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Reproductive solution for grade-two fluid model in two dimensions

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Abstract. We treat the existence of reproductive solution (weak periodic solution) of a second-grade fluid system in two dimensions, by using the Galerkin approximation method and compactness arguments.

1. Introduction

For a general incompressible fluid of grade 2, the Cauchy stress tensor is given by

$$\mathbf{T} = -p\mathbf{I} + \mu\mathbf{A}_1 + \alpha_1\mathbf{A}_2 + \alpha_2\mathbf{A}_1^2,$$

where $\mu \geq 0$ is the viscosity, α_1, α_2 are material coefficients, namely normal stress moduli, p is the pressure and $\mathbf{A}_1$, $\mathbf{A}_2$ are the first two Rivlin-Ericksen (see [8] or [9]) tensors defined by

$$\mathbf{A}_1 = \nabla\mathbf{u} + (\nabla\mathbf{u})^T,$$
$$\mathbf{A}_2 = \frac{d}{dt}\mathbf{A}_1 + \mathbf{A}_1\nabla\mathbf{u} + (\nabla\mathbf{v})^T\mathbf{A}_1.$$

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From the thermodynamical principles we have that $\alpha_1+\alpha_2$, and the requirement that the free energy be a minimum in equilibrium implies that $\alpha_1 \geq 0$. With all these conditions the equations of motion for an incompressible fluid of grade two are given by

$$\begin{cases} \dfrac{\partial}{\partial t}(\mathbf{u}-\alpha\Delta\mathbf{u}) - \nu\Delta\mathbf{u} + \mathrm{curl}(\mathbf{u}-\alpha\Delta\mathbf{u})\times\mathbf{u} + \nabla p = \mathbf{f} & \text{in } \Omega\times]0,T[, \\ \mathrm{div}\mathbf{u} = 0 & \text{in } \Omega\times]0,T[, \end{cases} \tag{1}$$

with homogeneous Dirichlet boundary conditions

$$\mathbf{u} = 0, \text{ on } \partial\Omega,$$

and initial condition

$$\mathbf{u}(0) = \mathbf{u}_0, \text{ in } \Omega.$$

Here, $\nu > 0$ represents the Kinematic viscosity and $\mathbf{f}$ the external forces.

The study of this kind of fluids was initiated by Dunn and Fosdick in [4] and by Fosdick and Rajapogal in [5]. The first successful mathematical analysis of (1) was done by Cioranescu and El Hacène in [1]. Another interesting work is due to Galdi and Sequeira [6], where the authors obtain some existence results.

Later Cioranescu and Girault in [2] establish existence, uniqueness and regularity of a global weak solution of (1) with small data $\mathbf{f}$ and $\mathbf{u}(0)$ and the same result on some interval for arbitrary data. The existence is obtained by applying Galerkin's method with a special basis.

In this paper we seek reproductive solutions of the two-grade fluid system, i.e. solutions of the following system:

$$\begin{cases} \dfrac{\partial}{\partial t}(\mathbf{u}-\alpha\Delta\mathbf{u}) - \nu\Delta\mathbf{u} + \mathrm{curl}(\mathbf{u}-\alpha\Delta\mathbf{u})\times\mathbf{u} + \nabla q = \mathbf{f} & \text{in } \Omega\times]0,T[, \\ \mathrm{div}\,\mathbf{u} = 0 & \text{in } \Omega\times]0,T[, \\ \mathbf{u} = \mathbf{0} & \text{on } \partial\Omega\times]0,T[, \\ \mathbf{u}(0) = \mathbf{u}(T), & \end{cases} \tag{2}$$

by supposing that $\mathbf{f}$ depends on the time t (notice that if $\mathbf{f}$ does not depend on t, the solution of the associated steady-state system of the second- grade fluid is actually a reproductive solution). As the reader can see, the usual initial condition has been changed by a time periodic condition.

The next theorem is the main result of this paper.

Theorem 1.1. *For any* $\mathbf{f} \in L^2(0,T;H(\mathrm{curl};\Omega)) \cap L^\infty(0,T;\mathbf{L}^2(\Omega))$*, there exists a weak solution of the two-grade fluid system* (2).

2. Preliminaries

Let Ω be a bounded domain of $\mathbb{R}^2$ of the class $\mathcal{C}^{2,1}$. To solve a grade 2 fluid system means to find a vector valued function $\mathbf{u} = (u_1, u_2)$ and a scalar function p defined on $\Omega\times]0, T[$ satisfying (2).

