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Remarks on positive and free boundary solutions to a singular equation

Juan Dávila[a,†], Marcelo Montenegro[b,‡,*]

[a] Universidad de Chile, Departamento de Ingeniería Matemática and CMM, Casilla 170, Santiago, Chile.
[b] Universidade Estadual de Campinas, IMECC, Departamento de Matemática, Campinas, SP, Brazil, CEP 13083-970.

Abstract. The equation $-\Delta u = \chi_{\{u>0\}}\big(-\frac{1}{u^\beta}+\lambda f(x,u)\big)$ in Ω with Dirichlet boundary condition on $\partial\Omega$ has a maximal solution $u_\lambda \geq 0$ for every $\lambda > 0$. For λ less than a constant λ^* the solution vanishes inside the domain, and for $\lambda > \lambda^*$ the solution is positive and stable. We obtain optimal regularity of u_λ even in the presence of the free boundary. If $0 < \lambda < \lambda^*$ the solutions of the singular parabolic equation $u_t - \Delta u + \frac{1}{u^\beta} = \lambda f(u)$ quench in finite time, and for $\lambda > \lambda^*$ the solutions are globally positively defined.
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Soluciones positivas y soluciones con frontera libre para ecuaciones singulares

Resumen. La ecuación $-\Delta u = \chi_{\{u>0\}}\big(-\frac{1}{u^\beta}+\lambda f(x,u)\big)$ en Ω con condición de frontera de tipo Dirichlet en $\partial\Omega$ posee una solución $u_\lambda \geq 0$ para $\lambda > 0$. Si λ es menor que una constante λ^* la solución es nula dentro de una región del dominio, y para $\lambda > \lambda^*$ la solución es positiva y estable. Obtenemos la regularidad óptima de u_λ aun con la frontera libre. Si $0 < \lambda < \lambda^*$ las soluciones de la ecuación parabólica singular $u_t - \Delta u + \frac{1}{u^\beta} = \lambda f(u)$ son nulas en tiempo finito, y para $\lambda > \lambda^*$ las soluciones son positivas y globalmente definidas.
Palabras claves: Ecuaciones singulares, frontera libre.

[*] Corresponding author: *E-mail*: msm@ime.unicamp.br.
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[‡] Partially supported by FAPESP and CNPq Brazil.

1. Introduction

We study the elliptic problem

$$\begin{cases} -\Delta u = g_\lambda(x,u) & \text{in } \Omega, \\ u \geq 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

on a smooth, bounded domain $\Omega \subset \mathbb{R}^n$ with a singular nonlinearity g_λ given by

$$g_\lambda(x,u) = \chi_{\{u>0\}}\Big(-\frac{1}{u^\beta} + \lambda f(x,u)\Big). \tag{2}$$

The constant β is positive, but we will mainly focus on the case $0 < \beta < 1$, $\lambda > 0$ is a parameter, $\chi_{\{u>0\}}$ is the characteristic function of the set $\{u > 0\}$ and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a function which is measurable in x, $f \geq 0$, $f \not\equiv 0$, and it is nondecreasing, concave and sublinear in the second variable u uniformly in x, that is,

$$\lim_{u\to\infty} \frac{f(x,u)}{u} = 0 \quad \text{uniformly for } x \in \Omega.$$

We also assume that $f_u(x,\cdot)$ is continuous on $(0,\infty)$ for a.e. $x \in \Omega$.

Equation (1) arises as limit of some equations modeling catalytic and enzymatic reactions (see [1] and [10] for an account).

Definition 1.1.

(i) Throughout the paper we use the notation

$$\delta(x) = \operatorname{dist}(x, \partial\Omega).$$

(ii) We say that $u \in H_0^1(\Omega)$, $u \geq 0$ is a solution of (1) if

$$\chi_{\{u>0\}}\Big(-\frac{1}{u^\beta} + \lambda f(x,u)\Big)\delta \in L^1(\Omega),$$

and

$$\int_\Omega \nabla u \nabla \varphi = \int_{\{u>0\}} \Big(-\frac{1}{u^\beta} + \lambda f(x,u)\Big)\varphi \quad \forall \varphi \in C_0^\infty(\Omega).$$

By a positive classical solution we mean a function $u \in C(\overline{\Omega}) \cap C^2(\Omega)$ which is positive in Ω and satisfies (1) in the usual sense.

This note is intended as a summary of results for the elliptic problem (1) as well as its parabolic counterpart. Complete proofs appeared in [6] and [8]. Here we have also included some remarks and detailed examples that are not in [6] and [8]. Further questions are addressed in [7], [16] and [9].

