Extension of absolute weak topologies and Riesz homomorphisms

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Abstract. Let L be a Riesz space and I an ideal of $L^{\sim}$. In this paper we show that if $I \subset L_n^{\sim}$ and I separates the points of L, then there exists a unique "largest" Riesz dual system $\langle L_I, L_I' \rangle$, called the largest enlargement of $\langle L, I \rangle$, which satisfies the statement given in [3, 5.2], and at the same time we give a sequential version of its result. From this, given in [1] theorem 23.33 extends to the argument on σ-laterally complete Riesz spaces and example 24.15 is generalized.

1. Introduction

Throughout this paper all Riesz spaces under consideration are assumed Archimedean. For notation and basic terminology concerning Riesz spaces not explained below, we refer to the books [1], [9] and [10].

A Riesz space L is *laterally complete* if every positive disjoint subset of L has the supremum in L, and *σ-laterally complete* if every positive disjoint sequence of L has the supremum in L. The Riesz space that is both laterally and Dedekind complete is a *universally complete* Riesz space. Similarly, σ-Dedekind and σ-laterally complete Riesz space is a *σ-universally complete* Riesz space. A *universal* (resp. *σ-universal*) *completion* of a Riesz space L is a universally (resp. σ-universally) complete Riesz space K having an order dense (resp. super order dense) Riesz subspace M that is Riesz isomorphic to L. As usual, identifying L with its image M in K, we shall treat L as a Riesz subspace of K. The universal completion of L denotes by L^u. The ideal generated by L in L^u is precisely the Dedekind completion L^{δ} of L. A Riesz space L is *almost σ-Dedekind complete* if it is Riesz isomorphic to a super order dense Riesz subspace of some σ-Dedekind complete Riesz space. By [1, 23.27] L is almost σ-Dedekind complete if and only if L has a σ-universal completion which is determined uniquely up to a Riesz isomorphism. The σ-universal completion of L denotes by L^s. If L is almost σ-Dedekind complete, then the ideal of L^s generated by L is precisely the σ-Dedekind completion L^{δ} of L.

For any Riesz space L, $L^{\sim}$ denotes the order dual of L. $L_n^{\sim}$ and $L_c^{\sim}$ denote the bands in $L^{\sim}$ consisting of all order continuous linear functionals and all σ-order continuous linear functionals on L resprctively. Let A be a nonempty subset of $L^{\sim}$. The *absolute weak topology* $|\sigma|(L,A)$ on L generated by A is a locally convex-solid Riesz topology generated by the Riesz semi-norms $x \rightarrow |f|(|x|)$ on L, where $x \in L$ and $f \in A$. If A is an ideal of $L^{\sim}$ separating the points of L, then $|\sigma|(L,A)$ is Hausdorff and the topological dual of $(L,|\sigma|(L,A))$ is precisely A([1,6.6]). Furthermore, the absolute weak topology $|\sigma|(L,A)$ defines a *Lebesgue topology* on L if $A \subset L_n^{\sim}$, and a *σ-Levi topology* if every increasing $|\sigma|(L,A)$-bounded net of L^+ has a supremum in L. Similarly, the topology $|\sigma|(L,A)$ is a *σ-Lebesgue topology* on L if $A \subset L_c^{\sim}$, and a *σ-Levi topology* if every increasing $|\sigma|(L,A)$-bounded sequence of L^+ has a supremum in L.

In [6] and [8] Labuda has shown that if (L,τ) is a Hausdorff locally solid Riesz space with the Fatou property, then there exists an essentially unique "largest" Nakano space $(L^\#,\tau^\#)$, called the largest enlargement of (L,τ), which has the property that $L \subseteq L^\# \subseteq L^u$ and $\tau = \tau^\# | L$, and has shown in [7] some sequential versions of the arguments given in [6]. The aim of this paper is to give the largest enlargement of a Riesz dual system and its sequential version, and characterize those results in terms of the absolute weak topologies.

2. The largest enlargement of $\langle L, L_n' \rangle$

Let L be a Riesz space. For each $x \in L^u$ put $L_x = L_{|x|} = \{y \in L \,\big|\, |y| \leq |x|\}$ and for any $f \in L^{\sim}$ define $L_f = \{x \in L^u : \sup(|f(y)| : y \in L_x) < \infty\}$. If $f \in L_n^{\sim}$, then L_f is expressible as the set consisting of all elements x of L^u such that $\sup_\alpha |f|(x_\alpha) < \infty$ for a net $\{x_\alpha\}$ of L^+ with $\theta \leq x_\alpha \uparrow |x|$ in L^u.

The proof of the following lemma is straightforward and we omit it.

LEMMA 1. For any $f \in L_n^{\sim}$, L_f is a solid Riesz subspace of L^u and $L^\delta \subseteq L_f \subseteq L^u$.