Since we are in two dimensions (see [7]), the curl operator is defined by

$$\text{curl } \mathbf{u} = \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2},$$

and if z is a scalar function, we define

$$z \times \mathbf{u} = (-zu_2, zu_1).$$

In what follows, the spaces in bold face represent spaces of bi-dimensional vector functions. We define the Hilbert spaces $\mathbf{H}$ and $\mathbf{V}$ in the following manner:

$$\mathbf{H} = \{\Psi \in \mathbf{L}^2(\Omega) \ : \ \text{div } \Psi = 0, \ \Psi \cdot \mathbf{n} = 0 \text{ on } \partial\Omega\},$$
$$\mathbf{V} = \{\mathbf{v} \in \mathbf{H}^1(\Omega) \ : \ \text{div } \mathbf{v} = 0, \ \mathbf{v} = \mathbf{0}, \text{ on } \partial\Omega\},$$
$$H(\text{curl};\Omega) = \{\mathbf{v} \in \mathbf{L}^2(\Omega) \ : \ \text{curl } \mathbf{v} \in \mathbf{L}^2(\Omega)\}.$$

For $\alpha \in \mathbb{R}^+$, we introduce the space (see [1] and [2])

$$\mathbf{V}_2 = \left\{\mathbf{v} \in \mathbf{V} \ : \ \text{curl } (\mathbf{v} - \alpha\Delta\mathbf{v}) \in \mathbf{L}^2(\Omega)\right\}, \tag{3}$$

equipped with the scalar product

$$(\mathbf{u}, \mathbf{v})_{V_2} = (\mathbf{u}, \mathbf{v}) + \alpha(\nabla\mathbf{u}, \nabla\mathbf{v}) + (\text{curl } (\mathbf{u} - \alpha\Delta\mathbf{u}), \text{curl } (\mathbf{v} - \alpha\Delta\mathbf{v})), \tag{4}$$

and associated norm and semi-norm

$$\|\mathbf{v}\|_{V_2} = (\mathbf{v}, \mathbf{v})_{V_2}^{1/2}, \qquad |\mathbf{v}|_{V_2} = \|\text{curl } (\mathbf{v} - \alpha\Delta\mathbf{v})\|_{\mathrm{L}^2(\Omega)}. \tag{5}$$

In the following lemma it is proved that the semi-norm $|\cdot|_{\mathbf{V}_2}$ is a norm in $\mathbf{H}^3$.

Lemma 2.1 ([1] p 182). *Let Ω be a bounded, simply-connected open set of $\mathbb{R}^2$ of the class $\mathcal{C}^{2,1}$. Then every $\mathbf{v} \in \mathbf{V}_2$ belongs to $\mathbf{H}^3(\Omega)$. Moreover, there exists $C > 0$ such that*

$$\|\mathbf{v}\|_{\mathbf{H}^3(\Omega)} \leq C\|\text{curl } (\mathbf{v} - \alpha\Delta\mathbf{v})\|_{L^2(\Omega)}.$$

An easy but tedious computation gives us the following equality:

$$\int_\Omega \text{curl } (\mathbf{u} - \alpha\Delta\mathbf{u}) \times \mathbf{u} \cdot \mathbf{v} dx = b(\mathbf{u}; \mathbf{u}, \mathbf{v}) - \alpha b(\mathbf{u}; \Delta\mathbf{u}, \mathbf{v}) + \alpha b(\mathbf{v}; \Delta\mathbf{u}, \mathbf{u});$$

here $b(\mathbf{u};\mathbf{v},\mathbf{w}) = \sum_{i,j=1}^{3} \int_\Omega \mathbf{u}_i \frac{\partial \mathbf{v}_j}{\partial x_i} \mathbf{w}_j dx$. From this, the variational formulation of the problem (1) is the following: Given $\mathbf{f} \in L^2(0,T;H(\text{curl};\Omega) \cap L^\infty(0,T;\mathbf{L}^2(\Omega))$ and $\mathbf{u}_0 \in \mathbf{V}_2$, find $\mathbf{u} \in L^\infty(0,T;\mathbf{V}_2)$ such that

$$(\mathbf{u}',\mathbf{v}) + \alpha(\nabla\mathbf{u},\nabla\mathbf{v}) + \nu(\nabla\mathbf{u},\nabla\mathbf{v}) + b(\mathbf{u};\mathbf{u},\mathbf{v}) - \alpha b(\mathbf{u};\Delta\mathbf{u},\mathbf{v}) + \alpha b(\mathbf{v},\Delta\mathbf{u},\mathbf{u}) = (\mathbf{f},\mathbf{v}),\ \forall \mathbf{v} \in \mathbf{V}. \quad (6)$$

