REMARK ON "A NOTE ON CONVERGENCE OF NEWTON'S METHOD"

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ABSTRACT

In our paper, we are giving a new proof of L.B.Rall's theorem [1]. In our proof we assume the open ball $U^* = \{x : \|x - x^*\| < (r-\sqrt{r})/(rB^*K)\}$, (where r is positive real number), where we generalize the radius by $(r-\sqrt{r})/(rB^*K)$, we will show that for the best possible bound for h, $r = 2$.

Introduction. Let X and Y be Banach spaces and P an operator $P : X \to Y$. If there exists a bounded linear operator L from X into Y such that at some point $x \in D_p$ $\| P(x+h) - P(x) - Lh \| = o(\| h \|)$, $h \in X$, then Lh is called the Fréchet differential of $P(x)$ at x and the operator L is called the Fréchet drivative of $P(x)$ at x and we write $L = P'(x)$.

Let $F : X \to X$ be a Fréchet differentiable operator. Newton's method is an attach to find a solution $x = x^*$ of the equation $F(x) = 0$, which consists of the construction of the sequence $\{x_m\}$ defined by,

$$x_{m+1} = x_m - [F'(x_m)]^{-1} F(x_m), \qquad m = 0, 1, 2, 3, \cdots \tag{1}$$

starting from some suitable chosen $x_0 \in X$. Sufficient conditions for the success of this procedure are given by the famous theorem of L. V. Kantorovich [4], [6]. Let us assume that the Fréchet derivative F' of F is Lipschitz continuous with constant K in some region U: that is

$$\| F'(x) - F'(y) \| \le K \| x - y \|, \qquad x, y \in U. \tag{2}$$

Then the hypotheses of the Kantorovich theorem are as follows:

(a) $[F'(x_0)]^{-1}$ exists and for constants B, η such that

$$\| [F'(x_0)]^{-1} \| \le B \quad , \quad \| [F'(x_0)]^{-1} F(x_0) \| \le \eta. \tag{3}$$

one has

$$h = BK\eta \le \frac{1}{2}, \tag{4}$$

(b) $U_0 \subset U$ where $U_0 = \{x : \| x - x_0 \| \le (1 - \sqrt{1-2h})(\eta / h)\}$. $\quad$ (5)

If these hypotheses are satisfied then the Newton sequence $\{x_m\}$ exists and converges to $x^* \in U_0$ such that $F(x^*) = 0$. The value of the constant h defined by (4) is significant for the study of the convergence of Newton's method.

We are giving the statement and proof of L. B. Rall's theorem [1].

Theorem.

If x^* is a simple zero of F,

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$$\| [F'(x_0)]^{-1} \| \le B^*$$

and

$$U_* = \left\{ x : \| x - x^* \| < \frac{1}{(B^* K)} \right\} \subset U, \tag{6}$$

then the hypotheses (a) with $h < \frac{1}{2}$ and (b) of the Kantorovich theorem are satisfied at each $x_0 \in U_*^*$ where $U^* = \{x : \| x - x^* \| < (2-\sqrt{2})/(2B^* K)\}$. (7)

Proof.

Let $U^* = \left\{ x : \| x - x^* \| < (r - \sqrt{r})/(rB^* K) \right\}$.

For $x_0 \in U^*$

$$\frac{r-\sqrt{r}}{rB^*K} = \frac{r-\sqrt{r}}{r\frac{1}{Kx^*}K} = (1 - \frac{\sqrt{r}}{r})x^*,$$

$$x_0 = x^* - (1 - \frac{\sqrt{r}}{r})x^* = \frac{x^*}{\sqrt{r}}, \text{ where } B^* = \frac{1}{Kx^*}.$$

$$\begin{aligned}
\| F'(x_0) - F'(x^*) \| &\le K \| x_0 - x^* \| < K \| -(1 - \frac{\sqrt{r}}{r})x^* \| \\
&< \left\{ (r - \sqrt{r})(\frac{Kx^*}{r}) \right\} \\
&< (r - \sqrt{r})/(rB^*) \\
&< \| [F'(x^*)]^{-1} \|^{-1}
\end{aligned} \tag{8}$$

so that $[F'(x_0)]^{-1}$ exists and

$$B = \frac{B^*}{1 - B^* K \| x_0 - x^* \|} = \left(\frac{\sqrt{r} K x^*}{r} \right)^{-1} \ge \| [F'(x_0)]^{-1} \| . \tag{9}$$