Let $\theta \leq f \in L_n^{\sim}$. For each $\theta \leq x \in L_f$ put $f^{\sim}(x) = \sup_\alpha f(x_\alpha)$, where $\{x_\alpha\} \subset L_x$ and $\theta \leq x_\alpha \uparrow x$ in L^u, and define $f^{\sim}$ by $f^{\sim}(x) = f^{\sim}(x^+) - f^{\sim}(x^-)$ for all $x \in L_f$. Since $x^+, x^- \in L_f$, it is simple to verify that $f^{\sim}$ is well-defined and positive linear. Furthermore, $f^{\sim} \in (L_f)_n^{\sim}$ holds. To this end, let $\theta \leq x_\alpha \uparrow x$ in L_f. For each x_α choose a net of L_{x_α} that increases to x_α. The net consisting of the finite suprema of these nets is a net $\{z_\gamma\}$ of L_x that increases to x. For any $\varepsilon > 0$ choose $z_\delta \in \{z_\gamma\}$ such that $f^{\sim}(x) - \varepsilon < f(z_\delta)$. It then follows that there exists some x_α with $z_\delta \leq x_\alpha$ and

$$f^{\sim}(x) - \varepsilon < f(z_\delta) \leq f^{\sim}(x_\alpha) \leq \sup_\alpha f^{\sim}(x_\alpha) \leq f^{\sim}(x).$$

This means that $f^{\sim}(x) = \sup_\alpha f^{\sim}(x_\alpha)$, and hence $f^{\sim} \in (L_f)_n^{\sim}$. In general, if $f \in L_n^{\sim}$, put $f = f^+ - f^-$ and define $f^{\sim}$ by $f^{\sim}(x) = f^{+\sim}(x) - f^{-\sim}(x)$ for all $x \in L_f$. Since $\theta \leq f^+, f^- \in L_n^{\sim}$ and $L_f = L_{|f|} = L_{f^+} \cap L_{f^-}$, $f^{\sim}$ is well-defined and it is not difficult to verify that $f^{\sim}$ is a unique order continuous extension of f to all of L_f. Thus $f^{\sim} \in (L_f)_n^{\sim}$ holds for all $f \in L_n^{\sim}$.

Let I be an ideal of $L_n^{\sim}$ and define $L_I = \{x \in L^u : L_x$ is $\sigma(L,I)$-bounded$\}$. Since L_x is a solid subset of L^u, L_I can denote as follows :

$$L_I = \{x \in L^u : L_x \text{ is } |\sigma|(L,I)\text{-bounded}\} \quad (= \cap\{L_f : f \in I\}).$$

By Lemma 1, it is clear that L_I is a Dedekind complete Riesz space in its own right and $L^\delta \subset L_I \subset L^u$. Put $L_I' = \{f^{\sim}|L_I : f \in I\}$. For each $f \in I$ denote again by $f^{\sim}$ the restriction

$f^{\sim}|L_I$ of $f^{\sim}$ to L_I. Since $(f+g)^{\sim} = f^{\sim} + g^{\sim}$ holds for all $\theta \leq f,\ g \in I$, we now have the following lemma.

LEMMA 2. Let L be a Riesz space and let I be an ideal of $L_n^{\sim}$. Then L_I' is an ideal of $(L_I)_n^{\sim}$.

Proof. From the argument above it is clear that L_I' is a vector subspace of $(L_I)_n^{\sim}$. In order to show that L_I' is a Riesz subspace of $(L_I)_n^{\sim}$, it is enough to prove that $(f^{\sim})^{+} = (f^{+})^{\sim}$ for every $f \in I$. To this end let $f \in I$ and $\theta \leq x \in L_I$. Then for $\{x_\alpha\} \subset L_x$ and $\theta \leq x_\alpha \uparrow x$ it follows that

$$f^{+}(x_\alpha) \leq \sup\{f^{\sim}(y) : \theta \leq y \leq x_\alpha,\ y \in L_I\} = (f^{\sim})^{+}(x_\alpha) \leq (f^{\sim})^{+}(x),$$

and so $(f^{+})^{\sim}(x) = \sup_\alpha f^{+}(x_\alpha) \leq (f^{\sim})^{+}(x)$. On the other hand,

$$(f^{\sim})^{+}(x) = \sup\{f^{\sim}(y) : \theta \leq y \leq x,\ y \in L_I\} \leq (f^{+})^{\sim}(x)$$

Hence $(f^{+})^{\sim}(x) = (f^{\sim})^{+}(x)$ holds for all $\theta \leq x \in L_I$. This means that L_I' is a Riesz subspace of $(L_I)_n^{\sim}$. It is simple to verify that L_I' is a solid subset of $(L_I)_n^{\sim}$. Thus L_I' is an ideal of $(L_I)_n^{\sim}$.

Let L be a Riesz space and let I be an ideal of $L^{\sim}$ separating the points of L. The pair $\langle L,I \rangle$ under its natural duality $\langle x,f \rangle (= f(x))$, where $x \in L$ and $f \in I$, is said to be a *Riesz dual system*. Let M be a Riesz subspace of L^u with $L \subset M \subset L^u$, and let I, J be the ideals of $L^{\sim}, M^{\sim}$ separating the points of L, M respectively.
A Riesz dual system $\langle M,J \rangle$ is said to be o-*continuous enlargement* of $\langle L,I \rangle$ if $I \subset L_n^{\sim}$, $J \subset M_n^{\sim}$ and $|\sigma|(M,J)\big|L = |\sigma|(L,I)$.