3. A priori estimates of the Galerkin solutions

By following the ideas given in [1] and [2] we consider the basis $\{\mathbf{w}_j\}_{j\in\mathbb{N}}$, the eigenfunctions of the problem: For $j \in \mathbb{N}$, $\mathbf{w}_j \in \mathbf{V}_2$ is the solution of

$$(\mathbf{w}_j,\mathbf{v})_{\mathbf{V}_2} = \lambda_j\{(\mathbf{w}_j,\mathbf{v}) + \alpha(\nabla\mathbf{w}_j,\nabla\mathbf{v})\},\ \forall \mathbf{v} \in \mathbf{V}_2, \quad (7)$$

where $(\cdot,\cdot)_{\mathbf{V}_2}$ is the scalar product in $\mathbf{V}_2$. Since the imbedding of $\mathbf{V}_2$ into $\mathbf{V}$ is compact, there exists a sequence of eigenvalues $(\lambda_j)_{j\geq 1}$ and a sequence of eigenfunctions $(\mathbf{w}_j)_{j\geq 1}$ that constitutes a basis of $\mathbf{V}_2$.

Lemma 3.1 ([2] p 326). *Let Ω be a bounded simply-connected open set of $\mathbb{R}^3$ with a boundary Γ of class $\mathcal{C}^{3,1}$. Then the eigenfunctions of the problem (7) belong to $\mathbf{H}^4(\Omega)$.*

For every $m \in \mathbb{N}$, we define $\mathbf{V}_2^m$ the vector space spanned by the first m eigenfunctions $\{\mathbf{w}_1,\ldots,\mathbf{w}_m\}$, and by P_m the orthogonal projection on $\mathbf{V}_2^m$ for the scalar product in $\mathbf{V}_2$. In order to construct a periodic solution of the problem (2) we will use Galerkin's discretization. Indeed, for $j \in \{1,2,\ldots,m\}$ we find

$$\mathbf{u}_m(t) = \sum_{j=1}^{m} c_j^m(t)\mathbf{w}_j,$$

solution of

$$(\mathbf{u}_m'(t),\mathbf{w}_j) + \alpha(\nabla\mathbf{u}_m'(t),\nabla\mathbf{w}_j) + \nu(\nabla\mathbf{u}_m(t),\nabla\mathbf{w}_j) + b(\mathbf{u}_m(t);\mathbf{u}_m(t),\mathbf{w}_j), -\alpha b(\mathbf{u}_m(t);\Delta\mathbf{u}_m(t),\mathbf{w}_j) + \alpha b(\mathbf{w}_j,\Delta\mathbf{u}_m(t),\mathbf{u}_m(t)) = (\mathbf{f}(t),\mathbf{w}_j), \quad (8)$$

$$\mathbf{u}_m(0) = P_m(\mathbf{u}_0). \quad (9)$$

By multiplying both sides of (8) by $c_j^m(t)$ and summing with respect to j, from the anti-symmetry of b we obtain the equality

$$\frac{1}{2}\frac{d}{dt}\left(\|\mathbf{u}_m(t)\|^2_{\mathbf{L}^2(\Omega)} + \alpha|\mathbf{u}_m|^2_{\mathbf{H}^1(\Omega)}\right) + \nu|\mathbf{u}_m|^2_{\mathbf{H}^1(\Omega)} = (\mathbf{f}(t),\mathbf{u}_m(t)).$$

Therefore, by integrating in time for $t \in [0,T]$ the above equality we obtain the following lemma.

