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Methodological contribution of K.W. Johansen to the structural analysis of long cylindrical roof shells. Beam method.

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ABSTRACT

Evaluating the state of stress of a cylindrical shell based on elastic analysis originally involved a series of hypotheses concerning the conditions around it and the structural material used. All these hypotheses involved either idealising a reality that was impossible to ascertain a priori, or referring to an ideal, homogeneous and isotopic material, when reinforced concrete does not have any of those properties. However, it was impossible to guarantee that the state of stress obtained in the shell represented the indisputable "real state" of the structure.

Thanks to the progress made in the study of plastic methods, a key figure in this context is the Danish engineer Knud Winstrup Johansen; in 1944, he published a pioneering article which set out the origin of the application of limit analysis to the structural calculation of long cylindrical

roof shells based on a plastic approach to equilibrium, enabling simple and accurate calculation of these types of structure, as will be discussed below.

Keywords: beam method; Limit analysis; long cylindrical shell; K.W. Johansen; plastic approach to equilibrium; structural calculation of long cylindrical shells.

NOTATION LIST

The following symbols are used in this paper:

A: Area of traction reinforcement;

H: resultant of the shear stresses;

L: Value of the resultant of external forces;

M: Positive moment;

M_{emp}: Moment of abutment;

m_P: Transverse moments due to the external load (P);

m_S: Transverse moments due to reactions in the pillars;

m_t: Transverse moments due to shear forces (t);

m_φ: Resulting transverse moment;

N: Resultant of the compressive stresses;

N_X: Normal forces in the direction of the shell;

$N_{x\varphi}$: Tangential forces in the cross-section of the shell;

P: Value of the external forces acting on the shell;

$\bar{P}$: Oblique resultant of the external forces;

Q: Resulting from the shear stress;

r: Radius value of the shell arc;

S: Value of reactions in the direction tangent to the curve;

S_l: Horizontal component of the reaction;

S_v: Vertical component of the reaction;

S_N: Normal component of the reactions in the pillars;

S_T: Tangential component of the reactions in the pillars;

t: Value of the shear stresses;

T_A : centre of mas;

T_D: centre of mas;

α: Angle between the arc and the horizontal;

σ: Maximum compressive stress in the concrete;

INTRODUCTION

The first long cylindrical reinforced concrete shell appeared in Germany in 1924, in the roof of a building that was to be used as a factory by the company Zeiss.

Until the 1920s, and in Germany in particular, the structural behaviour of thin reinforced concrete shells was studied as if they were "membranes". These studies, begun by R. Maillart and later formulated by H. Reissner (Reissner 1908) and F. Emperger (Emperger 1910), found that if the stresses in a thin but sufficiently rigid shell were only compressive, tensile and tangential, were all contained within the shell's thickness and there were no bending stresses at any point, the sheet only needed to be very thin - just a few centimetres thick - in order for its shape and support conditions to meet certain basic conditions. The structure therefore no longer solved the resistant problems in terms of a cross-section, but instead did so using the pure form; thereby satisfying the principle of economy of material that was so important to engineers and builders at that time. Reinforced concrete was obviously the material that complied with this mathematical model, due to its moldability, as it contained reinforcements to counteract tensile and shear stresses.

Meanwhile, reinforced concrete cylindrical shells became widespread types of structure able to cover large spans with a minimal use of material in construction after the First World War. As a result, a new construction system emerged, with a geometry that was ideal for covering utilitarian spaces, such as stations, warehouses, factories and hangars; in short, spaces with large spans which had previously been constructed using steel. In view of the demand for these new types of buildings, it became necessary to establish a mathematical foundation that was able to calculate them before they were constructed. Elastic Theory was applied, due to its widespread use in the structural calculations of the time.

The theory of elasticity began to be implemented in the structural calculation of long cylindrical shells in Germany in the 1930s, as a result of the work done by the engineers U. Finsterwalder (Finsterwalder 1928, 1932 and 1936) and Fr. Dischinger (Dischinger 1928, 1930, 1935 and 1936) and later by the Norwegian A. Aas Jakobsen (Jakobsen 1937, 1939, 1940 and 1941). The mathematical formulation provided by analytical theory, which was at that time widely used and referred too ideal, homogeneous and isotropic materials complying with Hooke's law, was also applied to the structural calculation of long cylindrical reinforced concrete shells, without any consideration being given to the characteristics of the new construction material used.

