(t,x)}(t,z) \text{ for } \lambda - \text{a.e. } (t,x) \in \mathcal{O}. \tag{27}$$

Note that $(t,z) \in \Psi^{-1}(t, X(t,x))$ is equivalent to $(t,z) \in \mathcal{O}$ and $X(t,z) = X(t,x)$, so we have $\Psi^{-1}(t, X(t,x)) = \mathcal{O} \cap [X_t x]$. Then we apply Proposition 2.4 in [19] to get that $\dot{X}(t,x) = \dot{X}(t,z)$ for *all* $(t,z) \in \mathcal{O} \cap [X_t x]$. According to (27), we get(26). QED.

Before proving the main theorem of this section, we need a measurability lemma.

Lemma 3.4. *Let $\alpha \in AC^2(0,T; \mathcal{P}_2(I\!R^3))$ for some $T > 0$. Then the set*

$$\mathcal{D}(\alpha) := \big\{(t,X) \in (0,T) \times I\!R^3 \ :\ X \text{ is an atom of } \alpha_t\big\} \tag{28}$$

is Borel.

Proof: Denote by C_c^+ the nonnegative cone of $C_c(I\!R^3)$. Consider, for every positive integer m and every $\xi \in C_c^+$, the set

$$\mathcal{D}_m^\xi := \left\{(t,X) \in (0,T) \times I\!R^3 \ :\ \int_{I\!R^3} \xi(Y) d\alpha_t(Y) \geq \frac{1}{m}\xi(X)\right\}.$$

Note that the absolute continuity in time of the left hand side of the above inequality [5] and the continuity in X of the right hand side imply that the difference is a continuous function of (t,X). Therefore, $\mathcal{D}_m^\xi$ is the nonnegative set of a continuous function, which makes it a closed subset of $(0,T) \times I\!R^3$. Thus,

$$\mathcal{D}_m := \left\{(t,X) \in (0,T) \times I\!R^3 \ :\ \int_{I\!R^3} \xi(Y) d\alpha_t(Y) \geq \frac{1}{m}\xi(X) \text{ for all } \xi \in C_c^+\right\}$$

is closed as well, by being an arbitrary intersection of closed sets. Since

$$\mathcal{D}(\alpha) = \cup_{m \geq 1} \mathcal{D}_m,$$

the proof is concluded. QED.

Now we prove the main result of this section: that weak (Eulerian) solutions in the physical space exhibiting some mild regularity in time cannot give rise to "very irregular" solutions in dual space.

Theorem 3.5. *Let (P,u) be a weak solution for the Semi-Geostrophic system in the physical space such that $\nabla P \in H^1(0,T; L^2(\Omega; I\!R^3))$. Then $\alpha_t := \nabla P_{t\#}\chi$ is atom-free for $\mathcal{L}^1$–a.e. $t \in (0,T)$.*

Proof: Set $\Psi(t,x) := (t, X(t,x))$. Since Ψ is a Borel map on $(0,T) \times \Omega$, due to Lemma 3.4 we infer that

$$\mathcal{O} := \Psi^{-1}\big(\mathcal{D}(\alpha)\big) \text{ is a Borel subset of } (0,T) \times \Omega.$$

One can see that

$$\mathcal{O} = \{(t,x) \in (0,T) \times \Omega \ :\ \mathcal{L}^3\big([X_t x]\big) > 0\}.$$

By Fubini's Theorem and by the convexity of the potentials whose gradients push χ forward to α_t, we infer that there exists a Borel subset $\mathcal{T} \subset (0,T)$ such that

$$\mathcal{O} = \cup_{t\in\mathcal{T}}\big(\{t\} \times \mathcal{O}_t\big), \tag{29}$$

where $\mathcal{O}_t$ is the union of all (at most countably many) convex subsets of Ω of positive $\mathcal{L}^3$ measure on each of which X_t is constant. Assume by contradiction that $\mathcal{L}^1(\mathcal{T}) > 0$. We further throw out of $\mathcal{T}$ the $\mathcal{L}^1$–negligible set of times at which

$$\int_\Omega \nabla\zeta(x)u(t,x)dx \neq \mathbf{0} \in I\!R^3 \tag{30}$$

for all $\zeta \in C_c^\infty(\Omega; I\!R^3)$ (via the separability of this space with respect to the sup norm), but we keep the notation $\mathcal{T}$ for the remaining subset. Consider $\mathcal{A}$ as the Borel subset of $(0,T)\times\Omega$ of full measure defined in Lemma 3.2, i.e. where the time pseudo-derivative of X in the sense of (25) exists. The set $\mathcal{A}\cap\mathcal{O}$ is a Borel set (which we still denote by $\mathcal{O}$) with $\mathcal{L}^4(\mathcal{O}) > 0$. According to Proposition 3.3, we infer there exists a Borel map $\mathbf{w} : (0,T)\times I\!R^3 \to I\!R^3$ such that (26) is satisfied. By taking $\phi(t,x) = \xi(t)\zeta(x)$ with $\xi \in C_c^\infty(0,T)$ and $\zeta \in C_c^\infty(\Omega; I\!R^3)$ in (22) we discover that

$$t \to \int_\Omega X(t,x)\cdot\zeta(x)dx \text{ is absolutely continuous}$$

and

$$\frac{d}{dt}\int_\Omega X(t,x)\cdot\zeta(x)dx = \int_\Omega \{X(t,x)\cdot[\nabla\zeta(x)u(t,x)] + J[X(t,x)-x]\cdot\zeta(x)\}dx$$

for a.e. $t\in(0,T)$ and every $\zeta \in C_c^\infty(\Omega; I\!R^3)$ (via the usual argument involving the separability of this space endowed with the sup–norm). Throwing out, if necessary, a negligible set of times, we conclude

$$\int_\Omega \dot{X}(t,x)\cdot\zeta(x)dx = \int_\Omega \{X(t,x)\cdot[\nabla\zeta(x)u(t,x)] + J[X(t,x)-x]\cdot\zeta(x)\}dx$$

for a.e. $t\in(0,T)$ and every $\zeta \in C_c^\infty(\Omega; I\!R^3)$. Choose such a t_0 that also lies in $\mathcal{T}$ and consider now only test functions $\zeta \in C_c^\infty(\omega_0; I\!R^3)$, where ω_0 is a connected component of $\mathcal{O}_{t_0}$ (which is a convex subset of Ω of positive $\mathcal{L}^3$–measure on which X_{t_0} is constant). Since $X_{t_0} \equiv \mathbf{c} \in I\!R^3$ in ω_0, we infer

$$\mathbf{w}(t_0,\mathbf{c})\cdot\int_{\omega_0}\zeta(x)dx = \mathbf{c}\cdot\int_{\omega_0}\nabla\zeta(x)u(t,x)dx + \int_{\omega_0}J[\mathbf{c}-x]\cdot\zeta(x)\}dx$$

for all $\zeta \in C_c^\infty(\omega_0; I\!R^3)$. Due to (30) the first term in the right hand side vanishes and since the equality holds for all $\zeta \in C_c^\infty(\omega_0; I\!R^3)$, we deduce

$$\mathbf{w}(t_0,\mathbf{c}) = J[\mathbf{c}-x] \text{ for a.e. } x\in\omega_0,$$

which contradicts the fact that ω_0 has nonempty interior. QED.

Remark 3.6. *Thus, in order to accommodate singular solutions in dual space, we see the need for defining Lagrangian solutions instead of Eulerian ones in the physical space. Whereas solutions in the dual space may come in any form or shape (from pure Dirac deltas to functions), only the absolutely continuous ones with respect to the Lebesgue measures have been so far known to give rise to Lagrangian solutions in the physical space [9], [14]. In the next section we discuss an extension to this notion and prove some existence results.*

4 Weak Lagrangian solutions in physical space for the case of singular initial data

In [9], Lagrangian solutions in the physical space with initial data $\nabla P_{0\#}\chi = \alpha_0 \in L^p(I\!R^3)$ $(p > 1)$, were constructed by the following procedure. First, time-stepping approximation from [10] combined with results of [1] yield existence of a solution (α, P) of the dual space system (2)–(4) with initial data (5), and a locally bounded map $\Phi : (0,T) \times I\!R^3 \to I\!R^3$ satisfying

$$\dot{\Phi} = U(\cdot, \Phi),\ \mathcal{L}^4 - \text{a.e. in } (0,T) \times \nabla P_0(\Omega),\quad \Phi(0,X) = X \text{ for } \alpha_0 - \text{a.e. } X \in I\!R^3, \tag{31}$$

such that

$$\alpha_t = \Phi_{t\#}\alpha_0 \quad \text{for all } t > 0.$$

There also exists a Borel map Φ^* such that Φ_t^* preserve $\mathcal{L}^3$ and $\Phi_t \circ \Phi_t^* = \mathrm{Id} = \Phi_t^* \circ \Phi_t$ a.e. in $I\!R^3$. The physical space flow is defined as

$$F_t := \nabla P_t^* \circ \Phi_t \circ \nabla P_0. \tag{32}$$

Then it is shown that (P, F) satisfies all the requirements of Definition 1.1.