Lemma 3.2 ([2] p 327)**.** *The solution* $\mathbf{u}_m(t)$ *of the problem* (8)-(9) *satisfies the following differential inequality for each* $t \in [0,T]$*:*

$$\begin{aligned}&\|\mathbf{u}_m(t)\|^2_{\mathbf{L}^2(\Omega)} + \alpha\|\nabla \mathbf{u}_m(t)\|^2_{\mathbf{L}^2(\Omega)}\\ &\quad\le e^{-\nu K t}\left(\|\mathbf{u}_m(0)\|^2_{\mathbf{L}^2(\Omega)} + \alpha\|\nabla \mathbf{u}_m(0)\|^2_{\mathbf{L}^2(\Omega)}\right) + \frac{\mathcal{P}^2}{\nu}\int_0^t e^{-\nu K(t-s)}\|\mathbf{f}(s)\|^2_{\mathbf{L}^2(\Omega)}ds,\end{aligned}$$

where $\mathcal{P} > 0$ *is the Poincaré constant and* $K = (\mathcal{P}^2 + \alpha)^{-1}$*.*

In order to obtain an estimation for the norm $\|\mathbf{u}_m\|_{\mathbf{V}_2}$, we adapt the proof of Theorem 4.4 in [2] and the proof of the differential inequality given in [1] p 189.

At first, we define the vector-valued function $\mathbf{F}(\mathbf{u}_m, \mathbf{u}_m)$ by

$$\begin{aligned}(\mathbf{F}(\mathbf{u}_m(t),\mathbf{u}_m(t)),\mathbf{v}) &= \nu(\nabla\mathbf{u}_m(t),\nabla\mathbf{v}) + b(\mathbf{u}_m(t);\mathbf{u}_m(t),\mathbf{v})\\ &\quad - \alpha b(\mathbf{u}_m(t);\Delta\mathbf{u}_m(t),\mathbf{v}) + \alpha b(\mathbf{v};\Delta\mathbf{u}_m(t),\mathbf{u}_m(t)),\end{aligned} \tag{10}$$

for every $\mathbf{v} \in \mathbf{H}^1_0(\Omega)$. For $1 \le m \le m$, by construction of $\mathbf{F}(\mathbf{u}_m, \mathbf{u}_m)$,

$$(\mathbf{u}'_m(t),\mathbf{w}_j) + \alpha(\nabla\mathbf{u}'_m(t),\nabla\mathbf{w}_j) + (\mathbf{F}(\mathbf{u}_m(t),\mathbf{u}_m(t)),\mathbf{w}_j) - (\mathbf{f}(t),\mathbf{w}_j) = 0. \tag{11}$$

From Lemma 3.1, $\mathbf{F}(\mathbf{u}_m(t),\mathbf{u}_m(t)) \in \mathbf{H}^1(\Omega)$.

Next for each t, let $\mathbf{v}_m(t) \in \mathbf{V}$ be solution of the Stokes equation

$$\mathbf{v}_m(t) - \alpha\Delta\mathbf{v}_m(t) + \nabla q_m(t) = \mathbf{F}(\mathbf{u}_m(t),\mathbf{u}_m(t)) - \mathbf{f}(t). \tag{12}$$

By classical regularity results, $\mathbf{v}_m(t) \in \mathbf{H}^3(\Omega)$ and then, $\mathrm{curl}(\mathbf{v}_m(t) - \alpha\Delta\mathbf{v}_m(t))$ belongs to $\mathbf{L}^2(\Omega)$. Therefore, $\mathbf{v}_m \in \mathbf{V}_2$.

By multiplying (12) by $\mathbf{w}_j$, we obtain

$$(\mathbf{v}_m(t),\mathbf{w}_j) + \alpha(\nabla\mathbf{v}_m(t),\nabla\mathbf{w}_j) = (\mathbf{F}(\mathbf{u}_m(t),\mathbf{u}_m(t)) - \mathbf{f}(t),\mathbf{w}_j),$$

thus (11) can be written

$$(\mathbf{u}'_m(t),\mathbf{w}_j) + \alpha(\nabla\mathbf{u}'_m(t),\nabla\mathbf{w}_j) + (\mathbf{v}_m(t),\mathbf{w}_j) + \alpha(\nabla\mathbf{v}_m(t),\nabla\mathbf{w}_j) = 0. \tag{13}$$

Multiplying equation (13) by $\lambda_j c^m_j(t)$ and adding for $j = 1,\ldots,m$, we get

$$(\mathbf{u}'_m,\mathbf{u}_m)_{\mathbf{V}_2} + (\mathbf{v}_m,\mathbf{u}_m)_{\mathbf{V}_2} = 0,$$

in other words,

$$(\text{curl}(\mathbf{u}'_m - \alpha\Delta\mathbf{u}'_m), \text{curl}(\mathbf{u}_m - \alpha\Delta\mathbf{u}_m)) + (\text{curl}(\mathbf{v}_m - \alpha\Delta\mathbf{v}_m), \text{curl}(\mathbf{u}_m - \alpha\Delta\mathbf{u}_m)) = 0.$$

