Convergence of the Modified Eigenvalues of the p-Laplacian

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ABSTRACT

In a previous paper [4] we proved that the first eigenvalues of the p-Lapalacian on manifold $M\backslash B_\varepsilon$ converge to zero as ε tends to zero. In this paper we prove that the modified higher eigenvalues of the p-Laplacian on manifold $M\backslash B_\varepsilon$ converge to the modified eigenvalues of that on manifold M.

1 Introduction

We recall the notation in the previous paper [4]. In this paper we assume $p \geq 2$, thus $L^p \subset L^2$. Let M be a compact Riemannian manifold with $\dim M = m$ and $\triangle_p$ the p-Laplacian acting on functions on M, where $\triangle_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$. Let M^* be a compact submanifold of M, and B_ε the tubular neighborhood of M^* of radius $\varepsilon > 0$, that is,

$$B_\varepsilon = \{x \in M \,;\, d(x, M^*) < \varepsilon\}.$$

Denote by $\triangle_{p,\varepsilon}$ the restriction of $\triangle$ to those functions on M vanishing identically in B_ε. Set

$$\Omega_\varepsilon = M\backslash B_\varepsilon, \partial\Omega_\varepsilon = \partial B_\varepsilon.$$

We consider the following Dirichlet problem

$$\begin{cases} \triangle_{p,\varepsilon} u(x) + \lambda(\varepsilon)|u(x)|^{p-2}u(x) = 0 & x \in \Omega_\varepsilon \\ u(x) = 0 & x \in \partial\Omega_\varepsilon. \end{cases}$$

In the case of $p=2$, many people have studied the asymptotic expansion of the eigenvalues $\lambda_{k,2}(\varepsilon)$ $(k = 1,2,\dots)$ for the 2-Laplacian of a manifold $M\backslash B_\varepsilon$ with the Dirichlet condition on the tubular neighborhood B_ε. The first eigenvalue $\lambda_{1,p}(\Omega_\varepsilon)$ of the p-Laplacian is defined as the least number λ for which the Dirichlet problem has a nontrivial solution $u \in W_0^{1,p}(\Omega_\varepsilon)$. Here the sobolev space $W_0^{1,p}(\Omega_\varepsilon)$ is the completion of $C_0^\infty(\Omega_\varepsilon)$ with respect to the Sobolev norm $\|u\|_{1,p} = \{\int_{\Omega_\varepsilon}(|u|^p + |\nabla u|^p)dv_g\}^{1/p}$. It can be characterized by

$$\lambda_{1,p}(\Omega_\varepsilon) = \inf_{u \neq 0} \frac{\int_{\Omega_\varepsilon} |\nabla u|^p \, dv_g}{\int_{\Omega_\varepsilon} |u|^p \, dv_g},$$

where u runs over $W_0^{1,p}(\Omega_\varepsilon)$ and dv_g denotes the volume element of M. For the higher eigenvalues, We will define the modified variational eigenvalues. Let $\{\phi_j\}(j = 1,2,\dots)$ be eigenfunctions of p-Laplacian associated with $\lambda_{j,p}(\Omega_\varepsilon)$ and $\|\phi\|_p = 1$. Set

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$$\lambda_{k,p}(\Omega_\varepsilon)=\inf_u\left\{\frac{\int_{\Omega_\varepsilon}|\nabla u|^p dv_g}{\int_{\Omega_\varepsilon}|u|^p dv_g},\int_{\Omega_\varepsilon} u\phi_j dv_g = 0\,(j=1,2,\dots,k-1)\right\}.$$

It can be characterized by

$$\lambda_{k,p}(\Omega_\varepsilon)=\inf_{V\in W_k}\sup_{u\in V\setminus\{0\}}\frac{\int_{\Omega_\varepsilon}|\nabla u|^p dv_g}{\int_{\Omega_\varepsilon}|u|^p dv_g}.$$