In order to see what can be expected in the case of general initial data, we consider an example in which α_0 is a point mass.

4.1 An example: a limit of Lagrangian solutions, with initial measures weakly converging to a point mass

We discuss the case $\alpha_0 = \frac{4}{3}\pi\delta_{z_0}$ for some $z_0 \in I\!R^3$ by considering a sequence of approximations

$$\alpha_0^{(\varepsilon)} = \frac{1}{\varepsilon^3}\chi_{B_{z_0}(\varepsilon)}, \quad \text{where } \varepsilon > 0,$$

where χ_A denotes the indicator function of the set A. This is a version of the counterexample in [14]. For notational simplicity, we drop the requirements that α_t be probability measures and that $\mathcal{L}^3(\Omega) = 1$; thus, α_t are measures of total mass equal to $\mathcal{L}^3(\Omega)$. We fix the domain in physical space to be the ball $\Omega = B_1(0) \subset I\!R^3$, which means $\chi = \mathcal{L}^3|_{B_1(0)}$. It is easy to see that $\alpha_0 = \nabla P_{0\#}\chi$ and $\alpha_0^{(\varepsilon)} = \nabla P_{0\#}^{(\varepsilon)}\chi$, where

$$\begin{aligned} P_0(x) &= z_0 \cdot x, \\ P_0^{(\varepsilon)}(x) &= z_0 \cdot x + \varepsilon\frac{|x|^2}{2}. \end{aligned} \tag{33}$$

Following the calculations in [14], we find that for each $\varepsilon > 0$, one Lagrangian solution $(P^{(\varepsilon)}, F^{(\varepsilon)})$ in the sense of Definition 1.1 with initial data $P_0^{(\varepsilon)}$, can be obtained as follows. Let $z(t)$ be determined by solving $\dot{z}(t) = Jz(t),\ z(0) = z_0$; thus, $z(t) = L_1(t)z_0$, where

$$L_c(t) = \begin{pmatrix} \cos ct & -\sin ct & 0 \\ \sin ct & \cos ct & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{for } c \in I\!R. \tag{34}$$

Then

$$\begin{aligned} P_t^{(\varepsilon)}(x) &= z(t) \cdot x + \varepsilon\frac{|x|^2}{2}, \\ F_t^{(\varepsilon)}(x) &= L_c(t)x,\ \text{with } c = 1 - \frac{1}{\varepsilon}. \end{aligned} \tag{35}$$

We also note that the Legendre transform (over Ω) of $P_t^{(\varepsilon)}$ is

$$(P_t^{(\varepsilon)})^*(X) = \begin{cases} \dfrac{|X - z(t)|^2}{2\varepsilon}, & \text{if } X \in B_{z(t)}(\varepsilon); \\ |X - z(t)| - \dfrac{\varepsilon}{2}, & \text{if } X \notin B_{z(t)}(\varepsilon), \end{cases}$$

and the flow map in the dual variables is

$$\Phi_t^{(\varepsilon)}(X) = z(t) + M_c(t)(X - z_0), \text{ with } c = 1 - \frac{1}{\varepsilon},$$

where

$$M_c = \begin{pmatrix} \cos ct & -\sin ct & 0 \\ \sin ct & \cos ct & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{36}$$

Expression (35) was obtained by (32) using $(P_t^{(\varepsilon)})^*$, $\Phi_t^{(\varepsilon)}$ given above.

Now we take limits as $\varepsilon \to 0+$. Clearly,

$$\nabla P_t^{(\varepsilon)} \to \nabla P_t \quad \text{uniformly in } \Omega \quad \text{for all } t \geq 0, \tag{37}$$

where

$$P_t(x) = z(t) \cdot x \quad \text{for } x \in \Omega,\ t \geq 0. \tag{38}$$

Note that by taking $t = 0$ in (38) we obtain the function P_0 in (33). On the other hand, as far as $F_t^{(\varepsilon)}$ is concerned, we see that

$$F_0^{(\varepsilon)}(x) = x \quad \text{for all } x \in \Omega,\ \varepsilon > 0,$$

yet, for any $t > 0$ the sequence $F_t^{(\varepsilon)}(\cdot)$ does not converge even in a weak sense. Then we consider a limit of $F^{(\varepsilon)}$ as functions of (t, x), and we note that, as $\varepsilon \to 0+$,

$$F^{(\varepsilon)} \rightharpoonup 0 \quad \text{weakly in } L^2([0,T] \times \Omega; I\!R^3) \text{ for any } T > 0. \tag{39}$$

(39) determines the limit $F^{(\varepsilon)}$ only up to the set of points (t, x) of $\mathcal{L}^4$-measure zero. However, since the initial measure α_0 for the limiting problem is a point mass, it is natural to expect that the flow map for such problem is sensitive to changes on sets of $\mathcal{L}^4$-measure zero. In particular, we cannot take the flow map of the form $F \equiv 0$ for the limiting problem, since we require $F(0, \cdot) \equiv \mathrm{Id}$. Thus, in the following lemma we note some properties which are satisfied by any map $\hat{F} : [0,T) \times \Omega \to \Omega$ which is a weak limit of $F^{(\varepsilon)}$. We first note the following: for P_t in (38) we have

$$\alpha_t := \nabla P_{t\#}\chi = \frac{4}{3}\pi\delta_{z(t)},$$

and the Kantorovich optimal plan $\gamma_t := \big(\nabla P_t \times \mathrm{Id}\big)_{\#}\chi$ is defined by $\gamma_t(A \times B) = \delta_{z(t)}(A)\chi(B)$ for all Borel $A \subset I\!R^3$, $B \subset \Omega$. Then (10) implies that the barycentric projection of γ_t onto α_t is

$$\bar{\gamma}_t(z(t)) = 0, \tag{40}$$

which defines $\bar{\gamma}_t(X)$ for α_t-a.e. $X \in I\!R^3$.

Lemma 4.1. *Let $T > 0$ and $\hat{F} : [0, T) \times \Omega \to \Omega$ be a Borel map such that*

$$F^{(\varepsilon)} \rightharpoonup \hat{F} \quad \textit{weakly in } \; L^2([0, T] \times \Omega; I\!R^3) \textit{ for any } \; T > 0$$

as $\epsilon \to 0^+$, where $F^{(\varepsilon)}$ is defined by (35). Then

i. $\hat{F}_{t\#}\chi = \dfrac{4}{3}\pi\delta_0$ *for a.e. $t > 0$. In particular, the map $\hat{F}_t$ does not preserve $\mathcal{L}^3|_\Omega$ for a.e. t.*

ii. $(P, \hat{F})$ *satisfies property (14) for a.e. $t > 0$. Here P is defined by (38).*

Proof: Assertion (i) follows from (39) and Fubini Theorem.

To show (ii), we fix any $t > 0$ and note that (38), (40) imply $\bar{\gamma}_t(\nabla P_t(x)) = \bar{\gamma}_t(z(t)) = 0$ for every $x \in \Omega$, which implies $(\bar{\gamma}_t \circ \nabla P_t)_{\#}\chi = \frac{4}{3}\pi\delta_0$. Now (ii) follows from (i). QED.