By taking curl in (12),

$$\text{curl}(\mathbf{v}_m - \alpha\Delta\mathbf{v}_m) = \text{curl}(\mathbf{F}(\mathbf{u}_m, \mathbf{u}_m) - \mathbf{f}),$$

and thus

$$(\text{curl}(\mathbf{u}'_m - \alpha\Delta\mathbf{u}'_m), \text{curl}(\mathbf{u}_m - \alpha\Delta\mathbf{u}_m)) + (\text{curl}(\mathbf{F}(\mathbf{u}_m, \mathbf{u}_m) - \mathbf{f}), \text{curl}(\mathbf{u}_m - \alpha\Delta\mathbf{u}_m)) = 0.$$

Using definition (10) we find

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|\text{curl}(\mathbf{u}_m - \alpha\Delta\mathbf{u}_m)\|^2_{\mathbf{L}^2(\Omega)} + (\text{curl}\mathbf{F}(\mathbf{u}_m, \mathbf{u}_m), \text{curl}(\mathbf{u}_m - \alpha\Delta\mathbf{u}_m)) \\ = (\mathbf{f}, \mathbf{u}_m) + (\text{curl}\mathbf{f}, \text{curl}(\mathbf{u}_m - \alpha\Delta\mathbf{u}_m)). \end{aligned} \tag{14}$$

Now, we will estimate the term:

$$T = (\text{curl}\mathbf{F}(\mathbf{u}_m, \mathbf{u}_m), \text{curl}(\mathbf{u}_m - \alpha\Delta\mathbf{u}_m)).$$

Since div $\mathbf{u}_m = 0$ and $\Omega \subseteq \mathbb{R}^2$, it is not so difficult to prove that

$$\text{curl}(\text{curl}(\mathbf{u}_m - \alpha\Delta\mathbf{u}_m) \times \mathbf{u}_m) = \mathbf{u}_m \cdot \nabla(\mathbf{u}_m - \alpha\Delta\mathbf{u}_m),$$

and

$$(\mathbf{u}_m \cdot \nabla(\mathbf{u}_m - \alpha\Delta\mathbf{u}_m), \text{curl}(\mathbf{u}_m - \alpha\Delta\mathbf{u}_m)) = 0,$$

and thus

$$\begin{aligned} T &= (-\nu\Delta\text{curl}\mathbf{u}_m, \text{curl}(\mathbf{u}_m - \alpha\Delta\mathbf{u}_m)) + (\mathbf{u}_m \cdot \nabla(\mathbf{u}_m - \alpha\Delta\mathbf{u}_m), \text{curl}(\mathbf{u}_m - \alpha\Delta\mathbf{u}_m)) \\ &= \frac{\nu}{\alpha}\|\text{curl}(\mathbf{u}_m - \alpha\Delta\mathbf{u}_m)\|^2_{\mathbf{L}^2(\Omega)} - \frac{\nu}{\alpha}(\text{curl}\mathbf{u}_m, \text{curl}(\mathbf{u}_m - \alpha\Delta\mathbf{u}_m)). \end{aligned}$$

Therefore, the equation (14) can be written

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|\text{curl}(\mathbf{u}_m - \alpha\Delta\mathbf{u}_m)\|^2_{\mathbf{L}^2} + \frac{\nu}{\alpha}\|\text{curl}(\mathbf{u}_m - \alpha\Delta\mathbf{u}_m)\|^2_{\mathbf{L}^2(\Omega)} \\ = (\text{curl}\mathbf{f}, \text{curl}(\mathbf{u}_m - \alpha\Delta\mathbf{u}_m)) + \frac{\nu}{\alpha}(\text{curl}\mathbf{u}_m, \text{curl}(\mathbf{u}_m - \alpha\Delta\mathbf{u}_m)). \end{aligned}$$