Here

$$W_k=\{V : V \text{ is a subspace of } W_0^{1,p}, \dim V\geq k, k\in N\}.$$

For $p>1, \lambda_{k,p}(\Omega_\varepsilon)\geq c_{k,p}$, where $\{c_{k,p}\}$ is the minimax sequence usually defined using genus γ, that is,

$$c_{k,p}(\Omega_\varepsilon)=\inf_{A\in\Sigma_k}\sup_{u\in A}\frac{\int_{\Omega_\varepsilon}|\nabla u|^p dv_g}{\int_{\Omega_\varepsilon}|u|^p dv_g},$$

where $\Sigma=\{A\in W_0^{1,p} : A=-A \text{ and } A \text{ is closed}\}$ and $\Sigma_k=\{A\in\Sigma : \gamma(A)\geq k\}$.
For $A\in\Sigma$ we define the genus $\gamma(A)$ as

$n=\gamma(A)=\min\{k\in\mathbf{N}$: there exists $h\in C^0(A,\mathbf{R}^k\setminus\{0\},)\, h(x)=-h(-x)$ i.e. h is an odd continuous map$\}$

$\gamma(A)=\infty$ if there exists no finite such n, and $\gamma(\phi)=0$. Clearly, $\lambda_{1,p}=c_{1,p}$ and
$\lambda_{k,p}\geq c_{k,p}$ for all $k\in\mathbf{N}$ since $W_k\subset\Sigma_k$ [1] [2].

2 Result

Theorem 1 *Given $N=1,2,\dots$, there exists a sequence of positive numbers $\{\varepsilon_l\}\downarrow 0$ as $l\to\infty$, N orthogonal p-eigenfunctions $\{\varphi_1,\dots,\varphi_N\}$ of the modified variational eigenvalues $\{\lambda_{1,p},\dots,\lambda_{N,p}\}$ on M, N orthogonal p-eigenfunctions $\{\varphi_1(\varepsilon_l),\dots,\varphi_N(\varepsilon_l)\}$ of the modified variational eigenvalues $\{\lambda_{1,p}(\varepsilon_l),\dots,\lambda_{N,p}(\varepsilon_l)\}$ on Ω_ε such that*

$$\lim_{l\to\infty}\varphi_j(\varepsilon_l)=\varphi_j \quad \textit{and} \quad \lim_{l\to\infty}\lambda_j(\varepsilon_l)=\lambda_j$$

in $L^p(M)$ for $j=1,\dots N$.

Proof: We prove this by induction. For $N=1$, the theorem is valid by theorem 2 in [4]. Assume that the theorem is valid for an positive integer N, that is, for $1\leq j\leq N$,

$$\lim_{l\to\infty}\|\varphi_j(\varepsilon_l)-\varphi_j\|_{1,p}=0$$

Let φ_{N+1} be an eigenfunction of p-Laplacian (briefly we call it a p-eigenfunction) associated with $\lambda_{N+1,p}$ with $\int_M|\varphi_{N+1}|^p dv_g=1$, and L^2 orthogonal to $\{\varphi_1,\dots,\varphi_N\}$. By lemma 1 in [4] there exitsts $\psi_l\in W_0^{1,p}(\Omega_{\varepsilon l})$ such that $\lim_{l\to\infty}\|\psi_l-\varphi_{N+1}\|_{1,p}=0$ and $\lim_{l\to\infty}\varepsilon_l=0$. We have

$$\lim_{l\to\infty}\int_{\Omega_{\varepsilon l}}|\nabla(\psi_l)|^p dv_g=\int_M|\nabla\varphi_{N+1}|^p dv_g=\lambda_{N+1,p}.$$

Set

$$f_l = \psi_l - \sum_{j=1}^{N} \langle \psi_l, \varphi_j(\varepsilon_l) \rangle \varphi_j(\varepsilon_l),$$

$$g_l = \frac{f_l}{\|f_l\|_p},$$

where $\|f_l\|_p = (\int_{\Omega_{\varepsilon l}} |f_l|^p dv_g)^{1/p}$.

