

Non-null Distribution of The Likelihood Ratio Statistic for Testing Multisample Compound Symmetry

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Abstract. The non-null distribution of the modified likelihood ratio test statistic Λ^* for testing multisample compound symmetry of q multivariate Gaussian models is derived. The non-null moments of Λ^* are obtained in terms of Lauricella's hypergeometric function. The non-null distribution is expressed in terms of H -function.

1. Introduction

Let $\{\mathbf{X}_1, A_1\}, \dots, \{\mathbf{X}_q, A_q\}$ be independently distributed, where \mathbf{X}_g and A_g are also independently distributed, $\mathbf{X}_g \sim N_m(\boldsymbol{\mu}_g, \Sigma_g)$ and $A_g \sim W_m(n_g, \Sigma_g)$, $g = 1, \dots, q$. Let $H_{vc(q)}$ denote the hypothesis of multisample compound symmetry, i.e.,

$$H_{vc(q)} : \Sigma_1 = \dots = \Sigma_q = \sigma^2[(1 - \rho)I_m + \rho J], \quad (1)$$

where $\sigma^2 > 0$, $\rho(-1/(m-1) < \rho < 1)$ are unknown constants, I_m is an identity matrix of order m and J is an $m \times m$ matrix having all its elements unity. The modified likelihood ratio statistic Λ^* can be expressed in terms of the following criterion (Nagar and Castañeda [6]):

$$\Lambda^* = \frac{(m-1)^{n_0(m-1)/2} (mn_0)^{n_0 m/2}}{\prod_{g=1}^q n_g^{n_g m/2}} \frac{\prod_{g=1}^q \det(A_g)^{n_g/2}}{\{\text{tr}(JA)\}^{n_0/2} [\text{tr}\{(mI_m - J)A\}]^{n_0(m-1)/2}}, \quad (2)$$

where $A = \sum_{g=1}^q A_g$ and $n_0 = \sum_{g=1}^q n_g$.

In the case $q = 1$, (1) is the usual Wilks' H_{vc} hypothesis for testing compound symmetry (intra-class correlation structure) of the covariance matrix of a multivariate normal model. The problems of testing compound symmetry, sphericity and circularity

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of the covariance matrix of a multivariate Gaussian model have been studied by many authors, e.g. see Wilks [13], Votaw [12], Olkin and Press [10], Nagar, Jain and Gupta [5], Nagar, Chen and Gupta [8], Nagar and Sánchez [7] and Nagar, Zarrazola and Bedoya [9].

In this article, we obtain the exact non-null distribution of the modified likelihood ratio test statistic Λ^* for testing multisample compound symmetry. In Section 2, the non-null moments are derived. In Section 3, the exact non-null distribution is obtained using the inverse Mellin transformation and the definition of H -function.

2. The Non-Null Moments

Since A_1, \dots, A_q are independent Wishart matrices (Anderson [1], Gupta and Nagar [3]), $A_g \sim W_m(n_g, \Sigma)$, we obtain the h^{th} non-null moment of Λ^* by integrating over Wishart densities. That is,

$$\begin{aligned} E(\Lambda^{*h}) &= \frac{(m-1)^{n_0(m-1)h/2} (mn_0)^{n_0mh/2}}{\prod_{g=1}^q n_g^{n_gmh/2}} \frac{1}{\prod_{g=1}^q \det(2\Sigma_g)^{n_g/2} \Gamma_m(n_g/2)} \\ &\times \int_{A_1 > 0} \cdots \int_{A_q > 0} \frac{\prod_{g=1}^q \det(A_g)^{n_g h/2}}{\{\text{tr}(JA)\}^{n_0 h/2} [\text{tr}\{(mI_m - J)A\}]^{n_0(m-1)h/2}} \\ &\times \prod_{g=1}^q \left[\det(A_g)^{(n_g - m - 1)/2} \text{etr} \left(-\frac{1}{2} \Sigma_g^{-1} A_g \right) \right] dA_1 \cdots dA_q. \quad (3) \end{aligned}$$