Then, we get the following inequality:

$$\frac{1}{2}\frac{d}{dt}\|\mathrm{curl}(\mathbf{u}_m-\alpha\Delta\mathbf{u}_m)\|^2_{\mathbf{L}^2}+\frac{\nu}{\alpha}\|\mathrm{curl}(\mathbf{u}_m-\alpha\Delta\mathbf{u}_m)\|^2_{\mathbf{L}^2(\Omega)}$$
$$\leq \|\mathrm{curl}\mathbf{f}\|_{\mathbf{L}^2(\Omega)}\|\mathrm{curl}(\mathbf{u}_m-\alpha\Delta\mathbf{u}_m))\|_{\mathbf{L}^2(\Omega)}+\frac{\nu}{\alpha}\|\mathrm{curl}\mathbf{u}_m\|_{\mathbf{L}^2(\Omega)}\|\mathrm{curl}(\mathbf{u}_m-\alpha\Delta\mathbf{u}_m)\|_{\mathbf{L}^2(\Omega)}$$
$$\leq \frac{1}{2}\left(\lambda\|\mathrm{curl}\mathbf{f}\|^2_{\mathbf{L}^2(\Omega)}+\frac{1}{\lambda}\|\mathrm{curl}(\mathbf{u}_m-\alpha\Delta\mathbf{u}_m))\|^2_{\mathbf{L}^2(\Omega)}\right)$$
$$+\frac{\nu}{2\alpha}\left(\varepsilon\|\mathrm{curl}\mathbf{u}_m\|^2_{\mathbf{L}^2(\Omega)}+\frac{1}{\varepsilon}\|\mathrm{curl}(\mathbf{u}_m-\alpha\Delta\mathbf{u}_m))\|^2_{\mathbf{L}^2(\Omega)}\right).$$

If we take $\varepsilon=2$ and $\lambda=\dfrac{2\alpha}{\nu}$ we have that

$$\frac{1}{2}\frac{d}{dt}\|\mathrm{curl}(\mathbf{u}_m-\alpha\Delta\mathbf{u}_m)\|^2_{\mathbf{L}^2}+\frac{\nu}{\alpha}\|\mathrm{curl}(\mathbf{u}_m-\alpha\Delta\mathbf{u}_m)\|^2_{\mathbf{L}^2(\Omega)}$$
$$\leq\frac{2\nu}{\alpha}\|\mathrm{curl}\mathbf{u}_m\|^2_{\mathbf{L}^2(\Omega)}+\frac{2\alpha}{\nu}\|\mathrm{curl}\mathbf{f}\|^2_{\mathbf{L}^2(\Omega)}.$$

But $\|\mathrm{curl}\mathbf{u}_m\|^2_{\mathbf{L}^2(\Omega)}\leq 2\|\nabla\mathbf{u}_m\|^2_{\mathbf{L}^2(\Omega)}$, thus

$$\frac{1}{2}\frac{d}{dt}\|\mathrm{curl}(\mathbf{u}_m-\alpha\Delta\mathbf{u}_m)\|^2_{\mathbf{L}^2}+\frac{\nu}{\alpha}\|\mathrm{curl}(\mathbf{u}_m-\alpha\Delta\mathbf{u}_m)\|^2_{\mathbf{L}^2(\Omega)}$$
$$\leq\frac{4\nu}{\alpha}\|\nabla\mathbf{u}_m\|^2_{\mathbf{L}^2(\Omega)}+\frac{2\alpha}{\nu}\|\mathrm{curl}\mathbf{f}\|^2_{\mathbf{L}^2(\Omega)}.$$

From Lemma 3.2

$$\frac{1}{2}\frac{d}{dt}\|\mathrm{curl}(\mathbf{u}_m(t)-\alpha\Delta\mathbf{u}_m(t))\|^2_{\mathbf{L}^2}+\frac{\nu}{\alpha}\|\mathrm{curl}(\mathbf{u}_m(t)-\alpha\Delta\mathbf{u}_m(t))\|^2_{\mathbf{L}^2(\Omega)}$$
$$\leq\frac{4\nu}{\alpha^2}(\|\mathbf{u}_m(0)\|^2_{\mathbf{L}^2(\Omega)}+\alpha\|\nabla\mathbf{u}_m(0)\|^2_{\mathbf{L}^2(\Omega)})+\frac{4\mathcal{P}^2}{\alpha\nu K}\|\mathbf{f}\|_{\mathbf{L}^\infty(0,T;\mathbf{L}^2(\Omega))}+\frac{2\nu}{\alpha}\|\mathrm{curl}\mathbf{f}(t)\|^2_{\mathbf{L}^2(\Omega)}.$$

From all this considerations, we have proved the following lemma.