Remark 4.2. *Lemma 4.1 suggests that for the case of singular initial measure α_0, the $\mathcal{L}^3$-measure preserving property of $F_{t\#}\chi = \chi$ of F_t should be replaced by (14) if we expect to have stability of weak Lagrangian solutions in physical space.*

Now we discuss a possible construction of a Lagrangian solution in physical space for the limiting initial data $P_0(x) = z_0 \cdot x$, i.e. with $\alpha_0 = \frac{4}{3}\pi\delta_{z_0}$, and compare it with the limit of $(P^{(\varepsilon)}, F^{(\varepsilon)})$ obtained above. We use P_t in (38) to see that $\alpha_t := \nabla P_{t\#}\chi = \frac{4}{3}\pi\delta_{z(t)}$. One can readily check that the property $\dot{z}(t) = Jz(t), \; z(0) = z_0$ implies that α is a weak solution of the SG in dual space, i.e. of the problem (2), (3), (9), (5).

In order to construct a flow map F_t in in physical space, let us go back to the definition of $F_t = \nabla P_t^* \circ \Phi_t \circ \nabla P_0$ from [9]. Since in the general case (when α_t is a measure) the place of ∇P_t^* is taken by the barycentric projection $\bar{\gamma}_t$ (see (13)), it is natural to ask if one can obtain a solution by putting

$$\begin{aligned} &F_0 \equiv \mathrm{Id}, \\ &F_t := \bar{\gamma}_t \circ \Phi_t \circ \nabla P_0 \quad \text{for } t > 0. \end{aligned} \tag{41}$$

Of course, for this function to be well-defined, one needs the Lagrangian flow in dual space Φ_t to be defined at z_0 for all $t \in [0, T]$. Thus, we solve

$$\dot{\Phi}(t, X) = J[\Phi(t, X) - \bar{\gamma}_t(\Phi(t, X))] \text{ with } \Phi(0, X) = X \tag{42}$$

for $X = z_0$. Using (40) we rewrite this as

$$\dot{\Phi}(t, z_0) = J\Phi(t, z_0) \text{ with } \Phi(0, z_0) = z_0.$$

Since $\dot{z}(t) = Jz(t), \; z(0) = z_0$, then uniqueness for the Cauchy problem associated to this ODE system implies

$$\Phi(t, z_0) = z(t).$$

Now, F_t is well-defined by (41), and explicitly is

$$F(t, x) = F_t(x) = \begin{cases} x, & \text{if } \; t = 0, \; x \in \Omega, \\ 0, & \text{if } \; t > 0, \; x \in \Omega. \end{cases} \tag{43}$$

From (39), (43) we conclude that, as $\varepsilon \to 0+$,

$$\begin{aligned} &F^{(\varepsilon)} \rightharpoonup F \quad \text{weakly in } \ L^2([0,T]\times\Omega;\mathbb{R}^3), \ \text{for any } \ T>0, \\ &F_0^{(\varepsilon)} \equiv F_0 \equiv \mathrm{Id} \quad \text{ in } \ \Omega, \quad \text{ for any } \varepsilon>0. \end{aligned} \tag{44}$$

We also notice that

$$F_{t\#}\chi = \frac{4}{3}\pi\delta_0 \quad \text{for all } \ t>0. \tag{45}$$

Moreover, we showed in the proof of Lemma 4.1 that

$$(\gamma_t \circ \nabla P_t)_{\#}\chi = \frac{4}{3}\pi\delta_0 \quad \text{for } \ t>0.$$

Then (45) implies that F_t satisfies the generalized measure-preserving property (14) for all $t>0$. Furthermore, since $\dot z(t) = Jz(t)$, $z(0)=z_0$, and $Z(t,x) := (\nabla P_t)(F(t,x)) = z(t)$, it follows that (P,F) satisfies equation (7) in the sense (8). Consequently, we have arrived to:

Lemma 4.3. *Let P be defined by (38). Then F defined by (41) has the explicit form (43), and (P,F) satisfies the following:*

(a) (P,F) is a weak limit (in the sense of (37), (44)) as $\varepsilon \to 0^+$, of a family of Lagrangian solutions $(P^{(\varepsilon)}, F^{(\varepsilon)})$ in the sense of Definition 1.1;

(b) (P,F) satisfies equation (7) in the sense (8);

(c) For all $t>0$, (P,F) satisfies property (14), which takes the form (45).

In other words, the limit (P,F) satisfies the properties of Lagrangian solution (as in Definition 1.1) except for the $\mathcal{L}^3$-measure preserving property of F_t, which is replaced by (14). An important remaining issue, the continuity of $t \to F_t(\cdot)$, will be clarified in the next section.

Remark 4.4. *The above computations of the Lagrangian flow in dual space highlight the difficulty in applying the theory of regular Lagrangian flows [1] to the case of singular initial measures α_0. One can extend $\bar\gamma_t$, defined by (40), to all $\mathbb{R}^3$ by using that $P_t^*(X) = |X - z(t)|$ is differentiable for $X \neq z(t)$, defining*

$$\bar\gamma_t(X) := \nabla P_t^*(X) = \frac{X - z(t)}{|X - z(t)|} \ \text{if } X \neq z(t).$$

Then one can easily check that $U(t,X) := J[X - \bar\gamma_t(X)]$ satisfies the conditions in [1], therefore a unique solution to the ODE (42) exists and is unique for Lebesgue-a.e. $X \in \mathbb{R}^3$ such that Φ_t preserves $\mathcal{L}^3$. However, when α_0 is singular, the set $\{X \in \mathbb{R}^3 \ : \ \Phi(t,X) \text{ exists and is unique}\}$ may not contain all (if any) of the support of α_0. In the case $\alpha_0 = \delta_{z_0}$, the solution $\Phi(\cdot, z_0)$ can be found explicitly by solving the ODE system, and this defines $\Phi(\cdot, X)$ for α_0-a.e. $X \in \mathbb{R}$. In general, for singular measures α_0, it is not clear how to find a suitable Lagrangian flow $\Phi(t,X)$ in the dual space that is defined for α_0-a.e X.

4.2 Definition of Lagrangian solutions with singular initial data, and existence results

In order to see the motivation for Definition 4.12 of Lagrangian solutions for the case of singular initial data given below, we first study properties of flows in physical space given by (41) in the case when $\alpha_0 = \nabla P_{0\#}\chi$ is possibly a singular measure, under the following assumptions:

$$
\begin{aligned}
&(i) \quad P_0 \in H^1(\Omega) \text{ is convex, } \alpha_0 = \nabla P_{0\#}\chi \in \mathcal{P}_2(I\!R^3);\\
&(ii) \quad \text{There exists a Borel map } \Phi : [0,T] \times I\!R^3 \to I\!R^3 \text{ such that (42) holds in the}\\
&\qquad \text{integral sense for } \alpha_0\text{-a.e. } X \in I\!R^3;\\
&(iii) \quad \text{The family of measures } \alpha_t := \Phi_{t\#}\alpha_0 \text{ for } t \in [0,T) \text{ is a solution in } \mathcal{D}'((0,T) \times I\!R^3)\\
&\qquad \text{for the equation (2) with } U \text{ defined by (9), (10), where } \gamma_t \text{ is the unique optimal}\\
&\qquad \text{transport plan between } \alpha_t \text{ and } \chi. \text{ This also defines a convex } P_t \text{ in } \Omega \text{ (for each } t) \text{ such}\\
&\qquad \text{that } \gamma_t := \big(\nabla P_t \times \mathrm{Id}\big)_{\#}\chi;\\
&(iv) \quad \alpha \in AC^\infty(0,\infty;\mathcal{P}_2(I\!R^3)).
\end{aligned}
\tag{46}
$$

Note that assumptions (46) say that there exists a Lagrangian solution of the dual problem with initial data P_0. In the case when $P_0 \in W^{1,\infty}(\Omega)$ and $\alpha_0 = \nabla P_{0\#}\chi \in L^p(I\!R^3)$ for $p \in (1,\infty]$ existence of Φ_t such that (46) is satisfied is shown in [9].