2. Existence of a maximal solution and its regularity

Theorem 2.1. *Assume $0 < \beta < 1$. Then there is a unique maximal solution u_λ to (1) for any $\lambda > 0$. Moreover, there exists $\lambda^* \in (0, \infty)$ such that for $\lambda > \lambda^*$ the maximal solution u_λ is positive in Ω, and belongs to $C(\overline{\Omega}) \cap C^{1,\mu}_{\text{loc}}(\Omega)$ for all $0 < \mu < 1$. We also deduce that $a\delta \leq u_\lambda \leq b\delta$ in Ω, where a, b are positive constants depending only on Ω, $\lambda > 0$ and f. If $f \in C^1(\overline{\Omega} \times [0, \infty))$ then actually u_λ is a classical solution.*

For $0 < \lambda \leq \lambda^*$ the maximal solution u_λ has optimal regularity $C(\overline{\Omega}) \cap C^{1,\gamma}_{\text{loc}}(\Omega)$ with $\gamma = \frac{1-\beta}{1+\beta}$, and for $0 < \lambda < \lambda^*$ the set $\{u_\lambda = 0\}$ has positive measure.

Particular cases of equation (1) were already considered in the literature. Díaz, Morel and Oswald [11] and Choi, Lazer and McKenna [4] studied the problem where f is bounded and depends only on x. They proved some results on existence, uniqueness and stability of solutions. Shi and Yao [18] studied the equation with $g_\lambda(x, u) = -K(x)/u^\beta + \lambda u^p$ with $0 < p < 1$, but only considered positive solutions. The weight K could change sign, but when $\inf_\Omega K > 0$ they found results similar to ours. Problems involving singular functions with different behavior from g_λ (more precisely, with the opposite sign in front of the singular term $u^{-\beta}$) were addressed by Crandall, Rabinowitz and Tartar [5], Mignot and Puel [15] and Gui and Lin [13].

Phillips [17] established interior $C^{1,\frac{1-\beta}{1+\beta}}$ estimates for local minimizers of the energy functional $\int \frac{1}{2}|\nabla u|^2 + (u^+)^{1-\beta}$ in the convex set $\{u \in H^1(\Omega) : u = 1 \text{ on } \partial\Omega\}$. He also showed with an example that the exponent $\frac{1-\beta}{1+\beta}$ is the best possible. One of the ideas behind his proof is that minimizers are preserved under a certain scaling. This is not exactly the case for our problem (1), which can be viewed as a perturbation by f of the minimization studied in [17]. Giaquinta and Giusti [12] provided a different proof for the result of Phillips, which applies only to minimizers of more general functionals, not necessarily scaling invariant.

We obtain the maximal solution u_λ as the (decreasing) limit of the maximal solutions $u_{\lambda,\epsilon}$ to

$$\begin{cases} -\Delta u + \dfrac{u}{(u+\epsilon)^{1+\beta}} = \lambda f(x, u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3}$$

as $\epsilon \to 0$. This approach is inspired by the work of Díaz [10].

First we show that $u_{\lambda,\epsilon}$ converges pointwisely to the maximal subsolution u of the following problem:

$$\begin{cases} -\Delta u + \chi_{\{u>0\}} \dfrac{1}{u^\beta} = \lambda f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{4}$$

Adapting the techniques of [17] we progressively regularize this maximal subsolution and obtain precise estimates of its derivatives. This approach allows us to verify that the function u satisfies (1) and we deduce that $u_\lambda = u$. A byproduct of these estimates is the uniform convergence $u_{\lambda,\epsilon} \to u$ in $\overline{\Omega}$ as $\epsilon \to 0$ (and not only a.e.).

Some additional properties related to problem (1) are listed below.

Remark 2.2.

(A) Any solution u satisfies $u^{-\beta}\chi_{\{u>0\}} \in L^1(\Omega)$ (and not just $u^{-\beta}\chi_{\{u>0\}}\delta \in L^1(\Omega)$).

(B) Set $u^* = u_{\lambda^*}$. Then u^* is positive a.e. in Ω, although it can vanish at some points in Ω (this makes sense because it is continuous). But $u^* > 0$ in Ω, when β satisfies an appropriate condition (see Theorem 3.2). The optimality of this situation is discussed in Examples 4.4 and 4.5.

(C) u^* is unique in the class of solutions which are positive a.e. in Ω. A similar result by Martel [14] deals with convex nonlinearities.

(D) For $\beta \geq 1$ and any $\lambda \geq 0$ there is no solution of (1) which is positive a.e. in Ω. This statement was already proved in less generality in [4].

(E) If $f \equiv 0$ there is no positive solution of (1).

3. Stability

The question of stability of the maximal solution u_λ for $\lambda \geq \lambda^*$ leads us to define, for a function $u \in L^1_{\text{loc}}(\Omega)$, $u > 0$ a.e. in Ω, the expression

$$\Lambda(u) = \inf_{\varphi \in C_0^\infty(\Omega)} \frac{\displaystyle\int_\Omega |\nabla\varphi|^2 - \big(\beta u^{-\beta-1} + \lambda f_u(x,u)\big)\varphi^2}{\displaystyle\int_\Omega \varphi^2} \tag{5}$$

(for a general $u > 0$ a.e. $\Lambda(u)$ makes sense, but can be $-\infty$). This is the first eigenvalue of the linearization of problem (1).