Replacing $\{\text{tr}(JA)\}^{-n_0 h/2}$ and $[\text{tr}\{(mI_m - J)A\}]^{-n_0(m-1)h/2}$ by their equivalent gamma integrals, namely

$$\begin{aligned} &\{\text{tr}(JA)\}^{-n_0 h/2} \\ &= \frac{1}{2^{n_0 h/2} \Gamma(n_0 h/2)} \int_0^\infty x^{n_0 h/2 - 1} \exp \left[-\frac{1}{2} x \text{tr}(JA) \right] dx, \text{Re}(h) > 0, \end{aligned}$$

and

$$\begin{aligned} &[\text{tr}\{(mI_m - J)A\}]^{-n_0(m-1)h/2} \\ &= \frac{1}{2^{n_0(m-1)h/2} \Gamma[n_0(m-1)h/2]} \int_0^\infty y^{n_0(m-1)h/2 - 1} \\ &\times \exp \left[-\frac{1}{2} y \text{tr}\{(mI_m - J)A\} \right] dy, \text{Re}(h) > 0, \end{aligned}$$

respectively, where $\text{Re}(\cdot)$ denotes the real part of (\cdot) in (3), and integrating out A using multivariate gamma integral (Gupta and Nagar [3, p. 18]) we have

$$E(\Lambda^{*h}) = \frac{(m-1)^{n_0(m-1)h/2} (mn_0)^{n_0mh/2}}{\prod_{g=1}^q n_g^{n_gmh/2}} \frac{1}{2^{n_0mh} \Gamma(n_0 h/2) \Gamma[n_0(m-1)h/2]}$$

$$\begin{aligned} & \times \prod_{g=1}^q \frac{2^{n_g m h/2} \Gamma_m[n_g(1+h)/2]}{\det(\Sigma_g)^{n_g/2} \Gamma_m(n_g/2)} \int_0^\infty x^{n_0 h/2-1} \int_0^\infty y^{n_0(m-1)h/2-1} \\ & \times \prod_{g=1}^q \det\left(\Sigma_g^{-1} + xJ + y(mI_m - J)\right)^{-n_g(1+h)/2} dx dy. \end{aligned} \quad (4)$$

It does not seem possible to evaluate this integral for an arbitrary Σ_g . However, some particular cases are of interest. Let $\Delta = myI_m + (x - y)J$. Let us consider the case where Δ and Σ_g can be simultaneously reduced to their canonical forms by the same orthogonal matrix Q . A sufficient condition for this is that Δ and Σ should commute. A sufficient condition for commutativity of Δ and Σ is that the sum of elements of each column of Σ_g (and Δ) be same. Obviously the subspace of positive definite matrices generated by using these conditions will contain matrices with intraclass correlation structure and many more. For example the matrix $\Sigma = \begin{pmatrix} 2a+1 & 0 & 0 \\ 0 & a+1 & a \\ 0 & a & a+1 \end{pmatrix}$, $2a + 1 > 0$, will also be in the subspace. The matrix Δ has two distinct roots mx and my with multiplicity 1 and $m - 1$, respectively. Further, let $\lambda_{g1}, \dots, \lambda_{gm}$ be the eigenvalues of Σ_g . Then

$$\begin{aligned} & \det\left(\Sigma_g^{-1} + myI_m + (x - y)J\right)^{-n_g(1+h)/2} \\ & = (\lambda_{g1}^{-1} + mx)^{-n_g(1+h)/2} \prod_{j=2}^m (\lambda_{gj}^{-1} + my)^{-n_g(1+h)/2}. \end{aligned} \quad (5)$$

Substituting (5) in (4) the non-null moment expression is rewritten as

$$\begin{aligned} E(\Lambda^{*h}) & = \frac{(m-1)^{n_0(m-1)h/2} (mn_0)^{n_0 m h/2}}{\prod_{g=1}^q n_g^{n_g m h/2}} \frac{1}{\Gamma(n_0 h/2) \Gamma[n_0(m-1)h/2]} \\ & \times \prod_{g=1}^q \frac{\Gamma_m[n_g(1+h)/2]}{\det(\Sigma_g)^{n_g/2} \Gamma_m(n_g/2)} I_1 I_2, \end{aligned} \quad (6)$$

where $\text{Re}[n_g(1+h)] > m - 1, g = 1, \dots, q$,

$$I_1 = \int_0^\infty x^{n_0 h/2-1} \prod_{g=1}^q (\lambda_{g1}^{-1} + mx)^{-n_g(1+h)/2} dx \quad (7)$$

and

$$I_2 = \int_0^\infty y^{n_0(m-1)h/2-1} \prod_{g=1}^q \prod_{j=2}^m (\lambda_{gj}^{-1} + my)^{-n_g(1+h)/2} dy. \quad (8)$$