In the theory of elasticity, based on the elastic behaviour of material, both the equilibrium between the internal and external forces and the compatibility of deformations had to be satisfied. In addition, solving the mathematical problem provided by the equilibrium equations became an indeterminate when the shell was attached to other deformable structural items, such as edge beams, other similar and contiguous shells, etc. In these cases, it was necessary to supplement the equilibrium equations provided by membrane theory with other equations generally derived from working theorems which considerably increased the complexity of the mathematical development.

By applying the theory of elasticity, it was thought to be possible to precisely determine the state of stress at each point in the shell, by solving eighth-order differential equations of great mathematical complexity. The solution to these equations determined the stresses and moments at all points in a long cylindrical shell, made of a homogeneous, isotropic and ideal material, as stipulated by Hooke's law. However, in practice it also became essential to introduce a number of hypotheses, which were so important that the fact that they were legitimate was neglected, provided that they did not contradict the real results.

However, it was practically impossible to apply the elastic theory of shells, as it involved solving complex eighth-order differential equations, based on unreal hypotheses about the surrounding conditions and the structural material used. All these hypotheses involved either idealising a situation that was impossible to ascertain *a priori*, or referred to an ideal, homogeneous and isotopic material, when reinforced concrete does not have any of those properties. To guarantee that the state of stress obtained in the shell represented the indisputable "real state" of the structure was always impossible. Consequently, some obvious and significant insurmountable inconsistencies between the results obtained from the elastic calculation and the results under real conditions and in tests began to appear.

A theoretical framework more appropriate to the one provided by elastic theory for the structural calculation of these types of structure was the study of the conditions in which the collapse of the structure occurred; or in other words, limit analysis. Although the "real" state of the structure could not be determined, its strength could be calculated accurately; it was also very insensitive to supposed shortcomings in the manufacture or execution, and to small variations in the conditions of the surroundings.

Plastic theory and the fundamental theorems which are derived from it are the most significant contribution to structural theory in the twentieth century (Gvozdev 1936 and Prager 1952). The cornerstone of limit analysis is the application of the study of possible states of equilibrium in the structure, the main corollary of the Security Theorem or Lower Limit Theorem.

In this context, in 1944 the Danish engineer Knud Winstrup Johansen published a very important article in which he carried out a structural analysis of a long cylindrical shell in a real roof. The calculation was based solely on the equilibrium equations approach, thereby enabling a simple and reliable calculation of these structural types (Johansen 1944). Other engineers such as the Hungarian G. Kazinczy (Kazinczy 1949) and the Dane H. Lundgren (Lundgren 1949) continued with this type of study, formulating a practical, clear and simple theory for application based on the equilibrium approach in the late 1940s (Lundgren 1949).

From 1950 until the appearance of computers, the application of limit analysis with the relevant study of systems of equilibrium was what really facilitated the calculation and construction of long cylindrical roof shells. This is how some engineers and architects worked when calculating these types of structures. This was the case with the engineer K. W. Johansen, as will be shown below.

RESEARCH SIGNIFICANCE

Reinforced concrete cylindrical shells have traditionally been designed according to either the membrane theory or the bending theory. The main assumption in both theories is that the shell material is linearly elastic which may be valid at low level working loads. For higher load levels and due to the material nonlinearity caused mainly by cracking of the concrete and plasticity of the reinforcement and the concrete; the assumption of linear elastic behavior is no longer applicable and a nonlinear analysis is required. Throughout the history of structures several studies have emerged that address this problem (Assan 2002, Suanno, Ferrari, Prates and E.E.T. 2003, Chen 2007, Chandrasekaran, Gupta and Carannante 2009, Jawad 2015 and Pophare and Jadhav 2017).

This study will not attempt to undermine the analytical method applied to these types of structure, but instead to fill a gap in the existing knowledge of the application of other simpler calculation methods, and specifically the so-called Beam Method. The present work tries to deepen in the methodological contribution of K. W. Johansen to the structural calculation of these typologies in a moment in which they were tried to be calculated, in a rigorous and "real" way, applying the mathematical mechanism of the theory of elasticity. It will be necessary to explain and demonstrate how the long cylindrical roof shells were analyzed using other simpler methods, as reliable as the analytical method, as the so-called beam method, based on the study of different equilibrium states, used by Johansen in the analysis of these structures.

ORIGIN AND APPLICATION OF LIMIT ANALYSIS TO THE STRUCTURAL ANALYSIS OF LONG CYLINDRICAL REINFORCED CONCRETE ROOF SHELLS.