LEMMA 3. Let $L \subset M \subset L^u$ and let $\langle L,I \rangle$ and $\langle M,J \rangle$ be Riesz dual systems defined by $I \subset L_n^{\sim}$ and $J \subset M_n^{\sim}$ respectively. If $\langle M,J \rangle$ is an o-continuous enlargement of $\langle L,I \rangle$, then $\langle L_I,L_I' \rangle$ is also an o-continuous enlargement of $\langle M,J \rangle$.

Proof. Let $\langle M,J \rangle$ be an o-continuous enlargement of $\langle L,I \rangle$. By [1, 6.6] the topological dual of $(M, |\sigma|(M,J))$ is precisely J. Similarly, $(L, |\sigma|(L,I))' = I$. Since I separates the points of L, it follows easily that L_I' separates the points of L_I and so by Lemma 2 $\langle L_I,L_I' \rangle$ is a Riesz dual system. Since $|\sigma|(M,J)\big|L = |\sigma|(L,I)$, $J\big|L = I$ (See the proof of [1, 6.6]) holds and L_x is $|\sigma|(L,I)$-bounded for any $x \in M$. Hence $M \subset L_I$, and $L_I'\big|M = J$ holds. Since $M^u = L^u$ by [1, 23.21], this means that $\langle L_I,L_I' \rangle$ is an o-continuous enlargement of $\langle M,J \rangle$.
From the argument above the Riesz dual system $\langle L_I,L_I' \rangle$ can be characterized as the largest o-continuous enlargement of a given Riesz dual system $\langle L,I \rangle$.

The following result satisfies the statements of proposition 5.2 given in [3] by Burkinshaw and Dodds.

THEOREM 4. Let L be a Riesz space and let I be an ideal of $L_n^{\sim}$ separating the points of L. Then the following holds.

(1) $|\sigma|(L_I, L_I')$ is a Hausdorff locally convex-solid, Lebesgue and Levi topology.

(2) $(L_I, |\sigma|(L_I, L_I'))$ is (L_I, L_I')-complete.

(3) $L_I = (L_I')_n^{\sim}$.

(4) Each order interval of L_I is $|\sigma|(L_I, L_I')$-compact.

Proof. (1) Since L_I' separates the points of L_I, it is immediate that $|\sigma|(L_I, L_I')$ is a Hausdorff locally convex-solid, Lebesgue topology. In order to show that $|\sigma|(L_I, L_I')$ is a Levi topology, let $\{x_\alpha\}$ be a $|\sigma|(L_I, L_I')$-bounded net of L_I with $\theta \leq x_\alpha \uparrow$ in L_I.
Since $|\sigma|(L_I, L_I')$ is Hausdorff, for any $\theta < y \in L_I$ there exist $\theta < f^{\sim} \in L_I'$ and $\varepsilon > 0$ such that $f^{\sim}(y) > \varepsilon$. Now, choose a $n \in \mathrm{N}$ satisfying $f^{\sim}(n^{-1}x_\alpha) < \varepsilon$ for all α. Then there exists $\theta < z \in L_I$ such that $n^{-1}x_\alpha \wedge y \leq y - z$ for all α. Indeed, if $n^{-1}x_\alpha \wedge y \uparrow y$ in L_I, then $f^{\sim}(n^{-1}x_\alpha \wedge y) \to f^{\sim}(y) \leq \varepsilon$ holds, contradicting the fact that $f^{\sim}(y) > \varepsilon$. Hence $(ny - x_\alpha)^+ = n(y - n^{-1}x_\alpha \wedge y) \geq nz \geq z > \theta$ for all α, that is, $\{x_\alpha\}$ is a dominable subset of L_I. From this, it it follows by [1, 23.10 and 23.13] that $\nu = \sup_\alpha x_\alpha$ exists in L^u. Let $\theta \leq x \in L_\nu$. Since $x_\alpha \wedge x \uparrow x$ in L_I and $\{x_\alpha\}$ is $|\sigma|(L_I, L_I')$-bounded, for each $f \in I$ it follows that there exists a positive number a_f (depending upon f) satisfying $|f^{\sim}|(x_\alpha \wedge x) \uparrow |f|(x) \leq a_f$. This means that L_ν is $|\sigma|(L, I)$-bounded. Hence $\nu \in L_I$ and the proof is finished.

(2) Since every locally convex-solid Lebesgue topology has a Fatou property [1, 11.1], $(L_I, |\sigma|(L_I, L_I'))$ is a Nakano space by (1). Hence the result follows from [1, 13.9].

(3) By (1), it follows immediately from [1, 9.4].

(4) It follows from (3) by applying [1, 19.14] to L_I' equipped with the topology $|\sigma|(L_I', (L_I')_n^{\sim})$, and the proof is complete.