Then g_l is orthogonal to $\{\varphi_1(\varepsilon_l), \varphi_2(\varepsilon_l), ..., \varphi_N(\varepsilon_l)\}$, so

$$g_l \in W_{N+1} = \{V : \text{subspace of } W_0^{1,p}(\Omega_\varepsilon) \text{ and } \dim V \geq N+1\}$$

and

$$\lim_{l \to \infty} \|g_l - \varphi_{N+1}\|_p = 0.$$

The variational principle implies that

$$\int_{\Omega_{\varepsilon l}} |\nabla g_l|^p dv_g \geq \lambda_{N+1,p}(\varepsilon_l) \tag{1}$$

for all l. Since $\int |\nabla g_l|^p dv_g \to \lambda_{N+1,p}$ as $l \to \infty$, we have

$$\lambda_{N+1,p} \geq \limsup_{l \to \infty} \lambda_{N+1,p}(\varepsilon_l). \tag{2}$$

Next we prove the following claim.

Claim 2

$$\liminf_{l \to \infty} \lambda_{N+1,p}(\varepsilon_l) \geq \lambda_{N+1,p}. \tag{3}$$

From (2) and (3) we have

$$\lim_{l \to \infty} \lambda_{N+1,p}(\varepsilon_l) = \lambda_{N+1,p}.$$

To prove claime, for each l, take $\varphi_{N+1}(\varepsilon_l)$ to be a normalized p-eigenfunction associated with $\lambda_{N+1,p}(\varepsilon_l)$ orthogonal to $\{\varphi_1(\varepsilon_l), ..., \varphi_N(\varepsilon_l)\}$. Since $C^\infty(M)$ is dence in $L_1^p(M)$, there exists $\psi_{N+1,l} \in C^\infty(M)$ such that

$$\|\psi_{N+1,l} - \varphi_{N+1}(\varepsilon_l)\|_{1,p} < \frac{1}{l}, \tag{4}$$

and

$$\int |\psi_{N+1,l}|^p dv_g = 1 \text{ for all } l.$$

The induction hypothesis implies

$$|\langle \varphi_{N+1}(\varepsilon_l), \varphi_j \rangle| = |\langle \varphi_{N+1}(\varepsilon_l), \varphi_j - \varphi_j(\varepsilon_l) \rangle| \leq \|\varphi_j - \varphi_j(\varepsilon_l)\|_{1,p} \to 0 \, (l \to \infty).$$

Thus we have

$$\lim_{l \to \infty} \langle \varphi_{N+1,l}, \varphi_j \rangle = 0 \qquad \text{for } j = 1, ..., N.$$

Now set

$$\phi_{N+1,l} = \sum_{j=1}^{\infty} d_{l,j}\varphi_j, \tag{5}$$

then

$$1 = \sum_{j=1}^{\infty} (d_{l,j})^2,$$

and

$$\lim_{l\to\infty} d_{l,j} = 0 \qquad \text{for } j = 1,\dots,N. \tag{6}$$

Let

$$f_{N+1,l} := \phi_{N+1,l} - \sum_{j=1}^{N} d_{l,j}\varphi_j,$$

$$g_{N+1,l} = \frac{f_{N+1,l}}{\|f_{N+1,l}\|_p}. \tag{7}$$

Then $g_{N+1,l}$ is orthogonal to $\{\varphi_1,\dots,\varphi_N\}$. The variational principle implies

$$\lambda_{N+1,p} \le \int |\nabla g_{N+1,l}|^p dv_g = \lambda_{N+1,p}(\varepsilon_l) + b_l.$$

While we have $b_l \to 0$ as $l \to \infty$ by (4) and (6). We therefore have

$$\lambda_{N+1,p} \le \liminf_{l\to\infty} \lambda_{N+1,p}(\varepsilon_l).$$

This completes the proof of the claim 2.