Theorem 3.1. *Assume* $0 < \beta < 1$. *For* $\lambda > \lambda^*$ *the maximal solution* u_λ *of* (1) *is stable, that is,* $\Lambda(u_\lambda) > 0$.

For $\lambda = \lambda^*$ *the solution* u^* *is weakly stable, in the sense that* $\Lambda(u^*) \geq 0$. *Conversely, if* u *is a solution of* (1) *for some* $\lambda \geq \lambda^*$ *such that* u *is positive a.e. and* $\Lambda(u) \geq 0$, *then* u *coincides with the maximal solution (i.e.,* $u = u_\lambda$*).*

The stability property allow us to obtain the positivity for u^* under some restrictions on β.

Theorem 3.2. *Let* $\beta \in (0,1)$. *If*

$$\frac{3\beta + 1 + 2\sqrt{\beta^2 + \beta}}{\beta + 1} > \frac{n}{2}, \tag{6}$$

then there exists $c > 0$ *depending only on* Ω, n *and* β *such that* $u^* \geq c\delta^{\frac{2}{1+\beta}}$. *In particular,* u^* *is positive in* Ω *(and not only a.e.).*

4. Remarks and examples

In this section we discuss the optimality of our results, and give examples illustrating various situations.

We start with the following observation. The stability of the maximal solution for $\lambda \geq \lambda^*$ implies that the map $\lambda \mapsto u_\lambda$ is continuous for $\lambda \geq \lambda^*$, for simplicity, considered as a map from $(0, \infty) \subset \mathbb{R}$ to $L^1(\Omega)$.

It is natural then to ask whether $\lambda \mapsto u_\lambda$ is continuous for all $\lambda > 0$. We can easily show that u_λ is continuous from the right. This follows from the characterization of u_λ as the unique maximal subsolution to (4). On the other hand, if $\lambda_k \nearrow \lambda$ with $\lambda_k < \lambda$, the increasing limit $u = \lim_{\lambda_k \nearrow \lambda} u_{\lambda_k}$ exists and is a subsolution of (4). But, is it the maximal one? The answer is negative in general, and examples can be easily constructed by applying the next proposition. For instance, take Ω to be the interval $(0,1)$ and $f(u) \equiv 1$. From Proposition 4.1 one concludes that $u_\lambda \equiv 0$ for all $0 < \lambda < \lambda^*$, but Theorem 3.2 says that $u^* > 0$ in Ω. Hence the branch $\lambda \mapsto u_\lambda$ of maximal solutions has a discontinuity at λ^*. In addition, it is easy to deduce that the branch $\lambda \mapsto u_\lambda$ is nondecreasing.

Proposition 4.1. *Assume* Ω *is an interval in* $\mathbb{R}$ *and that* f *depends only on* u. *Then, for any* $\lambda > 0$ *the maximal solution is either identically zero or positive in* Ω.

A similar statement can be found in [11], where they claim that if Ω is an interval in $\mathbb{R}$ and $f \equiv 1$, any minimizer of the corresponding energy is either zero or positive in Ω.

The previous proposition leads us to ask whether there are examples where the maximal solution u_λ is not identically zero for some $\lambda < \lambda^*$. The next construction provides an example in one dimension.

Example 4.2. *Here we consider $f = \chi_{(-A,A)}$ for some suitable $A > 0$ to be chosen later. First, fix $\eta > 1$ such that*

$$\eta - \frac{\eta^{1-\beta}}{1-\beta} > 0,$$

and consider the ODE

$$\begin{cases} -u'' = -u^{-\beta} + 1, \\ u(0) = \eta, \\ u'(0) = 0. \end{cases} \tag{7}$$

Standard results of ODE theory imply that u is defined on a maximal open interval, say $(-x_0, x_0)$ (at the end of this example we present a more explicit expression of u in the case $\beta = 1/2$). The solution u is symmetric with respect to 0 and is decreasing in the nonempty interval $x \in (0, x_0)$. Moreover, $\lim_{x \nearrow x_0} u(x) = 0$. Therefore there exists some $A > 0$ (unique) such that

$$\eta - \frac{\eta^{1-\beta}}{1-\beta} - u(A) = 0. \tag{8}$$

We fix $A > 0$ in this fashion and let $f = \chi_{(-A,A)}$. Note that the expression

$$\frac{1}{2}(u')^2 - \frac{u^{1-\beta}}{1-\beta} + u$$

is a constant in the interval $(0, A)$, and therefore condition (8) is equivalent to

$$\frac{1}{2}u'(A)^2 = \frac{u(A)^{1-\beta}}{1-\beta}.$$

Define

$$\alpha = \frac{2}{1+\beta}, \quad c = \big(\alpha(\alpha-1)\big)^{-1/(1+\beta)}, \quad B = \Big(\frac{1}{c}u(A)\Big)^{1/\alpha} + A,$$

and extend $u(x)$ by the formula

$$u(x) = \begin{cases} c(x+B)^\alpha, & x \in (-B, -A), \\ \text{solution of } (7), & x \in [-A, A], \\ c(B-x)^\alpha, & x \in (A, B), \\ 0, & x \notin (-B, B). \end{cases}$$