REMARK. Let L be a Dedekind complete Riesz space and let τ be a Hausdorff locally solid Fatou topology on L. Labuda showed in [6] and [8] that the largest enlargement $(L^{\#}, \tau^{\#})$ of (L, τ) has both the Levi property and the Fatou property. If τ is a Hausdorff locally convex-solid Lebesgue topology on L and L' its the topological dual, then we can show that $L_{L'} = L^{\#}$ and $L_{L'}{}'$ is the set of all $\tau^{\#}$-continuous linear functionals defined on $L_{L'}$.

The following results are immediate from Theorem 4.

COROLLARY 5. Let (L, τ) be a Hausdorff locally convex-solid Riesz space with the Lebesgue property and let L' be its topological dual. Then the largest enlargement $\langle L_I, L_I' \rangle$ of $\langle L_I, L' \rangle$ satisfies the statements (1) $\sim$ (4) of Theorem 4.

COROLLARY 6. Let (L, τ) be a Hausdorff locally convex-solid Riesz space with the Lebesgue property and let L' be its topological dual. Then $L_{L'} = L$ if and only if (L, τ) has the Levi property. In particular, if $L_{L'} = L$ holds, then (L, τ) satisfies the statements (i) $\sim$ (v) given in [3, Prop. 5.2].

3. The largest σ-enlargement of $\langle L, L_c' \rangle$

In this section we will give the σ-analogues of the arguments given in the previous section. Let L be an almost σ-Dedekind complete Riesz space and let L^S be its σ-universal completion. The ideal of L^S generated by L is precisely the σ-Dedekind completion L^σ of L and $L \subset L^\sigma \subset L^S$.

For each $x \in L^S$ put $L_x = L_{|x|} = \{y \in L : |y| \le |x|\}$ and for any $f \in L^\sim$ define $L_f = \{x \in L : \sup(|f(y)| : y \in L_x) < \infty\}$. L_f is a solid subset of L^S, and since L is super order dense in L^S, we have that

$$L_f = \{x \in L^S : \sup |f|(x_n) \text{ is bounded for a } \{x_n\} \subset L_x \text{ with } \theta \le x_n \uparrow |x| \text{ in } L^S\}.$$

Hence it is easily verified that L_f is a solid Riesz subspace of L^S and $L^\sigma \subset L_f \subset L^S$.

Let L be an almost σ-Dedekind complete Riesz space and let $\theta \le f \in L_c^\sim$. For each $\theta \le x \in L_f$ put $f^\sim(x) = \sup_n f(x_n)$, where $\{x_n\} \subset L_x$ and $\theta \le x_n \uparrow x$ in L^S, and define $f^\sim(x) = f^\sim(x^+) - f^\sim(x^-)$ for all $x \in L_f$. Using the argument similar to the previous section, $f^\sim$ is well-defined and it follows that $\theta \le f^\sim \in (L_f)_c^\sim$. Furthermore, if $f \in L_c^\sim$, let $f = f^+ - f^-$ and define $f^\sim$ by $f^\sim(x) = f^{+\sim}(x) - f^{-\sim}(x)$ for all $x \in L_f$. Since $\theta \le f^+, f^- \in L_c^\sim$ and $L_f = L_{|f|} = L_{f^+} \cap L_{f^-}$, it follows immediately that $f^\sim$ is a unique σ-order continuous extension of f. Thus $f^\sim \in (L_f)_c^\sim$ holds for all $f \in L_c^\sim$. Let I be an ideal of $L_c^\sim$ and define $L_I = \{x \in L^S : L_x$ is $\sigma(L,I)$-bounded$\}$ $(= \{x \in L^S : L_x$ is $|\sigma|(L,I)$-bounded$\} = \cap\{L_f : f \in I\})$. Put $L_I' = \{f^\sim|L_I : f \in I\}$. For each $f \in I$ we denote again by $f^\sim$ the restriction $f^\sim|L_f$ of $f^\sim$.

The following result is easily verified by replacing the word "net" with "sequence" used in Lemma 1 and Lemma 2.

LEMMA 7. Let L be an almost σ-Dedekind complete Riesz space and let I be an ideal of $L_c^\sim$. Then the following holds.

(1) L_I is a σ-Dedekind complete Riesz space in its own right and $L^\sigma \subset L_I \subset L^S$.

(2) L_I' is an ideal of $(L_I)_c^\sim$.

Let L be an almost σ-Dedekind complete Riesz space and let I be an ideal of $L_c^\sim$ separating the points of L. Then L_I' separates the points of L_I and it follows by Lemma 7 that $\langle L_I, L_I' \rangle$ is a Riesz dual system. Let M be a Riesz subspace with $L \subset M \subset L^S$. Let I, J be the ideals of $L^\sim, M^\sim$ separating the points of L, M respectively. A Riesz dual system $\langle M, J \rangle$ is called *σ-continuous enlargement* of $\langle L, I \rangle$ if $I \subset L_c^\sim$, $J \subset M_c^\sim$ and $|\sigma|(M,J)| L = |\sigma|(L,I)$. If $\langle M, J \rangle$ is a σ-continuous enlargement of $\langle L, I \rangle$, then it is follows by the argument similar to Lemma 3 that $\langle L_I, L_I' \rangle$ is also a σ-continuous enlargement of $\langle M, J \rangle$, where $M^S = L^S$ by [1, 23.27]. Hence $\langle L_I, L_I' \rangle$ is the largest σ-continuous enlargement of $\langle L, I \rangle$. We now have the following result which corresponds to Theorem 4.