Below it is convenient to work with a Borel representative of the barycentric projection $\bar{\gamma}$. That is why we prove the following lemma:

Lemma 4.5. *There exists a Borel measurable function defined on $(0,T) \times I\!R^3$ which for $\mathcal{L}^1$-a.e. $t \in (0,T)$ coincides with $\bar{\gamma}(t,X)$ for $\alpha(t,\cdot)$-a.e. $X \in I\!R^3$.*

Proof: Let $\vartheta \in \mathcal{P}\big((0,T) \times I\!R^3 \times \Omega\big)$ be the Borel probability given by

$$\int_0^T \int_{I\!R^3} \int_\Omega \varphi(t,X,y)\vartheta(dt,dX,dy) = \frac{1}{T}\int_0^T \int_{I\!R^3} \int_\Omega \varphi(t,X,y)\gamma(t,dX,dy)dt$$

for any continuous and bounded φ. Since $\alpha(t,\cdot)$ is the X-marginal of $\gamma(t,\cdot,\cdot)$, the (t,X) marginal of ϑ is $\tilde{\vartheta} \in \mathcal{P}\big((0,T) \times I\!R^3\big)$ given by

$$\int_0^T \int_{I\!R^3} \zeta(t,X)\tilde{\vartheta}(dt,dX) = \frac{1}{T}\int_0^T \int_{I\!R^3} \zeta(t,X)\alpha(t,dX)dt.$$

Thus, by disintegrating ϑ we get

$$\int_0^T \int_{I\!R^3} \int_\Omega \varphi(t,X,y)\vartheta(dt,dX,dy) = \frac{1}{T}\int_0^T \int_{I\!R^3} \left(\int_\Omega \varphi(t,X,y)\vartheta(t,X;dy)\right)\alpha(t,dX)dt,$$

where $\vartheta(t,X;\cdot)$ are Borel probabilities on Ω such that the map $(t,X) \to \vartheta(t,X;B)$ is Borel for any Borel set $B \subset \Omega$. In particular, the maps

$$(t,X) \to \int_\Omega \mathbf{f}(y)\vartheta(t,X;dy)$$

are Borel for all $\mathbf{f} \in C_b\big(\Omega; I\!R^3\big)$. By taking $\varphi(t,X,y) = u(t)\xi(X)\cdot y$ and using (10), we conclude that

$$\bar{\gamma}(t,X) = \int_\Omega y\vartheta(t,X;dy) \text{ for } \mathcal{L}^1-\text{a.e. } t \in (0,T) \text{ and } \alpha(t,\cdot)-\text{a.e. } X \in I\!R^3.$$

This finishes the proof. QED.

Remark 4.6. *In light of Lemma 4.5 and equation* (9)*, we see immediately that there exists a Borel measurable function defined on* $(0,T) \times I\!R^3$ *which for* $\mathcal{L}^1$*-a.e.* $t \in (0,T)$ *coincides with* $U(t,X)$ *for* $\alpha(t,\cdot)$*-a.e.* $X \in I\!R^3$*.*

Remark 4.7. *In view of the above lemma, from now on we shall use the notation* $\bar{\gamma}$ *to denote the Borel representative of the barycentric projection defined by* (10)*. Likewise, by following the same proof, it is easy to see that the barycentric projection of* $\big(\mathrm{Id} \times \nabla P_t\big)_{\#}\chi$ *onto its first marginal (namely,* χ*) can also be extended to a Borel map from* $(0,T) \times \Omega$ *into* $I\!R^3$*. Since this coincides with* $\nabla P_t(x)$ *for* $\mathcal{L}^4$*-a.e.* $(t,x) \in (0,T) \times \Omega$*, we shall assume in the remainder of the paper that* ∇P *is Borel measurable in both variables.*

We also note the following:

Lemma 4.8. *For* $(t,X) \in [0,T) \times I\!R^3$ *denote*

$$\Omega_{t,X} := \{x \in \Omega \;:\; \nabla P_t(x) \;\; \textit{exists and} \; \nabla P_t(x) = X\}.$$

Then

$$\bar{\gamma}_t(X) \in \Omega_{t,X} \;\; \textit{for } \alpha(t,\cdot)\textit{-a.e. } X \in I\!R^3. \tag{47}$$

In particular,

$$(\nabla P_t \circ \bar{\gamma}_t)(X) = X \;\; \textit{for every } t \in [0,T) \textit{ and } \alpha(t,\cdot)\textit{-a.e. } X \in I\!R^3. \tag{48}$$

Proof: Fix $t \in [0,T)$. Since $\gamma_t = (\nabla P_t \times \mathrm{Id})_{\#}\chi$ and $\pi_{1\#}\gamma_t = \alpha_t$, by disintegrating γ_t as in Theorem 5.3.1 [5], we get

$$\gamma_t = \int_{I\!R^3} \gamma_{t,X}\, d\alpha_t(X),$$

where $\gamma_{t,X}$ is a family of probability measures on Ω such that the map $X \to \gamma_{t,X}(B)$ is Borel for any Borel set $B \subset \Omega$, and

$$\gamma_{t,X}(I\!R^3 \setminus \Omega_{t,X}) = 0 \;\; \text{for } \alpha(t,\cdot)\text{-a.e. } X \in I\!R^3.$$

From (10) and the disintegration, we get

$$\bar{\gamma}_t(X) = \int_{\Omega} y\, d\gamma_{t,X}(y) = \int_{\Omega_{t,X}} y\, d\gamma_{t,X}(y) \;\; \text{for } \alpha(t,\cdot)\text{-a.e. } X \in I\!R^3.$$

The convexity of $P_t(\cdot)$ implies that $\Omega_{t,X}$ is a convex set (of dimension $k(t,X) \in \{0,1,2,3\}$) for every X such that $\Omega_{t,X} \neq \emptyset$. Thus,

$$\bar{\gamma}_t(X) \in \Omega_{t,X} \;\; \text{for } \alpha(t,\cdot)\text{-a.e. } X \in I\!R^3,$$

and for such X we get $(\nabla P_t \circ \bar{\gamma}_t)(X) = X$. QED.

The continuity of F in time was proved in [9], [14] in the case of absolutely continuous α_t. In general we cannot expect that, which is clear from the structure (41) of F_t, especially looking at the example when P_0 is linear, i.e. $\alpha_0 = \delta_{z_0}$ discussed above: streamlines in the physical space are concentrated on the barycenters $\bar{\gamma}_t(X)$ of the sets $\partial P_t^*(X)$ where $X \in \mathrm{supp}\,(\alpha_t)$. Thus it is natural to expect continuity of F_t "relative to ∇P_t". Indeed, we have the following:

Proposition 4.9. *Assume that (46) hold. Then the map F defined in* (41) *satisfies*

$$\lim_{t\to t_0}\int_\Omega \xi(\nabla P_t\circ F_t(x))\cdot F_t(x)dx=\int_\Omega \xi(\nabla P_{t_0}\circ F_{t_0}(x))\cdot F_{t_0}(x)dx \tag{49}$$

for all $t_0\in[0,\infty)$ (which we define as meaning $t\to 0^+$ if $t_0=0$) and all $\xi\in C_c(I\!R^3;I\!R^3)$.

Furthermore,

$$\lim_{t\to t_0}\int_\Omega \xi(\nabla P_{t_0}\circ F_{t_0}(x))\cdot F_t(x)dx=\int_\Omega \xi(\nabla P_{t_0}\circ F_{t_0}(x))\cdot F_{t_0}(x)dx, \tag{50}$$

and

$$\lim_{t\to 0^+}\int_\Omega \xi(\nabla P_0(x))\cdot F_t(x)dx=\int_\Omega \xi(\nabla P_0(x))\cdot x\,dx \tag{51}$$

for all $t_0\in[0,\infty)$ and all $\xi\in C_c(I\!R^3;I\!R^3)$.

Proof: The (unique) optimal transport plan between its X-marginal α_t and it y-marginal χ is $\gamma_t=(\nabla P_t\times \mathrm{Id})_{\#}\chi$. Let $t_0\in[0,\infty)$. Since $W_2(\alpha_t,\alpha_{t_0})\to 0$ as $t\to t_0$ (which we define as meaning $t\to 0^+$ if $t_0=0$), by stability of optimal plans [5], we infer that γ_t converges to γ_{t_0} in $\mathcal{P}_2(I\!R^3\times\Omega)$. We use the definition (41) of F_t, the fact that $\nabla P_{0\#}\chi=\alpha_0$ and $\Phi_{t\#}\alpha_0=\alpha_t$, and (10) to get for $t>0$

$$\int_\Omega \xi(\Phi_t\circ\nabla P_0(x))\cdot F_t(x)dx=\iint_{I\!R^3\times I\!R^3}\xi(X)\cdot y\gamma_t(dX,dy).$$

For $t=0$, we use $F_0=\mathrm{Id}$, $\Phi_0=\mathrm{Id}$, $\nabla P_{0\#}\chi=\alpha_0$ and (10) to get the equality displayed above. Now, by the continuity of $t\to\gamma_t$ proved above, we deduce for $t_0\in[0,\infty)$

$$\lim_{t\to t_0}\int_\Omega \xi(\Phi_t\circ\nabla P_0(x))\cdot F_t(x)dx=\int_{I\!R^3}\int_\Omega \xi(X)\cdot y\gamma_{t_0}(dX,dy)=\int_\Omega \xi(\Phi_{t_0}\circ\nabla P_0(x))\cdot F_{t_0}(x)dx. \tag{52}$$

Next we note that $(\Phi_t\circ\nabla P_0)_{\#}\chi=\alpha_t$. Combining this with (48), we obtain for every $t\in(0,T)$:

$$\nabla P_t\circ\bar{\gamma}_t\circ\Phi_t\circ\nabla P_0(x)=\Phi_t\circ\nabla P_0(x)\quad\text{for a.e. } x\in\Omega.$$

Then using (41) we obtain for every $t\in(0,T)$

$$\nabla P_t\circ F_t(x)=\Phi_t\circ\nabla P_0(x)\quad\text{for a.e. } x\in\Omega. \tag{53}$$

For $t=0$, we deduce (53) from $F_0=\mathrm{Id}$, $\Phi_0=\mathrm{Id}$. Now (52) implies (49).