Lemma 3.3. *The solution $\mathbf{u}_m$ of the problems* (8) *and* (9) *satisfies the a priori estimate for all $t\in[0,T]$:*

$$\|\mathrm{curl}(\mathbf{u}_m(t)-\alpha\Delta\mathbf{u}_m(t))\|^2_{\mathbf{L}^2(\Omega)}\leq e^{-\frac{\nu t}{\alpha}}\|\mathrm{curl}(\mathbf{u}_m(0)-\alpha\Delta\mathbf{u}_m(0))\|^2_{\mathbf{L}^2(\Omega)}$$
$$+\frac{2}{\alpha}(\|\mathbf{u}_m(0)\|^2_{\mathbf{L}^2(\Omega)}+\alpha\|\nabla\mathbf{u}_m(0)\|^2_{\mathbf{L}^2(\Omega)})+\frac{2\mathcal{P}^2}{\nu^2K}\|\mathbf{f}\|_{\mathbf{L}^\infty(0,T;\mathbf{L}^2(\Omega))}+\frac{2\alpha}{\nu}\|\mathbf{f}\|_{\mathbf{L}^2(0,T;H(\mathrm{curl};\Omega))}.$$

4. Proof of the Theorem 1.1

In this section, we prove the Theorem 1.1. To this end, at first we prove the existence of a sequence of Reproductive Galerkin solutions, by following the ideas given in [3], which converges to the reproductive solution of the grade two system fluid.

We define the operator $L^m(t) : [0, T] \to \mathbb{R}^m$ as

$$L^m(t) = (c_1^m(t), c_2^m(t), \dots, c_m^m(t)), \tag{15}$$

where $c_j^m(t)$ are the coefficients of the expansion of $\mathbf{u}_m$.

For every $(\xi_1, \xi_2, \dots, \xi_m) \in \mathbb{R}^m$, we define the following equivalent norms:

$$\|(\xi_1, \xi_2, \dots, \xi_m)\|^2_{a,\mathbb{R}^m} := \|\mathbf{u}\|^2_{\mathbf{L}^2(\Omega)} + \alpha\|\nabla\mathbf{u}\|^2_{\mathbf{L}^2(\Omega)},$$
$$\|(\xi_1, \xi_2, \dots, \xi_m)\|^2_{b,\mathbb{R}^m} := \|\mathrm{curl}(\mathbf{u} - \alpha\Delta\mathbf{u})\|^2_{\mathbf{L}^2(\Omega)},$$

where $\mathbf{u} = \xi_1\mathbf{w}_1 + \xi_2\mathbf{w}_2 + \cdots + \xi_m\mathbf{w}_m$.

We define the operator $\Phi^m : \mathbb{R}^m \to \mathbb{R}^m$ in the following manner: Given $L_0^m \in \mathbb{R}^m$, we define $\Phi^m(L_0^m) = L^m(T)$, where $L^m(t)$ is defined in (15). It is clear that Φ^m is continuous and we want to prove that it has a fixed point. In order to prove this result, we will use the Leray-Schauder Theorem. Indeed, it suffices to show that for all $\lambda \in [0, 1]$, the possible solution $L_0^m(\lambda)$ of the equation

$$L_0^m(\lambda) = \lambda\Phi^m(L_0^m(\lambda)) \tag{16}$$

are bounded independently of λ.

Since $L_0^m(0) = 0$, we will consider $\lambda \in (0, 1]$. In this case, (16) can be written as

$$\Phi^m(L_0^m(\lambda)) = \frac{1}{\lambda}L_0^m(\lambda).$$

Thus, by definition of Φ^m and Lemma 3.2, we obtain

$$\left\|\frac{1}{\lambda}L_0^m(\lambda)\right\|^2_{a,\mathbb{R}^m} \le e^{-\nu KT}\|L_0^m(\lambda)\|^2_{a,\mathbb{R}^m} + \frac{\mathcal{P}^2}{\nu}\int_0^T e^{\nu Ks}\|\mathbf{f}(s)\|^2_{L^2(\Omega)}ds,$$

which implies that

$$\|L_0^m(\lambda)\|^2_{a,\mathbb{R}^m} \le \frac{\dfrac{\mathcal{P}^2}{\nu}\displaystyle\int_0^t e^{\nu Ks}\|\mathbf{f}(s)\|^2_{L^2(\Omega)}ds}{1 - e^{-\nu KT}} = M_0.$$