Making the substitutions $z_1 = 1/(1 + \eta_1 mx)$ and $z_2 = 1/(1 + \eta_2 my)$ in (7) and (8) respectively, we obtain

$$I_1 = \frac{\eta_1^{n_0/2}}{m^{n_0 h/2}} \frac{\Gamma(n_0/2)\Gamma(n_0 h/2)}{\Gamma[n_0(1+h)/2]} F_D^{(q)} \left[\frac{n_0}{2}; \frac{n_1(1+h)}{2}, \dots, \frac{n_q(1+h)}{2}; \right. \\ \left. \frac{n_0(1+h)}{2}; 1 - \frac{\eta_1}{\lambda_{11}}, \dots, 1 - \frac{\eta_1}{\lambda_{q1}} \right] \quad (9)$$

and

$$I_2 = \frac{\eta_2^{n_0(m-1)/2}}{m^{n_0(m-1)h/2}} \frac{\Gamma[n_0(m-1)/2]\Gamma[n_0(m-1)h/2]}{\Gamma[n_0(m-1)(1+h)/2]} \\ \times F_D^{((m-1)q)} \left[\frac{n_0(m-1)}{2}; \frac{n_1(1+h)}{2}, \dots, \frac{n_1(1+h)}{2}, \dots, \frac{n_q(1+h)}{2}, \dots, \right. \\ \left. \frac{n_q(1+h)}{2}; \frac{n_0(m-1)(1+h)}{2}; 1 - \frac{\eta_2}{\lambda_{12}}, \dots, 1 - \frac{\eta_2}{\lambda_{1m}}, \dots, \right. \\ \left. 1 - \frac{\eta_2}{\lambda_{q2}}, \dots, 1 - \frac{\eta_2}{\lambda_{qm}} \right], \quad (10)$$

where $|1 - \eta_1/\lambda_{g1}| < 1$, $|1 - \eta_2/\lambda_{gj}| < 1$, $j = 2, \dots, m$, $g = 1, \dots, q$. The function $F_D^{(p)}$ is the Lauricella's hypergeometric function of p variables defined by the integral (Gupta and Nagar [2], Srivastava and Karlsson [11]),

$$F_D^{(p)} [a; b_1, \dots, b_p; c; x_1, \dots, x_p] \\ = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 u^{a-1} (1-u)^{c-a-1} \prod_{j=1}^p (1-x_j u)^{-b_j} du, \quad (11)$$

where $\text{Re}(a) > 0$ and $\text{Re}(c-a) > 0$. Expanding $(1-x_j u)^{-b_j}$ in power series and integrating out u term by term, the series expansion of $F_D^{(p)}$ is given by

$$F_D^{(p)} [a; b_1, \dots, b_p; c; x_1, \dots, x_p] \\ = \sum_{r_1, \dots, r_p=0}^{\infty} \frac{(a)_{r_1+\dots+r_p} (b_1)_{r_1} \cdots (b_p)_{r_p}}{(c)_{r_1+\dots+r_p} r_1! \cdots r_p!} x_1^{r_1} \cdots x_p^{r_p}, \quad (12)$$

where $|x_1| < 1, \dots, |x_p| < 1$ and the Pochhammer symbol $(a)_n$ is defined by $(a)_n = a(a+1)\cdots(a+n-1) = (a)_{n-1}(a+n-1)$ for $n = 1, 2, \dots$, and $(a)_0 = 1$.

Now, substituting (9) and (10) in (6) and writing multivariate gamma functions in terms of ordinary gamma function using

$$\Gamma_m(a) = \pi^{m(m-1)/4} \prod_{j=1}^m \Gamma\left(a - \frac{j-1}{2}\right), \text{Re}(a) > \frac{m-1}{2},$$