Furthermore, (31) implies

$$\Phi_t(X)=\Phi_{t_0}(X)+\int_{t_0}^t U(s,\Phi_s(X))ds \text{ for } \alpha_0-\text{a.e. } X\in I\!R^3 \text{ and all } 0\le t_0\le t.$$

This yields

$$\begin{aligned}|\xi(\Phi_t(X))-\xi(\Phi_{t_0}(X))|^2 &\le \|\nabla\xi\|_\infty^2\Big(\int_{t_0}^t |U(s,\Phi_s(X))|ds\Big)^2\\ &\le (t-t_0)\|\nabla\xi\|_\infty^2\int_{t_0}^t |U(s,\Phi_s(X))|^2ds\end{aligned}$$

for α_0-a.e. $X \in I\!R^3$. Note that $(0,T) \times I\!R^3 \ni (t,X) \to U(t, \Phi_t(X))$ is Borel (as composition of Borel maps). Next we use $\nabla P_{0\#}\chi = \alpha_0$ and $\Phi_{s\#}\alpha_0 = \alpha_s$ to get

$$\|\xi \circ \Phi_t \circ \nabla P_0 - \xi \circ \nabla \Phi_{t_0} \circ \nabla P_0\|_{L^2(\Omega;I\!R^3)} \leq \sqrt{t-t_0}\|\nabla\xi\|_\infty \left(\int_{t_0}^t \int_{I\!R^3} |U(s,Y)|^2 d\alpha_s(Y) ds\right)^{1/2}.$$

Then, using (53) we obtain

$$\|\xi \circ \nabla P_t \circ F_t - \xi \circ \nabla P_{t_0} \circ F_{t_0}\|_{L^2(\Omega;I\!R^3)} \leq \sqrt{t-t_0}\|\nabla\xi\|_\infty \left(\int_{t_0}^t \int_{I\!R^3} |U(s,Y)|^2 d\alpha_s(Y) ds\right)^{1/2}.$$

Since $\|U_t\|_{L^2(\alpha_t;I\!R^3)} \in L^\infty(0,\infty)$ and F is bounded uniformly in time-space, we use (49) and the inequality displayed above to deduce (50). For $t_0 = 0$ we use the fact that $F_0(x) = x$ for a.e. x to get (51) from (50). QED.

Remark 4.10. *In the case $\alpha_t \ll \mathcal{L}^3$ and $1 < p < \infty$, note that the strong L^p-continuity in time $\|F_t - F_{t_0}\|_{L^p(\Omega)} \to 0$ as $t \to t_0$ is equivalent to weak L^p-continuity, since all the maps F_t share the same L^p-norm (by being measure-preserving). In fact, we can further prove that when $\alpha_t \ll \mathcal{L}^3$ for all $t \geq 0$, (50) is equivalent to this weak L^p-continuity. Thus, our result above simply generalizes the L^p-continuity obtained in [9].*

Proof: Let $1 < q < \infty$ be such that $1/p + 1/q = 1$. First, one can easily check that

$$\zeta \in L^q(\alpha_{t_0}; I\!R^3) \Leftrightarrow \zeta \circ \Phi_{t_0} \circ \nabla P_0 \in L^q(\Omega; I\!R^3),\ \xi \in L^q(\Omega; I\!R^3) \Leftrightarrow \xi \circ \nabla P_0^* \circ \Phi_{t_0}^* \in L^q(\alpha_{t_0}; I\!R^3). \quad (54)$$

Thus, pick $\xi \in L^q(\Omega; I\!R^3)$ and a sequence $\{\xi_n\}_n \subset C_c^1(I\!R^3; I\!R^3)$ which converges to $\xi \circ \nabla P_0^* \circ \Phi_{t_0}^*$ strongly in $L^q(\alpha_{t_0}; I\!R^3)$, i.e. $\xi_n \circ \Phi_{t_0} \circ \nabla P_0$ converges strongly to ξ in $L^q(\Omega; I\!R^3)$. We have

$$\left|\int_\Omega \xi \cdot (F_t - F_{t_0}) dx\right| \leq \left|\int_\Omega \xi_n \circ \Phi_{t_0} \circ \nabla P_0 \cdot (F_t - F_{t_0}) dx\right| + C(p,\Omega)\|\xi_n \circ \Phi_{t_0} \circ \nabla P_0 - \xi\|_{L^q(\Omega;I\!R^3)}.$$

We choose n_0 sufficiently large such that the last term in the right hand side is sufficiently small. Then use (50) for $\xi = \xi_{n_0}$ to conclude that (50) implies the weak L^p-continuity for $t \to F_t$ at t_0. A similar argument can be used to prove the converse.

QED.

Proposition 4.9 motivates the following

Definition 4.11. *Let $P : [0,\infty) \times \Omega \to I\!R$ be such that $P \in C([0,\infty); H^1(\Omega))$ and $P(t,\cdot)$ is convex in Ω for each $t \in [0,\infty)$. A map $F : [0,\infty) \times \Omega \to \Omega$ is called weakly P–continuous if (50), (51) hold for all $t_0 \in (0,\infty)$ and all $\xi \in C_c(I\!R^3; I\!R^3)$.*

Furthermore, we note that considering the example of initial data in the dual space being a linear combination of Dirac masses (see also Proposition 4.14 below), the solution α_t at each time t is also a linear combination of Dirac masses concentrated in the time-dependent location, it is not clear whether distinct initial location of Dirac masses should imply that their locations are distinct at all times. Thus we cannot expect existence of the map F^* as in Definition 1.1(iii).

In light of the above, we are ready to generalize Definition 1.1 as follows:

Definition 4.12. *Let $P : [0,\infty) \times \Omega \to I\!R$ be such that $P \in C([0,\infty); H^1(\Omega))$ and $P(t,\cdot)$ is convex in Ω for each $t \in [0,\infty)$. Let $F : [0,\infty) \times \Omega \to \Omega$ be a weakly P-continuous Borel map. Denote by $\bar{\gamma}_t$ the barycentric projection of the measure $\gamma_t := \left(\nabla P_t \times \mathrm{Id}\right)_{\#}\chi$, defined in* (10)*, and set*

$$\alpha_t := \nabla P_{t\#}\chi, \quad \mu_t := \bar{\gamma}_{t\#}\alpha_t. \tag{55}$$

Then the pair (P,F) is called a weak Lagrangian solution of (1) in $[0,T) \times \Omega$ if

i. $F(0,x) = x$ and $P(0,x) = P_0(x)$ for all $x \in \Omega$;

ii. for any $t > 0$ the mapping $F_t = F(t,\cdot) : \Omega \to \Omega$ satisfies $F_{t\#}\chi = \mu_t$ and $F_{t\#}\mu_0 = \mu_t$;

iii. The function $Z : (0,T) \times \Omega \to I\!R^3$ defined by

$$Z_t = \nabla P_t \circ F_t \tag{56}$$

lies, along with F, in $L^\infty(0,T; L^2(\Omega; I\!R^3))$ and is a distributional solution of (7) in the sense of (8).

The following proposition gives sufficient conditions for the existence of weak Lagrangian solutions in physical space. Note that these conditions are exactly what was proved in [9] in the case $\alpha_t \in L^q(I\!R^3)$.