THEOREM 8. Let L be an almost σ-Dedekind complete Riesz space and let I be an ideal of $L_c^\sim$ separating the points of L. Then $|\sigma|(L_I, L_I')$ is a Hausdorff locally convex-solid, σ-Lebesgue and σ-Levi topology.

Proof. Every countable dominable subset of L^{s+} has a supremum in L^s [1, 23.23]. The proof is almost the same as that in Theorem 4.

COROLLARY 9. Let (L,τ) be an almost σ-Dedekind complete, Hausdorff locally convex-solid Riesz space with the σ-Lebesgue topology, and let L' be its topological dual. Then τ is a σ-Levi topology if and only if $L_{L'} = L$.

THEOREM 10. Let L be an almost σ-Dedekind complete Riesz space and let I be an ideal of $L_c^{\sim}$ separating the points of L. Then following statements are equivalent :

(1) L_I is sequentially $|\sigma|(L_I,L_I')$-complete.

(2) The order intervals of L^{σ} are sequentially $|\sigma|(L^{\sigma},L_I')$-complete.

Proof. (1) ⇒ (2) Since L^{σ} is a solid subspace of L^s, it is obvious.
(2) ⇒ (1) By Theorem 8 and [1, Chap. 2, Exer. 9] we need only to show that every order intervals of L_I is sequentially $|\sigma|(L_I,L_I')$-complete. To this end let $\{u_n\}$ be a $|\sigma|(L_I,L_I')$-Cauchy sequence of L_I^+ satisfying $\theta \le u \in L_I$ and $\theta \le u_n \le u$ for all n. Choose a sequence $\{x_i\}$ of L^+ with $\theta \le x_i \uparrow u$. Since $\theta \le u_n \wedge x_i \le x_i$ in L^{σ} and $|u_n \wedge x_i - u_m \wedge x_i| \le |u_n - u_m|$, it follows by (2) that $u_n \wedge x_i \to y_i\,(|\sigma|(L^{\sigma},L_I'))$ in L^{σ}. Clearly, $\theta \le y_i \uparrow\le u$. Hence $y = \sup_i y_i$ exists in L_I and $\theta \le y \le u$. On the other hand, since $\theta \le u_n - u_n \wedge x_i \le u - x_i$, it follows that $|y-u_n| \le (y-y_i) + |y_i - u_n \wedge x_i| + (u-x_i)$ for all i, n. From this it is simple to verify that $u_n \to y\,(|\sigma|(L_I,L_I'))$ in L_I. Thus every order intervals of L_I are sequentially $|\sigma|(L_I,L_I')$-complete, and the proof is finished.

4. σ-laterally complete spaces

Let L be a Riesz space and let $\pi \in L_n^{\sim}$ be a Riesz homomorphism on L. By Lemma 1 L_π is a solid Riesz subspace of L^u satisfying $L^{\delta} \subset L_\pi \subset L^u$, and π extends to an order continuous Riesz homomorphism on L_π. Indeed, let $x \wedge y = \theta$ in L_π and choose nets $\{x_\alpha\}$ and $\{y_\beta\}$ of L with $\theta \le x_\alpha \uparrow x$ and $\theta \le y_\beta \uparrow y$ in L_π. Since $x_\alpha \wedge y_\beta = \theta$ for all α, β, it is immediate that

$$\pi^{\sim}(x) \wedge \pi^{\sim}(y) = \sup_{\alpha,\beta} \pi(x_\alpha) \wedge \pi(y_\beta) = \sup_{\alpha,\beta} \pi(x_\alpha \wedge y_\beta) = 0\,.$$

Hence $\pi^{\sim}$ is a Riesz homomorphism on L_π, and $\pi^{\sim} \in (L_\pi)_n^{\sim}$. Similary, if L is almost σ-Dedekind complete and $\pi \in L_c^{\sim}$ is a Riesz homomorphism on L, then $\pi^{\sim}$ is a σ-order continuous Riesz homomorphism on L_π with $L^{\sigma} \subset L_\pi \subset L^s$.

We now have the following result.

THEOREM 11. (1) Let L be a Riesz space and let $\pi \in L_n^{\sim}$ be a Riesz homomorphism. Then $L_\pi = L^u$, that is, π extends to all of L^u as an order continuous Riesz homomorphism.

(2) Let L be an almost σ-Dedekind complete Riesz space and let $\pi \in L_c^{\sim}$ be a Riesz homomorphism. Then $L_\pi = L^s$, that is, π extends to all of L^s as a σ-order continuous Riesz homomorphism.