Now, from Lemma 3.3 and definition of Φ^m, we have that

$$\left\|\frac{1}{\lambda}L_0^m(\lambda)\right\|^2_{b,\mathbb{R}^m} \le e^{-\frac{\nu T}{\alpha}}\|L_0^m(\lambda)\|^2_{b,\mathbb{R}^m} + \frac{2}{\alpha}\|L_0^m(\lambda)\|^2_{a,\mathbb{R}^m}$$
$$+ \frac{2\mathcal{P}^2}{\nu^2 K}\|\mathbf{f}\|_{\mathbf{L}^\infty(0,T;\mathbf{L}^2(\Omega))} + \frac{2\alpha}{\nu}\|\mathbf{f}\|_{\mathbf{L}^2(0,T;H(\mathrm{curl};\Omega))},$$

then, we deduce that

$$\|L_0^m(\lambda)\|_{b,\mathbb{R}^m}^2 \leq \frac{\dfrac{2M_0}{\alpha} + \dfrac{2\mathcal{P}^2}{\nu^2 K}\|\mathbf{f}\|_{\mathbf{L}^\infty(0,T;\mathbf{L}^2(\Omega))} + \dfrac{2\alpha}{\nu}\|\mathbf{f}\|_{\mathbf{L}^2(0,T;H(\text{curl};\Omega))}}{1 - e^{-\frac{\nu T}{\alpha}}} = M_1,$$

for each $\lambda \in (0,1]$. This last estimate is independent of $\lambda \in [0,1]$ and $m \in \mathcal{N}$. Consequently, Leray-Shauder Theorem implies the existence of al least one fixed point of Φ^m, and then the existence of reproductive Galerkin solution $\mathbf{u}_m$. Moreover, since the previous estimates do not depend on $m \in \mathcal{N}$ and by Lemma 3.3, there exists $M \in \mathbb{R}$ independent of m such that

$$\|\mathbf{u}_m(t)\|_{\mathbf{V}_2} \leq M, \tag{17}$$

for each $t \in [0,T]$, it means that $(\mathbf{u}_m)_{m\geq 1}$ is bounded in $L^\infty(0,T;\mathbf{V}_2)$. By Lemma 2.1, we can write $\|\mathbf{u}_m(t)\|_{\mathbf{H}^3(\Omega)} \leq M$, for each $t \in [0,T]$.

It remains to pass to the limit with respect to m. This is a standard argument and we have only to prove that $(\mathbf{u}'_m)_{m\geq 1}$ is bounded in $L^\infty(0,T;\mathbf{V}'_2)$. In order to prove this bound, we use the arguments given in [1], p 190.

At first, we note that

$$|b(\mathbf{u}_m(t), \mathbf{u}_m(t), \mathbf{v})| \leq C\|\nabla \mathbf{u}_m\|_{\mathbf{L}^2(\Omega)}^2 \|\mathbf{v}\|_{\mathbf{V}_2},$$

which implies that there exists $T_m^1(t) \in \mathbf{V}'_2)$ such that

$$b(\mathbf{u}_m(t), \mathbf{u}_m(t), \mathbf{v}) = \langle T_m^1(t), \mathbf{v}\rangle, \ \forall \mathbf{v} \in \mathbf{V}_2,$$

and by estimate (17), we have that $(T_m^1)_{m\geq 1}$ is bounded in $L^\infty(0,T;\mathbf{V}'_2)$. In the same manner, there exists a bounded sequence $(T_m^2)_{m\geq 1}$ in $L^\infty(0,T;\mathbf{V}'_2)$ such that

$$b(\mathbf{u}_m(t), \Delta\mathbf{u}_m(t), \mathbf{v}) = \langle T_m^2(t), \mathbf{v}\rangle, \ \forall \mathbf{v} \in \mathbf{V}_2.$$

Finally, from equation (8), we can conclude that $\mathbf{u}'_m = T_m P_m$, where $(T_m)_{m\geq 1}$ is a bounded sequence of $L^\infty(0,T;\mathbf{V}'_2)$ and P_m is the projection of $\mathbf{V}_2$ on $\mathbf{V}_2^m$. This completes the proof.

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