The origin of plasticity studies.

Although the assumptions established by the theory of elasticity seemed reasonable, common sense suggested that a trivial defect or imperfection in the shell, or at least an unpredictable one, could not really affect its strength. There also came a point at which the contradictions arising between the constructive reality of these structural types and the results of the analytical calculation were inadmissible. In this respect, the conclusion was clear: calculating elastic stresses was not relevant for predicting the real strength of long cylindrical roof shells. As a result, the search began for new, simpler and more effective calculation methods, which would provide a more accurate response to the real characteristics of the structural material used: reinforced concrete.

Until well into the twentieth century, the most obvious and proven explanations for any structural phenomenon were discarded as "*unscientific*" if they were not accompanied by a corresponding formulaic deployment based on the theory of elasticity. However, non-linear ideas were not completely neglected even though they were treated as if they were a scientific curiosity when they were mentioned.

The Hungarian Gábor Kazinczy (1888-1964) was one of the first engineers who based the calculation of the plasticity of structures not only on theory, but also on empirical data obtained in tests. In 1914, he proved that the calculation of elastic stresses was not relevant in predicting a structure's real strength; if it was constructed using a ductile material, it was not dependent on the appearance of the threshold for elastic stress at a point on it, but instead due to the unacceptable increase in the deformations in it, due to the action of the loads (Kazinczy 2014).

As a result, plastic theory emerged like any new scientific theory; in other words, due to the shortcomings of the previous theory - in this case the theory of elasticity.

In this first study, Kazinczy only applied the concepts mentioned above to steel structures. Later, in 1933, he published another article on the plasticity of reinforced concrete, in which he proposed the concept of redistribution at moments of uniaxial bending, based on the plastic behaviour of both steel and concrete (Kazinczy 1933).

Three years later, in 1936, two significant events occurred in relation to the study of plastic methods: important papers on plasticity were contributed to the Second Congress of the IABSE (Maier-Leibnitz 1936 and Melan 1936) in Berlin, and theorems of plasticity were announced by the Russian A. A. Gvozdev (Gvozdev 1936). In addition to these events, a third took place in Denmark, related to the studies carried out by the Danish engineers K.W Johansen and H. Lundgren, concerning the application of limit analysis to the structural calculation of long cylindrical roofing shells.

Prior to the publication of the book *Cylindrical Shells* by H. Lundgren in 1949 (Lundgren 1949), the literature about the structural analysis of long cylindrical shells contained very few studies of the problem, apart from membrane theory (Reissner 1908 and Emperger 1910) and above all the theory of elasticity. However, there were various avenues towards the simplification of analytical tools for shell design, such as those proposed by the engineers W. Flügge (Flügge 1934), R. Vallette (Vallette 1934), H. Schorer (Schorer 1935), U. Finsterwalder (Finsterwalder 1932), and the Danish engineer A. Aas Jakobsen (Jakonbsen 1940).

The contributions by the Danish engineer Winstrup Knud Johansen on structural analysis and calculation of long cylindrical shells were crucial in this respect, since they mark the beginning of the application of limit analysis to these types of structure (Johansen 1944 and 1948).

K.W. Johansen and the application of limit analysis to the structural analysis of long cylindrical roof shells

In 1944, the Danish engineer K.W. Johansen (1901-1978), a professor and doctor of Construction Engineering and the President of the Technical University of Delft, published an article in Danish (Johansen 1944); concerning the analysis and structural calculation of a complete and asymmetrical long cylindrical shell in reinforced concrete. The shell is the roof of the General Broadcasting Corporation building, used by the film industry in Copenhagen, built in 1938. Because of its characteristics, the shell that forms the roof of this building could not have been resolved by methods based on the application of elastic theory. The calculation of this shell is the one that is developed next trying to demonstrate how these structural typologies were analyzed using other simpler method, as reliable as the analytical method, as the so-called beam method, based on the study of different equilibrium states.

The structure covers an area 16 m wide and 36 m long; it is divided in two spaces, one of 24 m and the other of 12, by means of a partition wall arranged across it. On the north side of the building, the shell is supported by pillars spaced 3.2 m apart; while on the south side of the building there is no support (Statsradiofonien 1946). The shell also is bounded at both ends by two transverse walls. The cross-section of the roof, which has a thickness of 12 cm, is roughly cylindrical but it has a ventilation channel inside.

The modelling of the cross-section of the shell prior to structural analysis is therefore as follows (Fig. 1).