Proof. (1) Since L_π is Dedekind complete, to see that $L_\pi = L^u$ we need only to show that L_π is laterally complete. To this end let $\{x_\alpha\}$ be disjoint subset of L_π^+. Let $\{y_\beta\}$ be the net consisting of the finite suprema of elements in $\{x_\alpha\}$. Then $x = \sup_\alpha x_\alpha$ exists in L^u and $\theta \leq y_\beta \uparrow x$. Since $\pi^\sim(x_\alpha) = 0$ holds except for at most one element of $\{x_\alpha\}$, it follows that for any $z \in L_x$,

$$\pi(z) = \pi^\sim(z) = \sup_\beta \pi^\sim(z \wedge y_\beta) \leq \sup_\beta \pi^\sim(y_\beta) \quad (< \infty).$$

This means that $x \in L_\pi^+$, and so L_π is laterally complete. Thus L_π is an universally complete Riesz space and hence it follows by [1, 23.18] that $L_\pi = L^u$.

(2) The proof is almost the same as that in (1).

Let L be a σ-laterally complete Riesz space. Then $L^\sim = L_c^\sim$ holds by [1, 23.7], and using the argument given by Fremlin in [4, 1.15], we can show that each $\theta \leq \varphi \in L^\sim$ is expressible as a sum of finite number of Riesz homomorphisms on L. Hence $L^\sim$ is a discrete Riesz space [9, 26].

COROLLARY 12. Let L be a Riesz space and let I be an ideal of $L_n^\sim$. Then the following are equivalent :

(1) $L_I = L^u$.

(2) Each $\theta \leq \varphi \in I$ is expressible as a sum of finite number of order-continuous Riesz homomorphisms on L.

Proof. From Lemma 1, Lemma 2 and Theorem 11 it is abvious.

COROLLARY 13. Let L be an almost σ-Dedekind complete Riesz space and let I be an ideal of $L_c^\sim$. Then the following are equivalent :

(1) $L_I = L^s$.

(2) Each $\theta \leq \varphi \in I$ is expressible as a sum of finite number of σ-order continuous Riesz homomorphisms of L.

Proof. From Lemma 7 and Theorem 11 it is obvious.

THEOREM 14. Let (L, τ) be a σ-laterally complete, Hausdorff locally (not necessarily convex) solid Riesz space and let L' be its topological dual. Then the following holds.

(1) $L_n^\sim \subset L' \subset L_c^\sim$.

(2) If τ satisfies the Lebesgue property, then $L' = L_n^\sim$.

Proof. (1) Since $L' \subset L_c^\sim$ holds by [1, 5.7 and 23.7], we need only to show that $L_n^\sim \subset L'$. To this end let $\theta \leq \varphi \in L_n^\sim$ and let C_φ be the carrier of φ, that is, $C_\varphi = N_\varphi^d$ and $N_\varphi = \{x : \varphi(|x|) = 0\}$. Note that φ is strictly positive on C_φ. Let $\{x_n\}$ be any disjoint sequence of C_φ^+. Since C_φ is σ-laterally complete in its own right, it follows for any sequence $\{a_n\}$ in R^+ that

$$x = \sup_n a_n x_n \text{ exists in } C_\varphi \text{ and } \varphi(x) = \Sigma_{n=1}^\infty a_n \varphi(x_n).$$

From this, it is simple to verify that $x_n = \theta$ except for finitely many $n \in \mathrm{N}$. Hence by [9, 26.10 and 26.11] C_φ has a basis $(e_1, e_2, \cdots, e_n)$ consisting of mutually disjoint atoms for some $n \in \mathrm{N}$, and so it is Riesz isomorphic to n-dimensional real number space R^n under the usual coordinatewise ordering. Put $u = \sup_{1 \leq i \leq n} e_i$ and let B_u be the band of L generated by u. Then $C_\varphi = B_u$ and it follows by [1, 23.4] that $L = B_u \oplus B_u^d = C_\varphi \oplus N_\varphi$. Now, assume that $x_\alpha \to \theta(\tau)$ in L, and put $x_\alpha = y_\alpha + z_\alpha$, where $y_\alpha \in C_\varphi$ and $z_\alpha \in N_\varphi$. Then it is immediate that $y_\alpha \to \theta(\tau)$ in C_φ. Since $C_\varphi \cong \mathrm{R}^n$, it follows by [5, P.142] that $\varphi(x_\alpha) = \varphi(y_\alpha) \to 0$. This means that φ is τ-continuous, and hence we have $L_n^\sim \subset L'$.

(2) If τ is a Lebesgue topology, then it is clear that $L' \subset L_n^\sim$. Hence $L' = L_n^\sim$ holds by (1).

The following result shows the extention of Theorem 23.33 given in [1].

THEOREM 15. Let L be a σ-laterally complete Riesz space carrying a Hausdorff locally convex-solid Lebesque topology τ and let L' be its topological dual. Furthermore, let $\langle L_{L'}, L'_{L'} \rangle$ be the largest enlargement of $\langle L, L' \rangle$. Then the following holds..