Now from (5) and (7),

$$g_{N+1,l} = \frac{\sum_{j=N+1}^{\infty} d_{l,j}\varphi_j}{\sqrt{1-\sum_{j=1}^{N}(d_{l,j})^2}}.$$

Let l_0 be the smallest integer such that $\lambda_{N+l_0+1,p} > \lambda_{N+1,p}$, and set

$$c_l = \sum_{j=N+l_0+1}^{\infty} (d_{l,j})^2,$$

$$a = \limsup_{l\to\infty} c_l.$$

Assume $a > 0$.

$$\lambda_{N+1,p} = \int_M |\nabla g_{N+1,l}|^p dv_g = -\int_M g_{N+1,l}\,\Delta_p\, g_{n+1} dv_g$$

$$= -\frac{1}{1-\sum_{j=1}^{N}(d_{l,j})^2}\int \sum d_{l,j}\varphi_j \sum d_{l,j}\Delta_p\varphi_j dv_g$$

$$= \frac{1}{1-\sum_{j=1}^{N}(d_{l,j})^2}\sum_{j=N+1}^{\infty}\lambda_{j,p}(d_{j,p})^2 \ (\text{because of } \int|\varphi_j|^p dv_g = 1)$$

$$>\frac{1}{1-\sum_{j=1}^{N}(d_{l,j})^2}\left\{\lambda_{N+1,p}\sum_{j=N+1}^{N+l_0}(d_{l,j})^2+\lambda_{N+l_0+1,p}\sum_{j=N+l_0+1}^{\infty}(d_{l,j})^2\right\}.$$

Let $\{l'\}$ be a subsequence of $\{l\}$ such that

$$a=\lim_{l'\to\infty} c_{l'}$$

Then

$$\lambda_{N+1,p}\geq\lambda_{N+1,p}(1-a)+\lambda_{N+l_0+1,p}a>\lambda_{N+1,p}$$

which implies a contradiction. So we get $a=0$. Thus

$$\lim_{l\to\infty}\varphi_{N+1}(\varepsilon_l)=\lim_{l\to\infty}\sum_{j=1}^{\infty}d_{l,j}\varphi_j=\sum_{j=N+1}^{N+l_0}d_{l,j}\varphi_j.$$

Let H be the subspace of $L^p(M)$ spanned by $\{\varphi_{N+1},\ldots,\varphi_{N+l_0}\}$, $H^\perp$ its othogonal complement in $L^p(M)$, and $P,P^\perp$ the respective projection operators associated with $H,H^\perp$. Then we have

$$\lim_{l\to\infty}P^\perp\varphi_{N+1}(\varepsilon_l)=0.$$

If, H is one-dimensional then we can see

$$\lim_{l\to\infty}\varphi_{N+1}(\varepsilon_l)=\varphi_{N+1}\quad\text{in } L^p(M).$$

(1) Since $\lambda_{1,p}(M)=0$ and $\lambda_{1,p}(\Omega_\varepsilon)$ always have multiplicity 1, we have

$$\lim_{l\to\infty}\varphi_1(\varepsilon_l)=\varphi_1.$$

(2) When all eigenspaces have multiplicity 1, we construct orthonormal bases $\{\varphi_1,\varphi_2\ldots\}$,$\{\varphi_1(\varepsilon_l),\varphi_2(\varepsilon_l)\ldots\}$ of the closed and boundary problems respectively such that

$$\lim_{l\to\infty}\varphi_j(\varepsilon_l)=\varphi_j\quad(j=1,2,\ldots).$$

(3) When $l_0>1$, we have a subsequence $\{l'\}\subset\{l\}$ and numbers $a_{N+1},\ldots,a_{N+l_0}$ such that

$$\lim_{l\to\infty}d_{l,j}=a_j\quad\text{for } j=N+1,\ldots N+l_0,$$

from which we have

$$\lim_{l\to\infty}\varphi_{N+1}(\varepsilon_l)=\sum_{j=N+1}^{N+l_0}a_j\varphi_j.$$

$\Delta_p\varphi_j+\lambda_N|\varphi_j|^{p-2}\varphi_j=0$ for all $j=N,..,N+l_0$, and $\sum(a_j)^2=1$. For the orthogonal set of p-eigenfunctions

$$\{\varphi_1,\ldots\varphi_N,\sum_{j=N+1}^{N+l_0}a_j\varphi_j\},$$

the induction $(N+1)$ is true.

(4) We have for each $j=1,2,\ldots$ a sequence $\varepsilon_l\to0$ such that $\lambda_{j,p}(\varepsilon_l)\to\lambda_{j,p}$ as $l\to\infty$.

References

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