If $R > B$, then u is a solution to

$$\begin{cases} -u'' = \chi_{\{u>0\}}(-u^{-\beta} + f(x)) & in\ (-R, R), \\ u(x) = 0 & x = \pm R. \end{cases} \tag{9}$$

So far, we have produced a nontrivial solution u corresponding to $\lambda = 1$.

We will see that the maximal solution $\bar{u}$ to (9) vanishes in $\Omega = (-R, R)$ if R is large enough. To accomplish this, let $\overline{B} = -\inf\{\, t \in (-R, -A) \mid \bar{u} > 0$ on $(t, 0)\,\} > 0$.

We are going to show that we have an a priori estimate for $\overline{B}$ independent of R, more precisely, that

$$\overline{B} \leq \left[\frac{1}{c}\Big(\frac{1-\beta}{2}A^2\Big)^{1/(1-\beta)}\right]^{1/\alpha} + A. \tag{10}$$

Therefore, by choosing R larger than the right hand side of (10) we see that the maximal solution has to vanish in $\Omega = (-R, R)$.

Now we derive (10). Integrate (9) over $(-A, 0)$ to get

$$\bar{u}'(0) - \bar{u}'(-A) = \int_0^A \bar{u}'' \geq -A.$$

By symmetry, $\bar{u}'(0) = 0$, and therefore we get the estimate

$$\bar{u}'(-A) \leq A. \tag{11}$$

Observe that on $(-\overline{B}, -A)$, $\bar{u}$ satisfies $\bar{u}'' = \bar{u}^{-\beta}$. Multiplying this equation by $\bar{u}'$ and integrating we find

$$\frac{1}{2}(\bar{u}')^2 - \frac{\bar{u}^{1-\beta}}{1-\beta} = D \qquad on\ (-\overline{B}, -A),$$

where D is a constant. Since $\bar{u}(-\bar{B}) = 0$ we must have $D \geq 0$, and this implies that

$$\frac{\bar{u}^{1-\beta}}{1-\beta} \leq \frac{1}{2}(\bar{u}')^2 \qquad on\ (-\overline{B}, -A). \tag{12}$$

It is not difficult then to check that

$$\bar{u}(x) \geq c(x + \overline{B})^{\alpha}, \quad \forall x \in (-\overline{B}, -A). \tag{13}$$

In particular, combining (13) at $x = -A$, (12) and (11) we get

$$c(\overline{B} - A)^{\alpha} \leq \bar{u}(-A) \leq \Big(\frac{1-\beta}{2}\bar{u}'(-A)^2\Big)^{1/(1-\beta)} \leq \Big(\frac{1-\beta}{2}A^2\Big)^{1/(1-\beta)},$$

from which (10) *follows.*

When $\beta = 1/2$ *we have a more explicit expression of the solution of problem* (7). *Multiplying the equation by* u' *and integrating one finds* $u' = (4u^{1/2} - 2u + c)^{1/2}$, *where* $c > 0$ *is a constant depending only on* β *and* η. *Set* $h(s) = (4s^{1/2} - 2s + c)^{-1/2}$ *and integrate it from* 0 *to* ξ. *We obtain*

$$H(\xi) = \sqrt{c} - \sqrt{2}\arcsin\Big(\frac{\sqrt{2}}{\sqrt{2+c}}\Big) - \sqrt{c + 4\sqrt{\xi} - 2\xi} - \sqrt{2}\arcsin\Big(\frac{\sqrt{2}(1-\sqrt{\xi})}{\sqrt{2+c}}\Big).$$

Our equation transforms into $(H(u(x)))' = -1$ *for* $x > 0$. *Integrating and applying the inverse function* H^{-1} *we obtain* $u(x) = H^{-1}(H(\eta) - x)$, $0 < x < H(\eta)$. *We remark that when* $\beta = 1/2$, *it is proved in* [4] *that there is a unique correspondence between* $\eta \geq 4$ *and* x_0, *where* $u(x_0) = 0$ *and* u *solves* (7). *Thus* $u(x) = H^{-1}(H(\eta) - x)$ *is the maximal solution in* $(-x_0, x_0)$ *with* $f \equiv 1$ *and it is stable.* ☑