(1) $L_{L'} = L^u$.

(2) L^u is a discrete Riesz space.

(3) L^u is Riesz isomorphic to some (Riesz space of the form) R^{X}.

(4) $\tau = \sigma(L, L') = |\sigma|(L, L')$ and τ extends to a locally convex-solid Lebesque topology on L^u.

(5) L^u is Riesz isomorphic to the topological completion $\hat{L}$ of (L, τ).

Proof. (1) It is immediate from Corollary 12 and Theorem 14.

(2) It is enough to show that if $\theta < x \in L^u$ then there exists a discrete element e of L^u with $\theta < e \leq x$. Since $L' = L_n^\sim$ holds by Theorem 14, it is immediate that $L_{L'}' = (L_{L'})_n^\sim = (L^u)_n^\sim$, and for any $\theta < x \in L^u$ there exists a $\theta \leq \varphi \in (L^u)_n^\sim$ such that $\varphi(x) > 0$. Using the argument given in the proof of Theorem 14, it follows that $L^u = C_\varphi \oplus N_\varphi$, and C_φ has a basis $\{e_1, e_2, \cdots, e_n\}$ consisting of pairwise disjoint discrete elements for some $n \in \mathrm{N}$. Hence there exists at least one element e_i satisfying $x \wedge e_i > \theta$. Then $e = x \wedge e_i$ is a discrete element of L^u with $\theta < e \leq x$, and this means that L^u is a discrete Riesz space.

(3) By (2) the result follows immediately from [1, 2.17].

(4) Note first that each $\varphi \in L'$ is expressible as a linear combination of finite number of order-continuous Riesz homomorphisms on L. Hence it is clear that $|\sigma|(L, L') = \sigma(L, L') \leq \tau$. Conversely, to show that $\tau \leq |\sigma|(L, L')$, we assume that for a net $\{x_\alpha\}$ of L^+, $x_\alpha \to \theta(|\sigma|(L, L'))$ in L but $x_\alpha \to \theta(\tau)$ does not hold. Then there exists a τ-neighborhood V of zero such that for any α there exists $\beta \geq \alpha$ with $x_\beta \notin V$. Choose a normal sequence $\{V_n\}$ consisting of τ-closed neighborhoods of zero with $V_1 + V_1 \subset V$ and $V_{n+1} + V_{n+1} \subset V_n$ for all n, and put $N = \cap\{V_n : n \in \mathrm{N}\}$. Each V_n is order closed, and so N is a band of L. Using the argument given in [2, 3.3] it follows that every disjoint system consisting of strictly positive elements of N^d in L is at most countable. Since L^u is discrete by (2) and L is order dense in L^u, by Zorn's lemma there exists an at most countable

complete disjoint system $\{e_i\}$ consisting of discrete elements of N^d in L. For each $i \in \mathrm{N}$ choose a Riesz homomorphism π_i on L such that $\pi_i \in L'$ and $\pi_i(e_i) = 1$, and put $\varphi_n = \Sigma_{i=1}^n \pi_i$ for each $n \in \mathrm{N}$. Since L has the principal projection property and N is a band of L, it is immediate that $L = B \oplus N$, where B denots the band of L generated by $u = \sup \{e_n : n \in \mathrm{N}\} \in L$. Denote by B^o and N^o the polar of B and N respectively. Then $L' = B^o \oplus N^o$ holds by [1, 19.5]. Since L' is a discrete space, for any $\Psi \in N^o$ it is not difficult to verify that there exist some $n \in \mathrm{N}$ and $a > 0$ such that

$$|\Psi(x)| \leq |\Psi|(|x|) \leq a\varphi_n(|x|) \quad \text{for all } x \in B.$$

Hence $|\sigma|(B,N^o)$ generates a metrizable Lebesque topology τ' on B, and $|\sigma|(B,L') = |\sigma|(B,N^o)$ holds. For each α put $x_\alpha = y_\alpha + z_\alpha$, where $y_\alpha \in B$ and $z_\alpha \in N$. It then follows immediately that $y_\alpha \to \theta(\tau')$. Hence there exists a subsequence $\{y_n\}$ of $\{y_\alpha\}$ such that $y_n \to \theta(\tau')$ and $x_n \notin V$. For each i and each n, let $r_n^i = \pi_i(y_n)$ and put $s_n^i = \sup_{n \leq k} r_k^i$. Since $r_n^i \to 0$ as $n \to \infty$, $\{s_n^i : n, i \in \mathrm{N}\} \subset \mathrm{R}^+$ and $w_n = \sup_i s_n^i e^i$ exists in L for all n. It is not difficult to verify that $\theta \leq y_n \leq w_n$ and $w_n \downarrow \theta$ in L. Hence it follows that $y_n \to \theta(\tau)$, and so there exists a some n_0 such that $x_n = y_n + z_n \in V_1 + V_1 \subset V$ for all $n \geq n_0$. But this is a contradiction. Thus $\tau \leq |\sigma|(L,L')$ holds and hence we have $\tau = |\sigma|(L,L')$. From this, it follows that τ exists to a $|\sigma|(L^u,(L^u)')$-topology on L^u and $\left(L^u, |\sigma|(L^u,(L^u)')\right)$ has the Lebesgue property.