Using the fundamental theorem of calculus, we have

$$\begin{aligned}
F(x^*) - F(x_0) &= \int_0^1 F'(x_0 + \theta(x^* - x_0))(x^* - x_0) d\theta \\
&= F'(x_0)(x^* - x_0) + \int_0^1 [F'(x_0 + \theta(x^* - x_0)) - F'(x_0)](x^* - x_0) d\theta .
\end{aligned} \tag{10}$$

As $F(x^*) = 0$,

$$-[F'(x_0)]^{-1} F(x_0) = (x^* - x_0) + [F'(x_0)]^{-1} \int_0^1 [F'(x_0 + \theta(x^* - x_0)) - F'(x_0)](x^* - x_0) d\theta \tag{11}$$

and thus

$$\| F'[(x_0)]^{-1} F(x_0) \| \le \left\{ 1 + BK \| x^* - x_0 \| \int_0^1 \theta d\theta \right\} \| x^* - x_0 \| \tag{12}$$

from which (9) may be used to obtain

$$\eta = \frac{1 - \frac{1}{2} B^* K \| x^* - x_0 \|}{1 - B^* K \| x^* - x_0 \|} \| x^* - x_0 \| = \frac{\frac{1}{2}(1 - \frac{1}{r}) x^*}{\frac{1}{\sqrt{r}}} \ge \| [F'(x_0)]^{-1} F(x_0) \| . \tag{13}$$

It follows that from $x_0 \in U^*$

$$h = BK\eta = \left(\frac{Kx^*}{\sqrt{r}} \right)^{-1} K \frac{\frac{1}{2}(1 - \frac{1}{r}) x^*}{\frac{1}{\sqrt{r}}} = (\frac{r}{2} - \frac{1}{2}) . \tag{14}$$

From (14) it is eveident that for $r > 2$ the bounds for h are very crude but for $r = 2$ the bound becomes the best bound which satisfies Kantorovich theorem.

From the above proof we can conclude that the generalization of the radius of U^* by $(r - \sqrt{r})/(rB^* K)$, (r is a positive real number) is not possible.

Example〔1〕Consider the quadratic operator

$$F(x) = \frac{1}{2} K (x^2 - x^{*2}),$$

where $K > 0$. Therefore,

$U^* = \{x : \| x - x^* \| < (r - \sqrt{r})/(rB^* K)\}$ and r is a positive real number.

We have $\frac{r - \sqrt{r}}{rB^* K} = \frac{r - \sqrt{r}}{r \frac{1}{Kx^*} K} = (1 - \frac{\sqrt{r}}{r}) x^*$ and for

$$x_0 = x^* - (1 - \frac{\sqrt{r}}{r}) x^* = \frac{x^*}{\sqrt{r}}, \quad \text{where } B^* = \frac{1}{kx^*},$$

$$B = \frac{B^*}{1 - B^* K \| x_0 - x^* \|} = \left(\frac{\sqrt{r} K x^*}{r} \right)^{-1} \ge \| [F'(x_0)]^{-1} \| ,$$

and

$$\eta = \frac{1 - \frac{1}{2} B^* K \| x_0 - x^* \|}{1 - B^* K \| x_0 - x^* \|} \| x_0 - x^* \| = \frac{\frac{1}{2}(1 - \frac{1}{r}) x^*}{\frac{1}{\sqrt{r}}} \ge \| [F'(x_0)]^{-1} F(x_0) \| .$$

From which $h = BK\eta = \frac{\frac{1}{2}(1 - \frac{1}{r})}{\frac{1}{r}} = (\frac{r}{2} - \frac{1}{2})$

Hence $h = \frac{1}{2}$ if $r = 2$.

References

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