It is natural to ask whether or not there is a characterization for the maximal solution u_λ in terms of stability when $0 < \lambda < \lambda^*$, similarly to Theorem 3.1. The situation in the range $0 < \lambda < \lambda^*$ is more delicate, because the maximal solution u_λ vanishes in parts of the domain, and therefore a solution to (1) vanishes on a set of positive measure. In the same spirit, whenever $\lambda \geq \lambda^*$ one may ask whether the characterization of the maximal solution given in Theorem 3.1 is valid for any solution (not known a priori to be positive a.e.). One possible approach would be to say that a solution $u \in C(\Omega)$ to (1) is weakly stable if

$$\int_\omega \frac{\partial g_\lambda}{\partial u}(x,u)\varphi^2 \leq \int_\omega |\nabla\varphi|^2 \quad \forall \varphi \in C_0^\infty(\omega), \tag{14}$$

where ω is the open set

$$\omega = \{x \in \Omega \mid u(x) > 0\}.$$

Assume now that $u \in C(\Omega)$ is a weakly stable solution of (1) in the sense of relation (14). Is it true that it has to be the maximal solution? It turns out that the answer is negative in general; the next example clarifies our ideas.

Example 4.3. *Let* Ω *be the interval* $(-2, 2)$. *We shall construct a function* $f = f(x)$ *and a continuous solution* u *to* (1) *in* Ω *with* $\lambda = 1$, *such that* $u > 0$ *in* $(-2,0)\cup(0,2)$, *but* $u(0) = 0$. *Moreover* u *satisfies the condition* (14), *but* u *is not the maximal solution. Indeed, first note that* $\lambda^* \leq 1$ *because* $u > 0$ *a.e. If* $\lambda^* = 1$ *then by Remark 2.2* (*C*) *(uniqueness of* u^**) we would infer that* $u^* = u$, *which is* *not possible by Theorem 3.2. Hence $\lambda^* < 1$, and then u cannot be the maximal solution u_λ, because $u(0) = 0$ and $u_\lambda \geq a\delta$ (with $a > 0$).*

The details of the construction of f and u are as follows. Let $w_\epsilon : \mathbb{R} \to \mathbb{R}^+$ be a family of smooth convex functions such that

$$\begin{array}{ll} w_\epsilon(x) = |x| & \text{for } |x| > \epsilon, \\ 0 < w_\epsilon(x) \leq \epsilon & \text{for } |x| \leq \epsilon, \\ |w'_\epsilon(x)| \leq 1 & \text{for all } x \in \mathbb{R}. \end{array}$$

Let

$$u_\epsilon(x) = c(1 - w_\epsilon)^\alpha, \quad x \in (-1, 1),$$

where (as before) $\alpha = \frac{2}{1+\beta}$ and $c > 0$ is defined by $c^{-\beta-1} = \alpha(\alpha - 1)$. A similar computation to the one in the previous example shows that

$$-u''_\epsilon + \frac{1}{u_\epsilon^\beta} = f_\epsilon \quad \text{in } (-1, 1),$$

where

$$f_\epsilon = c^{-\beta}(1 - w_\epsilon)^{-\alpha\beta}\left(1 - c^{\beta+1}\alpha(\alpha - 1)(w'_\epsilon)^2\right) + c\alpha(1 - w_\epsilon)^{\alpha-1}w''_\epsilon. \tag{15}$$

We claim that for $\epsilon > 0$ sufficiently small u_ϵ is weakly stable in $(-1, 1)$ in the sense of relation (14), i.e.

$$\beta \int_{-1}^{1} u_\epsilon^{-1-\beta}\varphi^2 \leq \int_{-1}^{1} \varphi'^2 \quad \forall \varphi \in C_0^\infty(-1, 1). \tag{16}$$

Indeed,

$$\beta u_\epsilon^{-1-\beta} = \beta c^{-\beta-1}(1 - w_\epsilon)^{-2} = \beta\alpha(\alpha - 1)(1 - w_\epsilon)^{-2} \leq \frac{1}{4}(1 - w_\epsilon)^{-2}.$$

Therefore

$$\begin{aligned} \beta \int_{-1}^{1} u_\epsilon^{-1-\beta}\varphi^2 &\leq \frac{1}{4}\int_{-1}^{1} \frac{\varphi^2}{(1 - w_\epsilon)^2} \\ &= \frac{1}{4}\int_{-1}^{1} \frac{\varphi^2}{(1 - |x|)^2} + \frac{1}{4}\int_{-\epsilon}^{\epsilon} \left((1 - w_\epsilon)^{-2} - (1 - |x|)^{-2}\right)\varphi^2 \\ &= \frac{1}{4}\int_{-1}^{1} \frac{\varphi^2}{(1 - |x|)^2} + \frac{1}{4}\int_{-\epsilon}^{\epsilon} \frac{(1 - w_\epsilon)(1 - |x|)}{(1 - w_\epsilon)^2(1 - |x|)^2}(w_\epsilon - |x|)\varphi^2 \\ &\leq \frac{1}{4}\int_{-1}^{1} \frac{\varphi^2}{(1 - |x|)^2} + C\epsilon\int_{-1}^{1} \varphi^2. \end{aligned}$$