The cross-section is established by a circular arc $\widehat{ABD}$, with a radius of 9.22 m. At point B on the arc, it tangentially touches another arc $\widehat{BC}$, with a radius of 14 m. In turn, a vertical straight section $\widehat{CD}$ starts at point *C*, thus closing its cross-section together with a concrete ogee as a cantilever roof.

Analysis method.

The hypotheses formulated prior to the calculation are as follows:

1. Model the cross-section of the shell, dividing it into two parts due to its uneven structural behaviour. Due to its stiffness, Johansen relates the structural behaviour of the part of the shell considered closed and defined by points *B*, *C* and *D* to that of a concrete beam. Meanwhile, due to its low stiffness, he considers the part defined by points *A* and *B* as similar to a membrane, i.e. where stresses act solely on its plane. As for the value of the reactions, on the supports of the shell's edge A, it will comply with the following expression:

$$S = Prcos\alpha \qquad [1]$$

This expression being similar to the one referring to the value of normal union forces in the direction tangent to the curve: $N_{\phi} = Prcos\phi$,

Where *P* is the value of the external forces acting on the shell and $N_{\emptyset}$ is the value of the normal force in the direction tangent to the curve: $N_{\phi} = P_r.r$;

P_r is the value of the normal load: $P_r = Pcos\phi$ and S is the value of reactions in the direction tangent to the curve.

In 1928, Dischinger and Finsterwalder wrote on this subject; the components of the load *P* acting in a distributed manner along the cylindrical shell in the direction of the three axes (Dischinger and Finsterwalder 1928).

2. The second hypothesis refers to the rupture lines, or limits. Johansen says that the contribution of the concrete is not solely in terms of its tensile strength; while the acceptable stress in the iron reinforcements was that of the creep within a safety coefficient. In other words, he introduces the real and fundamental characteristics of the structural materials used, in contrast to the provisions of elastic theory.

3. Finally, Johansen says that in statically indeterminate conditions, the moments will be distributed in accordance with the reinforcement made, i.e. in accordance with the theory of plasticity.

These assumptions, set out by Johansen and applied to the structural calculation of a long cylindrical roofing shell, form the basis for limit analysis with its equilibrium approach.

Having established the hypotheses, the structural analysis method used is based on the long cylindrical shell's behaviour being similar to that of a reinforced concrete beam (Fig. 2), which has the following calculation process.

Longitudinal calculation of the shell:

First, he obtains the value of the external forces acting on the shell. These actions consist of the permanent load, with a value of $340\, kg/m^2$, and the variable load of $100\, kg/m^2$. The resulting value is therefore $L = 9{,}80\, t/m$, acting at a distance of 1,06 m from point B (Fig. 3). Likewise, the reactions in the pillars, located on the north side of the building and in the direction tangential to the curve, will have a value of $S = 2{,}70\, t/m$ [1] for an angle α= 48.62º at the start of the shell at point A, and will have horizontal component, $S_l = 2{,}00\, t/m$ and vertical component, $S_v = 1{,}80\, t/m$.

Combining the horizontal component, in a polygon of forces (top right in Fig. 3), with the resultant of the external forces L, gives an oblique resultant with a value of $\bar{P} = 7{,}40\, t/m$.

While the part of the cylindrical shell defined by points A and B rests on pillars, the other part of the shell, defined by points *B*, *C* and *D*, has a free end. This part of the roof consists of two spaces; each one is defined by the transverse end wall and the partition wall located in the middle of the building.

As a result of these characteristics, Johansen proposes that the part of the shell defined by points *B*, *C* and *D* should be calculated as if it was a continuous concrete beam with two spaces; but with a plastic approach to the equilibrium, respecting the characteristics of the structural material used. It is here where the simplicity and validity of the method proposed truly lies.

The moment of abutment is obtained directly with a value of:

$$M_{emp} = \frac{3}{32}.\,7{,}40\,\frac{t}{m}\,(24\ m)^2 = 400\ tm \qquad [2]$$

This moment of abutment it would be greater than the real one because in fact the shear stress has a value greater than the theoretical case of the beam. Johansen estimates the value of this moment of abutment, according to the theory of elasticity, at $370tm$; while the largest positive moment will be $M = 365tm$.