we get

$$\begin{aligned}
 & E(\Lambda^{*h}) \\
 &= \frac{(m-1)^{n_0(m-1)h/2} n_0^{n_0mh/2}}{\prod_{g=1}^q n_g^{n_gmh/2}} \frac{\Gamma(n_0/2)\Gamma[n_0(m-1)/2]}{\Gamma[n_0(1+h)/2]\Gamma[n_0(m-1)(1+h)/2]} \\
 &\times \prod_{g=1}^q \left[\det(\Sigma_g)^{-n_g/2} \prod_{j=1}^m \frac{\Gamma[n_g(1+h)/2 - (j-1)/2]}{\Gamma[(n_g-j+1)/2]} \right] \eta_1^{n_0/2} \eta_2^{n_0(m-1)/2} \\
 &\times F_D^{(q)} \left[\frac{n_0}{2}; \frac{n_1(1+h)}{2}, \dots, \frac{n_q(1+h)}{2}; \frac{n_0(1+h)}{2}; 1 - \frac{\eta_1}{\lambda_{11}}, \dots, 1 - \frac{\eta_1}{\lambda_{q1}} \right] \\
 &\times F_D^{((m-1)q)} \left[\frac{n_0(m-1)}{2}; \frac{n_1(1+h)}{2}, \dots, \frac{n_1(1+h)}{2}, \dots, \frac{n_q(1+h)}{2}, \right. \\
 &\quad \dots, \frac{n_q(1+h)}{2}; \frac{n_0(m-1)(1+h)}{2}; 1 - \frac{\eta_2}{\lambda_{12}}, \dots, 1 - \frac{\eta_2}{\lambda_{1m}}, \\
 &\quad \left. \dots, 1 - \frac{\eta_2}{\lambda_{q2}}, \dots, 1 - \frac{\eta_2}{\lambda_{qm}} \right], \tag{13}
 \end{aligned}$$

where $|1 - \eta_1/\lambda_{g1}| < 1$, $|1 - \eta_2/\lambda_{gj}| < 1$, $j = 2, \dots, m$, $g = 1, \dots, q$ and $\text{Re}[n_g(1+h)] > m-1$, $g = 1, \dots, q$. Substituting $q = 1$, $n_1 = n$, $\Sigma_1 = \Sigma$, $\lambda_{1i} = \lambda_i$ and $\eta_2 = \eta$ in (13) and simplifying, we get the h^{th} non-null moment of the modified Wilks' Λ_{vc}^* statistic as

$$\begin{aligned}
 E(\Lambda_{vc}^{*h}) &= \frac{(m-1)^{n(m-1)h/2} \Gamma[n(m-1)/2]}{\Gamma[n(m-1)(1+h)/2]} \prod_{j=2}^m \frac{\Gamma[n(1+h)/2 - (j-1)/2]}{(\eta^{-1}\lambda_j)^{n/2} \Gamma[(n-j+1)/2]} \\
 &\times F_D^{(m-1)} \left[\frac{n(m-1)}{2}; \frac{n(1+h)}{2}, \dots, \frac{n(1+h)}{2}; \right. \\
 &\quad \left. \frac{n(m-1)(1+h)}{2}; 1 - \frac{\eta}{\lambda_2}, \dots, 1 - \frac{\eta}{\lambda_m} \right].
 \end{aligned}$$

Also, substituting $\Sigma_1 = \dots = \Sigma_q = \sigma^2[(1-\rho)I_m + \rho J]$ in (13), the h -th null moment of Λ^* for testing $H_{vc(q)}$ is obtained as

$$\begin{aligned}
 E(\Lambda^{*h}) &= \frac{(m-1)^{n_0(m-1)h/2} n_0^{n_0mh/2}}{\prod_{g=1}^q n_g^{n_gmh/2}} \frac{\Gamma(n_0/2)\Gamma[n_0(m-1)/2]}{\Gamma[n_0(1+h)/2]\Gamma[n_0(m-1)(1+h)/2]} \\
 &\times \prod_{g=1}^q \prod_{j=1}^m \frac{\Gamma[n_g(1+h)/2 - (j-1)/2]}{\Gamma[(n_g-j+1)/2]}.
 \end{aligned}$$

3. Non-Null Distribution

Expanding the Lauricella's hypergeometric functions in (13) using (12), and simplifying the resulting expression, one obtains