Proposition 4.13. *Assume that (46) hold. Define F by* (41)*. Then the pair (P,F) is a weak Lagrangian solution in physical space in the sense of Definition 4.12.*

Furthermore, if there exists a Borel map $\Phi^* : [0,T] \times I\!R^3 \to I\!R^3$ such that $\Phi^*_{\#}\alpha_t = \alpha_0$ for all $t \in [0,T)$, then

$$\begin{array}{l}\textit{there exists a Borel mapping } F^* : [0,T] \times \Omega \to \Omega \textit{ satisfying} \\ F^*_{t\#}\mu_t = \mu_0 \textit{ and } F^*_t \circ F_t = \mathrm{Id}\ \mu_0\textit{-a.e., and } F_t \circ F^*_t = \mathrm{Id}\ \mu_t\textit{-a.e.}\end{array} \tag{57}$$

Proof: Proposition 4.9 implies that F is weakly P-continuous.

From (48) and since $\nabla P_{t\#}\chi = \alpha_t$, we have:

$$\nabla P_t \circ \bar{\gamma}_t \circ \nabla P_t(x) = \nabla P_t(x) \text{ for every } t \in (0,T) \text{ and } \chi - \text{a.e. } x \in \Omega. \tag{58}$$

Now, in order to prove (i) of Definition 4.12 for F, we find that

$$\bar{\gamma}_0 \circ (\Phi_0 \circ \nabla P_0 \circ \bar{\gamma}_0) \circ \nabla P_0(x) = \bar{\gamma}_0 \circ \nabla P_0(x) \text{ for } \chi\text{-a.e. } x \in \Omega,$$

due to (58) for $t = 0$ and the hypothesis on Φ_0. Since $\mu_0 = \bar{\gamma}_0 \circ \nabla P_{0\#}\chi$, then (i) follows.

To prove (ii) we check that for $t > 0$

$$F_{t\#}\chi \;=\; (\bar{\gamma}_t \circ \Phi_t \circ \nabla P_0)_{\#}\chi = (\bar{\gamma}_t \circ \Phi_t)_{\#}\alpha_0 = \bar{\gamma}_{t\#}\alpha_t = \mu_t,$$

where we used the definitions (55), and that $\Phi_{t\#}\alpha_0 = \alpha_t$. Similarly, for $t > 0$

$$\begin{aligned} F_{t\#}\mu_0 &= (\bar{\gamma}_t \circ \Phi_t \circ \nabla P_0)_{\#}\mu_0 \\ &= (\bar{\gamma}_t \circ \Phi_t \circ \nabla P_0 \circ \bar{\gamma}_0)_{\#}\alpha_0 \\ &= (\bar{\gamma}_t \circ \Phi_t)_{\#}\alpha_0 \\ &= \bar{\gamma}_{t\#}\alpha_t = \mu_t, \end{aligned}$$

where we used in the second line that $(\nabla P_0 \circ \bar{\gamma}_0)_{\#}\alpha_0 = \alpha_0$ by (48) for $t = 0$.

Next we prove (iii) of Definition 4.12. We first note that Z and F are Borel (therefore, Lebesgue) measurable as compositions of Borel maps (see Lemma 4.5). By the definition of $\bar{\gamma}$, we have

$$\left| \int_{I\!R^3} \xi(X) \cdot \bar{\gamma}_t(X) d\alpha_t(X) \right| \leq \left(\int_{I\!R^3} |\xi(X)|^2 d\gamma_t(X,y) \right)^{1/2} \left(\int_{\Omega} |y|^2 d\gamma_t(X,y) \right)^{1/2}$$
$$= \|\xi\|_{L^2(\alpha_t;I\!R^3)} \left(\int_{\Omega} |y|^2 dy \right)^{1/2}.$$

It follows that for all $t \in (0,T)$ we have

$$\bar{\gamma}_t \in L^2(\alpha_t; I\!R^3) \text{ with } \|\bar{\gamma}_t\|_{L^2(\alpha_t;I\!R^3)} \leq R_0(\mathcal{L}^3(\Omega))^{1/2}, \tag{59}$$

where $0 < R_0 < \infty$ is large enough such that $\Omega \subset B(0, R_0)$. Thus, there exists $C \in I\!R$ independent of t such that

$$\int_{\Omega} |F_t(x)|^2 dx = \int_{I\!R^3} |\bar{\gamma}_t \circ \Phi_t|^2 d\alpha_0 = \int_{I\!R^3} |\bar{\gamma}_t|^2 d\alpha_t = \|\bar{\gamma}_t\|^2_{L^2(\alpha_t;I\!R^3)} \leq C.$$

Also, using that $F_{t\#}\chi = \mu_t$ as proved above, and also using (48), we have for $Z_t = \nabla P_t \circ F_t$:

$$\int_{\Omega} |Z_t(x)|^2 dx = \int_{\Omega} |\nabla P_t(y)|^2 d\mu_t(y) = \int_{\Omega} |\nabla P_t(\bar{\gamma}_t(X))|^2 d\alpha_t(X)$$
$$= \int_{I\!R^3} |X|^2 d\alpha_t(X) = \int_{\Omega} |\nabla P_t(x)|^2 dx \leq \tilde{C},$$

where $\tilde{C} < \infty$ is a constant coming from the fact that $P \in C([0,T]; H^1(\Omega))$. Thus, both Z and F belong to $L^\infty(0,T; L^2(\Omega; I\!R^3))$. Next we note that assumption (ii) of (46) implies that for for χ-a.e. $x \in \Omega$, the function $\dot{\Phi}(\cdot, \nabla P_0(x))$ is a weak solution of the problem

$$\dot{\Phi}(t, \nabla P_0(x)) = J[\Phi(t, \nabla P_0(x)) - \bar{\gamma}_t(\Phi(t, \nabla P_0(x)))] \quad \text{with} \quad \Phi(0, \nabla P_0(x)) = \nabla P_0(x). \tag{60}$$

From (53), for each $t \in [0,T)$ the equality $Z_t(x) = \Phi_t \circ \nabla P_0$ holds for a.e. $x \in \Omega$. Thus using the integrability of Z and F proved above, using a function $\varphi \in C_0^1([0,T) \times I\!R^3)$ in the weak form of (60) and integrating with respect to x, we get (8).

Finally, in order to prove (57) under the additional assumption of the existence of Φ^*, we set

$$F_0^*(x) = x, \quad F_t^*(x) := \bar{\gamma}_0 \circ \Phi_t^* \circ \nabla P_t \text{ for } t > 0.$$

Then property (57) is obvious for $t = 0$. Thus we fix $t > 0$ and and compute

$$F^*_{t\#}\mu_t = (\bar{\gamma}_0 \circ \Phi_t^* \circ \nabla P_t \circ \bar{\gamma}_t)_{\#}\alpha_t = (\bar{\gamma}_0 \circ \Phi_t^*)_{\#}\alpha_t = \bar{\gamma}_{0\#}\alpha_0 = \mu_0. \tag{61}$$

To finish proving (57), note that $F_t^* \circ F_t(y) = y$ for μ_0-a.e. $y \in \Omega$ amounts to

$$F_t^* \circ F_t \circ \bar{\gamma}_0 \circ \nabla P_0(x) = \bar{\gamma}_0 \circ \nabla P_0(x) \tag{62}$$

for χ-a.e. $x \in \Omega$. Since

$$(F_t^* \circ F_t \circ \bar{\gamma}_0)(\nabla P_0(x)) = (\bar{\gamma}_0 \circ \Phi_t^* \circ \nabla P_t \circ \bar{\gamma}_t \circ \Phi_t \circ \nabla P_0 \circ \bar{\gamma}_0)(\nabla P_0(x))$$

and $\nabla P_t \circ \bar{\gamma}_t$, $\nabla P_0 \circ \bar{\gamma}_0$, $\Phi_t^* \circ \Phi_t$ are all equal to the identity on the corresponding domains, we deduce (62).
QED.

In the case of discrete measures we can prove that the construction works. Even though not explicitly present in [4], a simple argument inserted in the proof of Theorem 7.4 from said reference yields that the solutions to the Hamiltonian ODE constructed there are convex combinations of point masses provided that the initial measures are of the same form (coefficients of the convex combinations are time-invariant). Our Hamiltonian in (21) satisfies all the requirements for Theorem 7.4 [4] to apply (see Lemma 7.6 [4]).