We use now the following improvement of Hardy's inequality (see Brezis and Marcus [3]*): let B be the unit ball in $\mathbb{R}^n$, $n \geq 1$ and $\delta(x) = \operatorname{dist}(x, \partial B) = 1 - |x|$. Then*

$$\inf_{\varphi \in C_0^\infty(B)} \frac{\int_B |\nabla\varphi|^2 - \frac{1}{4}\int_B \varphi^2/\delta^2}{\int_B \varphi^2} > 0. \tag{17}$$

We conclude that for ϵ small enough (16) *holds. From now on we fix this $\epsilon > 0$.*

Observe that by Theorems 2.1 *and* 3.1*, we have $\lambda^* = 1$ and $u^* = u_\epsilon$ for the problem*

$$\begin{cases} -u_\epsilon'' = \chi_{\{u_\epsilon > 0\}}\Big(-\dfrac{1}{u_\epsilon^\beta} + \lambda f_\epsilon(x)\Big) & \text{in } (-1,1), \\ u_\epsilon = 0 & \text{at } x = \pm 1. \end{cases}$$

Define

$$u(x) = \begin{cases} u_\epsilon(x-1) & \text{for } x \in (0,2), \\ u_\epsilon(x+1) & \text{for } x \in (-2,0), \end{cases}$$

and

$$f(x) = \begin{cases} f_\epsilon(x-1) & \text{for } x \in (0,2), \\ f_\epsilon(x+1) & \text{for } x \in (-2,0). \end{cases}$$

Then u is continuous (even $C^{1,\gamma}(\overline{(-2,2)})$, $\gamma = \frac{1-\beta}{1+\beta}$) and it is a solution of

$$\begin{cases} -u'' = \chi_{\{u > 0\}}\Big(-\dfrac{1}{u^\beta} + \lambda f(x)\Big) & \text{in } (-2,2), \\ u = 0 & \text{at } x = \pm 2, \end{cases}$$

and satisfies the condition (14). ☑

Theorem 3.2 reveals to be somewhat optimal regarding the behavior of u^* near the boundary in view of the example that follows.

Example 4.4. *There exists a function $f = f(x)$ such that problem*

$$\begin{cases} -\Delta u + \dfrac{1}{u^\beta} = f(x) & \text{in } A := \{r \,:\, R < r < 1\}, \\ u = 0 & \text{on } \partial B_1, \\ u = c(1-R)^\alpha & \text{on } \partial B_R, \end{cases} \tag{18}$$

has a solution $u \sim \delta^{\frac{2}{1+\beta}}$ near ∂B_1.

In fact, given $0 < \beta < 1$, let $\alpha = \frac{2}{1+\beta}$ and choose $c > 0$ such that $c^{-\beta-1} = \alpha(\alpha-1)$. The function $u = c(1-r)^\alpha$, $r = |x|$, is a solution of equation (18) *with*

$$f(x) = f(r) = c\alpha(1-r)^{\alpha-1} > 0.$$

We claim that the first eigenvalue of the linearized operator is positive, that is,

$$\inf_{\varphi \in C_0^\infty(A)} \frac{\int_A |\nabla\varphi|^2 - \beta u^{-\beta-1}\varphi^2}{\int_A \varphi^2} > 0. \tag{19}$$

Indeed

$$\begin{aligned}\beta\int_A u^{-1-\beta}\varphi^2 &= \beta\int_A (c(1-r)^\alpha)^{-1-\beta}\varphi^2\\ &= \beta c^{-1-\beta}\int_A (1-r)^{-\alpha(-1-\beta)}\varphi^2\\ &= \beta\alpha(\alpha-1)\int_A (1-r)^{-2}\varphi^2.\end{aligned}$$

Observe that $\beta\alpha(\alpha-1) \le \frac{1}{4}$ *(with equality only if* $\alpha = 3/2$ *i.e.* $\beta = 1/3$*). By inequality* (17) *we deduce that* (19) *holds.*

It is worth to mention that the methods applied for (1) *can be used for* (18)*. This indicates that the extremal function* u^* *can not satisfy an estimate of the form*

$$u^* \ge c\delta^\gamma$$

for an exponent γ *smaller than* $\frac{2}{1+\beta}$*. In this sense, the conclusion of Theorem 3.2 is optimal.* ☑

The next example shows that in Theorem 3.2, condition (6) on β and n is almost optimal.

Example 4.5. Let B be the unit ball of $\mathbb{R}^n$. If

$$\frac{3\beta+1+2\sqrt{\beta^2+\beta}}{\beta+1} < \frac{n}{2}, \tag{20}$$

then there exists $f = f(x) \in C^\infty(B) \cap L^\infty(B)$ with $f \ge 0$ such that the solution $u^* = u_{\lambda^*}$ satisfies $u^* > 0$ in $B \setminus \{0\}$ and $u^*(x) = c|x|^\alpha$ for x near the origin, where $\alpha = \frac{2}{1+\beta}$ and $c > 0$.