(5) By (4), the result now follows immediately from [1, 24.3].

COROLLARY 16 ([1, 23.33]). If a laterally complete Riesz space L admits a Hausdorff locally convex-solid Lebesque topology, then $L = L_{L'}$ and L is Riesz isomorphic to some R^{X}.

Proof. By Theorem 15 is a discrete Riesz space and $L^u = L_{L'}$. Let $\{e_\alpha : \alpha \in \mathrm{X}\}$ be a complete disjoint system of $L_{L'}$ consisting of discrete elements. Since L is order dense in $L_{L'}$, it is immediate that $\{e_\alpha : \alpha \in \mathrm{X}\} \subset L$. Furthermore, since L is laterally complete, it follows easily that $L^u = L$ and L is Riesz isomorphic to R^{X}.

Finally, we close this paper by generalizing example 24.15 given in [1] (compare [1, 20.26]).

THEOREM 17. Let L be a σ-laterally complete Riesz space carrying a Hausdorff locally covex-solid Lebesgue topology τ, and let L' be its topological dual. If L has the countable sup property, then the following holds.

(1) L is τ-sequentially complete.

(2) L is Riesz isomorphic to a σ-ideal A of some R^{X} such that
$A = \{f \in \mathrm{R}^{\mathrm{X}} : (x \in \mathrm{X} : f(x) \neq 0) \text{ is at most countable}\}$.

(3) (L,τ) (which must be necessarily $(L, \sigma(L,L'))$) has the σ-Levi Property.

Proof. (1) Note first that by Theorem 15 L is a discrete Riesz space in its own right. Choose a complete disjoint system $\{e_\alpha : \alpha \in \mathrm{X}\}$ consisting of discrete elememts of

L. For each $\theta \leq x \in L$, $x = \sup\{P_\alpha(x) : \alpha \in X\}$ holds in L, where P_α denotes the projection of L onto the band of L generated by e_α. By the assumption there exists a subsequence $\{e_\alpha : n \geq 1\}$ of $\{e_\alpha : \alpha \in X\}$ such that $x = \sup\{P_n(x) : n \geq 1\}$. Now, let $\{x_n\}$ be a τ-Cauchy sequence of L. For each n there exists a subset $\{\lambda_{n,\alpha} : \alpha \in X\} \subset R$ such that $x_n = \sup\{P_\alpha(x_n) : \alpha \in X\} = \sup\{\lambda_{n,\alpha} e_\alpha : \alpha \in X\}$, where $\lambda_{n,\alpha} = 0$ except for at most countable many indices $\alpha \in X$. For each $\alpha \in X$ choose a Riesz homomorphism π_α of L satisfying $\pi_\alpha \in L'$ and $\pi_\alpha(e_\alpha) = 1$. Then for $\varepsilon > 0$ and $\alpha \in X$ there exists some $n_0 \in N$ such that

$$|\pi_\alpha(x_n - x_m)| = |\lambda_{n,\alpha} - \lambda_{m,\alpha}| < \varepsilon \quad \text{for } m,\ n \geq n_0 .$$

Hence $\lambda_{n,\alpha} \to \lambda_\alpha$ holds in R with $\lambda_\alpha = 0$ except for at most countable many indices $\alpha \in X$, and so $x = \sup\{\lambda_\alpha e_\alpha : \alpha \in X\}$ exists in L. For any $\varphi \in L'$ choose Riesz homomorphisms π_{α_i} on L such $\varphi = \Sigma_{i=1}^n r_i \pi_{\alpha_i}$, where $\{r_i : 1 \leq i \leq n\} \subset R$ and $\{\alpha_i : 1 \leq i \leq n\} \subset X$. Then it follows that for any $\varepsilon > 0$ there exists some $m_0 \in N$ such that

$$|\varphi(x - x_m)| = \left|\Sigma_{i=1}^n r_i \lambda_{\alpha_i} - \Sigma_{i=1}^n r_i \lambda_{m,\alpha_i}\right| \leq \varepsilon \Sigma_{i=1}^n |r_i| \text{ for all } m \geq m_0 .$$

This means that L is $\sigma(L,L')$-sequentially complete, and hence it follows by Theorem 15 (4) that L is τ-sequentially complete.

(2) From the argument given in (1), each $\theta \leq x \in L$ is expressible as $x = \sup\{P_\alpha(x) : \alpha \in X\} = \sup\{P_\alpha(x) : n \geq 1\}$ for some $\{e_n : n \geq 1\} \subset \{e_\alpha : \alpha \in X\}$. Since $x \wedge e_\alpha = \theta$ holds except for at most countable many indices $\alpha \in X$, the result now follows immediately from the argument given in [1, 2.17].

(3) It follows immediately from the argument used in (1).

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