Proposition 4.14. *Let* $\mathbf{x_0} = (x_0^1, ..., x_0^n) \in I\!R^{3n}$ *be arbitrary for some integer* $n \geq 1$, *and let*

$$\bar{\mu} = \sum_{i=1}^{n} c_i \delta_{x_0^i} \tag{63}$$

be a convex combination of the Diracs at these points, i.e. nonnegative constants c_i *satisfy* $\sum_{i=1}^n c_i = 1$. *Then the solution constructed in Theorem 7.4 [4] for the initial-value problem associated to the Hamiltonian ODE as in Definition 2.1 for the Hamiltonian in* (21) *is of the form*

$$\alpha_t = \sum_{i=1}^{n} c_i \delta_{x^i(t)}, \tag{64}$$

where $[0, T] \ni t \to \mathbf{x}(t)$ *is in* $W^{1,\infty}(0, T; I\!R^{3n})$ *and* $\mathbf{x}(0) = \mathbf{x}_0$.

Proof: Let m be a positive integer and set $h = T/m$. Then take $w_0^m := -J\nabla H(\bar{\alpha})$, where $\nabla H(\alpha)$ denotes the element of $\partial H(\alpha)$ with least $L^2(\alpha; I\!R^3)$-norm, and set, for all $t \in [0, h]$,

$$\alpha_t^m = (\mathrm{Id} + t w_0^m)_\# \bar{\alpha}, \;\; \nu_t^m = (\mathrm{Id} + t w_0^m)_\# (w_0^m \bar{\alpha}) \text{ and } w_t^m := \frac{\mathrm{d}\nu_t^m}{\mathrm{d}\alpha_t^m},$$

where we used the fact (see Lemma 7.1 [4]) that $\nu_t^m \ll \alpha_t^m$ to get the Radon-Nykodim derivative w_t^m. On the next time subinterval $[h, 2h]$ one defines α_t^m and w_t^m similarly by using α_h^m and w_h^m instead of $\bar{\alpha}$ and w_0^m, and $t - h$ instead of t. In general, the construction can be extended to $[kh, (k + 1)h]$ for $k = 0, ..., m - 1$ by using α_{kh}^m and w_{kh}^m, $t - kh$ instead of t, and repeating the steps above. It is proved in [4] that the paths of measures $t \to \alpha_t^m$ are uniformly bounded in $\mathcal{P}_2(I\!R^3)$ and uniformly Lipschitz continuous. For a subsequence $m_j \to \infty$ we have a limiting $t \to \alpha_t$, which is shown to satisfy the Hamiltonian ODE with $\alpha_0 = \bar{\alpha}$. Since

$$\alpha_t^{m_j} = [\mathrm{Id} + (t - kh) w_{kh}^{m_j}]_\# \alpha_{kh}^{m_j}$$

for $t \in [kh, (k + 1)h]$, we deduce that all probabilities $\alpha_t^{m_j}$ are convex combinations of n Dirac masses if $\alpha_{kh}^{m_j}$ is (with same coefficients). This is true for all $k = 0, ..., m_j - 1$, so we deduce that it holds for $\alpha_t^{m_j}$ all $t \in [0, T]$. The uniform bounds on α_t^m mentioned above translate into the uniform $L^\infty(0, T; I\!R^{3n})$ bounds on $t \to \mathbf{x}^{m_j}(t) = (x^{m_j,1}(t), ..., x^{m_j,n}(t))$ (where $x^{m_j,i}(t)$ are the points in the support of $\alpha_t^{m_j}$). Furthermore, the uniform Lipschitz continuity of the paths $t \to \alpha_t^m$ in the Wasserstein space $\mathcal{P}_2(I\!R^3)$ gives a finite constant $C > 0$ for which $W_2(\alpha_t^{m_j}, \alpha_s^{m_j}) \leq C|t-s|$ for all t, $s \in [0, T]$. Now fix $t_0 \in (0, T)$. Note that $\mathbf{x}^m$ are piecewise linear and continuous in time, thus it is clear that

$$C^2|t - t_0|^2 \geq W_2^2(\alpha_t^{m_j}, \alpha_{t_0}^{m_j}) = \sum_{i=1}^{n} c_i |x^{m_j,i}(t) - x^{m_j,i}(t_0)|^2 \text{ for all } t \text{ close enough to } t_0.$$

Thus, the vector functions $t \to \mathbf{x}^{m_j}(t)$ are uniformly Lipschitz. By Ascoli-Arzela's theorem, a subsequence converges in the sup norm to a function $\mathbf{x} \in W^{1,\infty}(0,T;\mathbb{R}^{3n})$, which implies the limiting measures α_t found above must have the form (64) for all $t \in [0,T]$. This follows from

$$W_2^2(\alpha_t^{m_j}, \beta_t) \le \sum_{i=1}^n c_i |x^{m_j,i}(t) - x^i(t)|^2 \to 0 \text{ as } j \to \infty,$$

where β_t is the convex combination of the Dirac masses at $x_i(t)$ with coefficients $c_1, ..., c_n$. QED.

So far we are not aware of any reason why distinct initial x_0^k should give rise to distinct $x^k(t)$ at all later times. As a consequence, existence of the map Φ^* is uncertain (since transport maps from an average of n points masses to one of m point masses exist if and only if $n \ge m$).

Proposition 4.15. *Let n be a positive integer and*

$$\alpha_t := \sum_{i=1}^n c_i \delta_{x^i(t)}, \text{ for } t \in [0,T)$$

be the solution of SG in dual space constructed in Proposition 4.14 with initial data

$$\alpha_0 := \sum_{i=1}^n c_i \delta_{x_0^i},$$

where x_0^i are arbitrary in $\mathbb{R}^3$, $i = 1, ..., n$, and c_i are nonnegative and $\sum_{i=1}^n c_i = 1$. Then the map $\Phi^n : [0,T] \times \mathbb{R}^3 \to \mathbb{R}^3$ given by $\Phi^n(t,X) = X$ if $X \neq x_0^i$, and $\Phi^n(t, x_0^i) = x^i(t)$ for $t \in [0,T]$, $i = 1, ..., n$ satisfies (46), *(ii)–(iii).*

Proof: It is obvious that Φ^n is Borel and Φ^n satisfies $\Phi^n_{t\#}\alpha_0 = \alpha_t$. To show that Φ^n solves (42) we start from the fact that α_t solves the system (17), (18). In fact, the Hamiltonian (21) has the property that $\mathcal{T}_\alpha \mathcal{P}_2(\mathbb{R}^3) \cap \partial H(\alpha) = \{\bar{\gamma} - \mathrm{Id}\}$ (see, e.g. [4]), so α_t solves (13) in the sense of distributions. This is equivalent to

$$t \to \int_{\mathbb{R}^3} \xi(X) d\alpha_t(X) \text{ is absolutely continuous}$$

and for a.e. $t \in (0,T)$ we have

$$\frac{d}{dt} \int_{\mathbb{R}^3} \xi(X) d\alpha_t(X) = \int_{\mathbb{R}^3} \nabla\xi(X) \cdot U(t,X) d\alpha_t(X) \text{ for all } \xi \in C_c^1(\mathbb{R}^3),$$

for U given in (9). According to Proposition 4.14, we have that $t \to x^i(t)$ is in $W^{1,\infty}(0,T;\mathbb{R}^3)$ for all $i = 1, ..., n$, which implies

$$\sum_{i=1}^n c_i \nabla\xi(x^i(t)) \cdot \dot{x}^i(t) = \sum_{i=1}^n c_i \nabla\xi(x^i(t)) \cdot U(t, x^i(t)) \text{ for all } \xi \in C_c^1(\mathbb{R}^3).$$

This leads to the desired conclusion. QED.

We are now in a position to formulate:

Corollary 4.16. *Let P_0 be the maximum of finitely many affine functions: for some integer $n \geq 1$ and $a_i \in I\!R^3$, $b_i \in I\!R^1$ for $i = 1, \dots, n$*

$$P_0(x) = \max_{i=1,\dots,n} (a_i \cdot x + b_i), \quad \text{for } x \in \Omega.$$

Then there exists a weak Lagrangian solution for (1) in the sense of Definition 4.12 with $P(0, \cdot) = P_0$ a.e. in Ω.