In what follows we describe the explicit construction. Let $v(x) = c|x|^\alpha$ where the constant $c > 0$ is chosen so that $\alpha(\alpha+n-2) = c^{-1-\beta}$. Then it is easy to verify that v satisfies

$$\Delta v = \frac{1}{v^\beta} \quad \text{in } \mathbb{R}^n.$$

Let $0 < R < 1$ to be fixed later and let $h(r)$ be a smooth function defined for $r \in [0,1]$ such that

$$\begin{aligned} &h(r) = 0 && \forall r \in [0,R],\\ &0 \le h(r) \le c && \forall r \in (R,1],\\ &h(1) = c, && \\ &h'(r) \ge 0,\ h''(r) \ge 0 && \forall r \in [0,1]. \end{aligned}$$

Then $\Delta h = h'' + \frac{n-1}{r}h' \ge 0$ in $\mathbb{R}^n$. Set $u(x) = v(x) - h(|x|)$. We find

$$\begin{aligned} -\Delta u &= -v^{-\beta} + \Delta h \\ &= -u^{-\beta} + f(x), \end{aligned}$$

where

$$f(x) = u^{-\beta} - v^{-\beta} + \Delta h \ge 0.$$

Similarly as done before, we check now that u is weakly stable if R and h are chosen appropriately. Let $\varphi \in C_0^\infty(B)$ and consider

$$\begin{aligned} I &= \beta \int_B u^{-1-\beta}\varphi^2 \\ &= \beta\alpha(\alpha+n-2)\int_B r^{-2}\varphi^2 + \beta\int_B \Big((cr^\alpha - h)^{-1-\beta} - (cr^\alpha)^{-1-\beta}\Big)\varphi^2 \qquad (21)\\ &= I_1 + I_2. \end{aligned}$$

We estimate I_1 first. A computation shows that condition (20) is equivalent to

$$\beta\alpha(\alpha+n-2) < \frac{(n-2)^2}{4}.$$

Thus by Hardy's inequality with the weight r^{-2},

$$I_1 \le (1-\epsilon)\int_B |\nabla\varphi|^2$$

for some $\epsilon > 0$ depending only on β and n. To estimate I_2 observe that since $h \equiv 0$ on $[0,R]$ we have

$$I_2 \le \beta \int_{B\setminus B_R} (cr^\alpha - h)^{-1-\beta}\varphi^2. \qquad (22)$$

We can choose h in such a way that $cr^\alpha - h \ge \frac{1}{C}\delta$, where $\delta(x) = \mathrm{dist}(x,\partial B) = 1 - |x|$, and the constant C is independent of R. In this way $(cr^\alpha - h)^{-1-\beta} \le C\delta^{-1-\beta}$, and therefore, using Hardy's inequality with weight δ^{-2}, we get

$$\begin{aligned} I_2 &\le C \int_{B\setminus B_R} \delta^{-1-\beta}\varphi^2 \\ &\le C\Big(\int_B \delta^{-2}\varphi^2\Big)^{(1+\beta)/2}\Big(\int_{B\setminus B_R}\varphi^2\Big)^{(1-\beta)/2} \\ &\le \frac{\epsilon}{2}\int_B |\nabla\varphi|^2 + C(\epsilon)\int_{B\setminus B_R}\varphi^2. \end{aligned}$$

But

$$\int_{B\setminus B_R}\varphi^2 \le C(1-R)^2\int_B |\nabla\varphi|^2 \quad \forall\varphi\in C_0^\infty(B),$$

where C is independent of φ. Hence, by choosing $R<1$ with $1-R$ small enough we obtain

$$I_2 \le \epsilon \int_B |\nabla\varphi|^2. \tag{23}$$

Combining (21), (22) and (23) we conclude that

$$\beta\int_B u^{-1-\beta}\varphi^2 \le \int_B |\nabla\varphi|^2 \quad \forall\varphi\in C_0^\infty(B).$$

By Theorem 3.1 u is the maximal solution on B with data f. Moreover, $\lambda^*=1$ in this situation. ☑

5. The parabolic problem

We are also interested in studying the singular parabolic problem,

$$\begin{cases} u_t - \Delta u = g_\lambda(x,u) & \text{in } \Omega\times(0,\infty), \\ u = 0 & \text{on } \partial\Omega\times(0,\infty), \\ u(0)=u_0 & \text{in } \Omega. \end{cases} \tag{24}$$

The quantity λ^* given in Theorem 2.1 is a critical parameter for the elliptic problem (1), but we will see that it also serves as a borderline for existence of global positive solutions of (24) with a suitable fixed initial data u_0. More precisely, global positive solutions exist for $\lambda\ge\lambda^*$ (see Theorem 5.5). But for $0<\lambda<\lambda^*$ the solutions of (24) vanish in finite time (and hence, the term $u^{-\beta}$ blows up in finite time), in a sense which we make precise later on. This kind of interplay between stationary and evolution problem was undertaken in Brezis *et al.* [2] for $g_\lambda(x,u)=\lambda f(u)$ with f positive, increasing and convex. They established the existence of globally defined solutions and solutions blowing-up in finite time in terms of a similar critical constant for the corresponding elliptic problem.