$$\begin{aligned}
 E(\Lambda^{*h}) &= \\
 &= \frac{(m-1)^{n_0(m-1)h/2} n_0^{n_0 m h/2}}{\prod_{g=1}^q n_g^{n_g m h/2} \prod_{g=1}^q \det(\Sigma_g)^{n_g/2}} (\eta_1 \eta_2^{m-1})^{n_0/2} \\
 &\times \sum_{s_{11}, \dots, s_{qm}=0}^{\infty} \frac{\Gamma(n_0/2 + \sum_{g=1}^q s_{g1}) \Gamma[n_0(m-1)/2 + \sum_{g=1}^q \sum_{j=2}^m s_{gj}]}{\prod_{g=1}^q \prod_{j=1}^m \Gamma[(n_g - j + 1)/2]} \\
 &\times \frac{\prod_{g=1}^q \prod_{j=2}^m \{\Gamma[n_g(1+h)/2 - (j-1)/2] \Gamma[n_g(1+h)/2 + s_{gj}]\}}{\Gamma[n_0(1+h)/2 + \sum_{g=1}^q s_{g1}] \Gamma[n_0(m-1)(1+h)/2 + \sum_{g=1}^q \sum_{j=2}^m s_{gj}]} \\
 &\times \left[\prod_{g=1}^q \frac{\Gamma[n_g(1+h)/2 + s_{g1}]}{\Gamma^{m-1}[n_g(1+h)/2]} \right] \prod_{g=1}^q \left[\frac{(1 - \eta_1 \lambda_{g1}^{-1})^{s_{g1}}}{s_{g1}!} \prod_{j=2}^m \frac{(1 - \eta_2 \lambda_{gj}^{-1})^{s_{gj}}}{s_{gj}!} \right].
 \end{aligned}$$

Finally, using the inverse Mellin transform and the above moment expression, the density function of Λ^* is obtained as

$$\begin{aligned}
 f(\lambda^*) &= \frac{(\eta_1 \eta_2^{m-1})^{n_0/2}}{\prod_{g=1}^q \det(\Sigma_g)^{n_g/2}} \sum_{s_{11}, \dots, s_{qm}=0}^{\infty} \prod_{g=1}^q \left[\frac{(1 - \eta_1 \lambda_{g1}^{-1})^{s_{g1}}}{s_{g1}!} \prod_{j=2}^m \frac{(1 - \eta_2 \lambda_{gj}^{-1})^{s_{gj}}}{s_{gj}!} \right] \\
 &\times \frac{\Gamma(n_0/2 + \sum_{g=1}^q s_{g1}) \Gamma[n_0(m-1)/2 + \sum_{g=1}^q \sum_{j=2}^m s_{gj}]}{\prod_{g=1}^q \prod_{j=1}^m \Gamma[(n_g - j + 1)/2]} \\
 &\times \frac{1}{\lambda^*} H_{q(m-1)+2, 2mq-q}^{2mq-q, 0} \left[c\lambda^* \left| \begin{array}{l} (a_1, A_1), (a_2, A_2), \\ \{(b_g, B_g), g = 1, \dots, q\}, \\ \{(a_{gj}, A_{gj}), j = 2, \dots, m, g = 1, \dots, q\} \\ \{(b_{1gj}, B_{1gj}), (b_{2gj}, B_{2gj}), j = 2, \dots, m, g = 1, \dots, q\} \end{array} \right. \right],
 \end{aligned}$$

where $0 < \lambda^* < 1$,

$$c = \frac{\prod_{g=1}^q n_g^{n_g m/2}}{(m-1)^{n_0(m-1)/2} n_0^{n_0 m/2}},$$

$$(a_1, A_1) = \left(\frac{n_0}{2} + \sum_{g=1}^q s_{g1}, \frac{n_0}{2} \right),$$

$$(a_2, A_2) = \left(\frac{n_0(m-1)}{2} + \sum_{g=1}^q \sum_{j=2}^m s_{gj}, \frac{n_0(m-1)}{2} \right),$$

$$(a_{gj}, A_{gj}) = \left(\frac{n_g}{2}, \frac{n_g}{2} \right), j = 2, \dots, m, g = 1, \dots, q,$$

$$(b_g, B_g) = \left(\frac{n_g}{2} + s_{g1}, \frac{n_g}{2} \right), g = 1, \dots, q,$$

$$(b_{1gj}, B_{1gj}) = \left(\frac{n_g}{2} + s_{gj}, \frac{n_g}{2} \right), j = 2, \dots, m, g = 1, \dots, q,$$

$$(b_{2gj}, B_{2gj}) = \left(\frac{n_g - j + 1}{2}, \frac{n_g}{2} \right), j = 2, \dots, m, g = 1, \dots, q,$$

and $H_{p,q}^{m,n}$ is an H -function. For properties and results on H -function the reader is referred to Mathai and Saxena [4].

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