5 Return to dual space and conservation of energy

We show that weak Lagrangian solutions give rise to solutions in dual space.

Theorem 5.1. *Let (P, F) be a weak Lagrangian solution of (1) in the sense of Definition 4.12 and set $\alpha_t := \nabla P_{t\#}\chi$. Then α is a distributional solution of (13).*

Proof: We need to show that for every $\varphi \in C_0^\infty((0,T) \times I\!R^3)$

$$\int_0^T \int_{I\!R^3} (\partial_t \varphi + J[X - \bar{\gamma}_t(X)] \cdot \nabla \varphi) d\alpha_t(X) dt = 0. \tag{65}$$

By the density argument, it is sufficient to show that for

$$\varphi(t, X) = \zeta(t)\xi(X), \quad \text{for all } \xi \in C_0^\infty(I\!R^3),\ \zeta \in C_0^\infty(0,T).$$

Fix such ξ, ζ.

From (55), $(\nabla P_t \circ \bar{\gamma}_t)_\# \alpha_t = \nabla P_{t\#}\mu_t$. Then (48) yields

$$\nabla P_{t\#}\mu_t = \alpha_t.$$

Then $F_{t\#}\chi = \mu_t$ implies $Z_{t\#}\chi = (\nabla P_t \circ F_t)_\# \chi = \nabla P_{t\#}\mu_t = \alpha_t$, i.e.

$$Z_{t\#}\chi = \alpha_t. \tag{66}$$

Now we calculate using (66):

$$\int_0^T \int_\Omega \zeta'(t)\xi(Z_t(x)) dx dt = \int_0^T \int_{I\!R^3} \zeta'(t)\xi(X) d\alpha_t dt. \tag{67}$$

On the other hand, we can show that integrating by parts in t and using the regularity of Z, F and equation (8), we get

$$\int_0^T \int_\Omega \zeta'(t)\xi(Z_t(x)) dx dt = -\int_0^T \int_\Omega \zeta(t) \nabla \xi(Z_t(x)) \cdot J[Z_t(x) - F_t(x)] dx dt. \tag{68}$$

Indeed, let $\eta^\varepsilon(t, x)$ be the family of standard mollifiers in time-space. We extend $Z(t, x)$ to $I\!R^1 \times I\!R^3$ by defining it to be zero outside of $(0,T) \times \Omega$, and define $Z^\varepsilon = \eta^\varepsilon * Z(t, x)$ on $(0,T) \times I\!R^3$, where the convolution is with respect to (t, x). Then $Z^\varepsilon \in C^\infty((0,T) \times I\!R^3)$. Also, (8) implies that the distributional derivative $\partial_t Z(t, x)$ in $(0,T) \times I\!R^3$ is $J(Z - F)$ (extended by zero outside of $[0,T] \times \Omega$), and the functions Z, $Z - F$ are in $L^2((0,T) \times \Omega; I\!R^3)$ by (iii) of Definition 4.12. Let $[a, b] \subset (0,T)$ be such that $\text{supp}(\zeta) \subset [a, b]$. Then for sufficiently small ε,

$\partial_t Z^\varepsilon = J(Z-F)*\eta_\varepsilon(t,x)$ in $(a,b)\times I\!R^3$, and $(Z^\varepsilon, \partial_t Z^\varepsilon) \to (Z, J(Z-F))$ in $L^2((a,b)\times\Omega; I\!R^3\times I\!R^3)$ as $\varepsilon \to 0$. Then, integrating by parts to get

$$\int_0^T \int_\Omega \zeta'(t)\xi(Z_t^\varepsilon(x))dxdt = -\int_0^T \int_\Omega \zeta(t)\nabla\xi(Z_t^\varepsilon(x)) \cdot \partial_t Z_t^\varepsilon(x)dxdt, \tag{69}$$

and using that $\xi, \nabla\xi, D^2\xi$ are bounded, we get that the left and right hand sides of the above equality converge to the left and right hand sides of (68), respectively. Indeed, denoting by R_ε and R the right-hand sides of (69) and (68) respectively, we have

$$\begin{aligned}|R - R_\varepsilon| \le &\int_a^b \int_\Omega \Big(\|D^2\xi\|_{L^\infty(I\!R^3)} |Z_t^\varepsilon(x) - Z_t(x)||Z_t(x) - F_t(x)| \\ &\qquad + \|D\xi\|_{L^\infty(I\!R^3)} |\partial_t Z_t^\varepsilon(x) - J[Z_t(x) - F_t(x)]| \Big) dxdt \\ &\le C\Big(\|Z_t(x) - F_t(x)\|_{L^2((0,T)\times\Omega)} \|Z_t^\varepsilon - Z_t\|_{L^2((a,b)\times\Omega)} \\ &\qquad + \|\partial_t Z_t^\varepsilon(x) - J[Z_t(x) - F_t(x)]\|_{L^2((a,b)\times\Omega)} \Big) \to 0.\end{aligned}$$

Convergence of the left-hand sides is proved similarly. This shows (68).

Next, (48) implies

$$(\bar\gamma_t \circ \nabla P_t \circ \bar\gamma_t)(X) = \bar\gamma_t(X) \qquad \text{for every } t \in (0,T) \text{ and } \alpha(t,\cdot)\text{-a.e. } X \in I\!R^3.$$

Using $\bar\gamma_{t\#}\alpha_t = \mu_t$, we get

$$\bar\gamma_t \circ \nabla P_t(x) = x \text{ for every } t \in (0,T) \text{ and } \mu_t\text{-a.e. } x \in \Omega.$$

In view of $F_t\#\chi = \mu_t$, we have

$$\bar\gamma_t \circ Z_t(x) = \bar\gamma_t \circ \nabla P_t \circ F_t(x) = F_t(x) \text{ for every } t \in (0,T) \text{ and } \chi\text{-a.e. } x \in \Omega.$$

Then we can rewrite (68) as

$$\int_0^T \int_\Omega \zeta'(t)\xi(Z_t(x))dxdt = -\int_0^T \int_\Omega \zeta(t)\nabla\xi(Z_t(x)) \cdot J[Z_t(x) - \bar\gamma_t \circ Z_t(x)]dxdt,$$

and using (66) to change variables in the right-hand side, we get

$$\int_0^T \int_\Omega \zeta'(t)\xi(Z_t(x))dxdt = -\int_0^T \int_{I\!R^3} \zeta(t)\nabla\xi(X) \cdot J[X - \bar\gamma_t(X)]d\alpha_t(X)dt,$$

Combining with (67), we get

$$\int_0^T \int_{I\!R^3} \zeta'(t)\xi(X)d\alpha_t dt = -\int_0^T \int_{I\!R^3} \zeta(t)\nabla\xi(X) \cdot J[X - \bar\gamma_t(X)]d\alpha_t(X)dt$$

which is (65) in the case $\varphi(t,X) = \zeta(t)\xi(X)$. QED.

We finish with an observation concerning energy conservation along weak Lagrangian solutions of (1).

Corollary 5.2. *Let (P,F) be a weak Lagrangian solution of (1) in the sense of Definition 4.12. Then*

$$\text{the function } [0,T) \ni t \to \int_\Omega \left|y - \nabla P_t(y)\right|^2 dy \text{ is constant.} \tag{70}$$

Proof: Due to (59), we infer that the dual-space velocity $U(t,X) = X - \bar{\gamma}_t(X)$ satisfies

$$\|U(t,\cdot)\|_{L^2(\alpha_t;I\!R^3)} \leq R_0[\mathcal{L}^3(\Omega)]^{1/2} + \left(\int_{I\!R^3} |X|^2 d\alpha_t(X)\right)^{1/2}.$$

But

$$\int_{I\!R^3} |X|^2 d\alpha_t(X) = \|\nabla P_t\|^2_{L^2(\Omega;I\!R^3)} \text{ for all } t \in [0,T),$$

which, since $P \in C([0,\infty); H^1(\Omega))$, implies the local boundedness in time of the $L^2(\alpha_t; I\!R^3)$–norm of the velocity $U(t,\cdot)$, boundedness required by Theorem 5.2 in [4]. Furthermore, it follows that the path $t \to \alpha_t$ lies in $AC^2(0,T;\mathcal{P}_2(I\!R^3))$, see e.g. [4, page 24]. Thus, in light of Theorem 5.1, we can apply Theorem 5.2 in [4] to conclude. QED.

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