For simplicity, we will consider in this section the function f depending only on u. We are still assuming that $f \geq 0$, $f \not\equiv 0$ and f is nondecreasing, concave and sublinear.

We begin stating the existence of a local solution in time.

Theorem 5.1. *Let $0 < \beta < 1$. Then the parabolic problem (24) has a local solution defined in an interval $(0, T)$, provided that the initial data u_0 is bounded and $u_0 \geq c\delta^\alpha$ for some $c > 0$ and $1 < \alpha < \frac{2}{1+\beta}$. Moreover, u belongs to $L^\infty(\Omega \times (0,T)) \cap C^1(\Omega \times (0,T))$ and satisfies $u \geq c'\delta^\alpha$ in $(0,T)$ for some $c' > 0$ (T and c' depend on c and α).*

The locally defined solution is unique in an adequate class.

Theorem 5.2. *Suppose $\beta \in (0,1)$ and assume $u_0 \in L^\infty(\Omega)$ and $u_0 \geq c\delta^\alpha$ for some $c > 0$ and $1 < \alpha < \frac{2}{1+\beta}$. Then the local solution u is unique in the set*

$$\mathcal{M} = \big\{u \in L^\infty(\Omega \times (0,T)) : \forall S \in (0,T) \text{ there exists } c > 0 \\ \text{such that } u(t) \geq c\delta^\alpha \text{ for } t \in (0,S)\big\}.$$

A function $u \in \mathcal{M}$ is regarded as a solution to (24) if

$$u(t) = T(t)u_0 + \int_0^t T(t-s) g_\lambda(u(s))\, ds, \tag{25}$$

where $T(t)$ is the heat semigroup in Ω with zero Dirichlet boundary condition. Note that since $u \geq c\delta^\alpha$, we obtain $u^{-\beta} \leq c\delta^{-\alpha\beta}$, but $\alpha\beta < \frac{2\beta}{1+\beta} < 1$. In particular, $u^{-\beta} \in L^\infty((0,T), L^p(\Omega))$ for some $p > 1$, hence (25)) makes sense in $L^p(\Omega)$.

The above result is an immediate consequence of the following comparison principle.

Lemma 5.3. *Let $\beta \in (0,1)$ and let u, $v \in L^\infty(\Omega \times (0,T))$ be a subsolution and a supersolution of (24), respectively, on $(0,T)$ in the sense of the semigroup relation (25) (in particular we assume that $g_\lambda(u(t)), g_\lambda(v(t)) \in L^1(\Omega \times (0,T))$). Furthermore, assume that there exists $c > 0$ and $1 < \alpha < \frac{2}{1+\beta}$ such that the supersolution v satisfies*

$$v(t) \geq c\delta^\alpha \qquad \text{for } t \in (0,T).$$

Then

$$u(t) \leq v(t) \qquad \text{for } t \in (0,T).$$

The main ingredient in the proof of Lemma 5.3 is a version of the smoothing effect for the heat semigroup $T(t)$ with weights involving powers of δ.

Proposition 5.4. *For any $q > 0$ there is a constant $C = C(q,\Omega) > 0$ such that*

$$\|T(t)\left(\delta^{-q}\varphi\right)\|_{L^2(\Omega)} \leq Ct^{-q/2}\|\varphi\|_{L^2(\Omega)} \quad \forall\varphi \in L^2(\Omega).$$

The local solution of Theorem 5.1 is globally defined if the initial data is greater than a solution of (1).

Theorem 5.5. *Assume that $0 < \beta < 1$ and that the elliptic problem (1) has a solution w which is positive a.e. Then, for any initial data $u_0 \in L^\infty(\Omega)$ satisfying $u_0 \geq w$, $u_0 \geq c\delta^\alpha$ for some $c > 0$ and $1 < \alpha < \frac{2}{1+\beta}$, the solution of the parabolic problem (24) is global, in the sense that*

$$\sup\left\{T > 0 \,|\, \exists c > 0 \; u(t) \geq c\delta^\alpha \; \forall t \in (0,T)\,\right\} = \infty.$$

We present next a converse result to the previous theorem.

Theorem 5.6. *Assume that $0 < \beta < 1$ and that the parabolic problem (24) has a positive global classical solution. Then the elliptic problem (1) has a solution which is positive a.e.*

Theorem 5.5 is sharp with respect to β.

Corollary 5.7. *If $\beta \geq 1$ there is no positive global classical solution of (24).*

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