After obtaining the moments, Johansen places the neutral axis in the cross-section of the shell in order to obtain the normal stress and compressive forces. As the cross-section of the shell is unable to withstand torques, which is common in open sections, Johansen suggests placing the neutral axis based on a single assumption: that the resultant of the tangential forces has an equal magnitude and an opposite direction to that resulting from the shear stress *Q*; which in turn must be equal to the value obtained for the oblique resultant of the loads (Fig. 3). In other words, starting from a condition of equilibrium of forces, Johansen establishes the location of the neutral line, obtaining a solution to the structural problem as a result; but it is not the only one, because another positioning of the neutral line would obtain another state of equilibrium that would address other assumptions.

The true location of the neutral axis (line n-n' in Fig. 4) is determined empirically, i.e. after performing various tests only considering the equilibrium of tensile and compressive forces. Once the neutral axis has been located, it is easy to determine the reinforcement area required, and the concrete stress and shear stress.

The point furthest from the neutral axis in the point designated B', and as such that point will have the greatest compressive stress. Meanwhile, iron rods are placed around the shell, at the centres of mass T_A and T_D.

As a result, the neutral axis is positioned at a distance of 0.80 m from the centre of mass of the tensioned axis T_D, and 0.43 m from point B', where the most compressed axis in the shell is located (Fig. 3).

For these distances and the rod stress, provided by Johansen in this article with a value of $1200\,kg/cm^2$ at the point T_D, the maximum compressive stress in the concrete, σ, is determined as follows:

$$\sigma = \frac{1200kg/cm^2}{15} . \frac{0{,}43m}{0{,}80m} = 43\,kg/cm^2$$

Having obtained the value of the maximum compressive stress for concrete, Johansen then calculates the module, the direction and the point of application of the resultant of the compressive stresses *N*, corresponding to a value of $N = 193t$; indicating that the bending plane must contain the force vector $\bar{P}$, meaning that the moment must be perpendicular to this plane. The torque, constituted by the result of the tensile and compression forces, must therefore be contained on the same plane that contains the force $\bar{P}$.

On this basis, the point of application is obtained by the intersection of two lines. One is the line connecting the resultants of the tractions, i.e. the line $\overline{T_A - T_B}$; while the second is a line parallel to the direction of Q, which passes through the point B', i.e. the line resulting from the compressions.

The distance between points B' and T (Fig. 3) determines the lever arm between the torque, forming a distance of $190cm$. The value of the positive moment, *M*, will therefore be:

$$M = 193t.1{,}90m = 367tm \qquad [3]$$

Knowing the tension relative to the steel used, the necessary area of traction reinforcement, A, will be determined by the expression:

$$A = \frac{N}{\sigma}, \qquad [4]$$

where, *N* is the resultant of the compressive stresses and σ is the value of the maximum compressive stress in the concrete.

In short, the procedure followed by Johansen related to the longitudinal calculation of the long cylindrical shell, based on successive results obtained by the equilibrium of forces is as follows:

1. Obtaining the value of the external loads.
2. Calculation of the positive and negative bending moments.
3. Obtaining the location of the neutral axis neutral axis in the cross-section of the shell by the equilibrium of forces.
4. Calculation of the maximum compressive stress of the concrete.
5. Obtaining the values for the normal forces of tension and compression.
6. Calculation of the longitudinal reinforcements in the shell.

Transverse calculation of the shell.

After obtaining the necessary longitudinal reinforcement in the shell, Johansen performs the transverse calculation. To do so, it is first necessary to obtain the value of the resultant of the tangential forces.

Calculation of the transverse forces:

To do this, Johansen uses the beam theory, where the normal forces in the direction of the shell, N_x, are concentrated in a single generatrix, called the beam, and applied at the centre of mass in the area concerned; while the tangential forces, $N_{x\varphi}$, in the cross-section of the shell remain constant within each interval between two beams (Bredt 1896).

In the beam method, for a cylindrical cross-section, the resultant of the shear stresses *H* is placed on a line parallel to the bowstring at a distance $h = {}^{4}/_{3}\, f$; measured from the bowstring (Fig. 4a) where f is its height measured from the bowstring. Meanwhile, the resultant of the shear forces *H* in this range would have a value of *KH*, where *K* is the value of the bowstring of the arc (Fig. 4b).

The beam method therefore determines the location of each beam, the value of the normal forces in each one, and the tangential forces between the beams.

On this basis, Johansen divides the cross-section of the shell analysed into three different arcs: $\widehat{AB'}$, $\widehat{B'D}$ and $\widehat{B'CD}$; calling the tangential forces in each of the three sections:

$$H_A, H_c, H_D,$$

meaning the resultant of each one would be (Fig. 3):

$$9{,}3H_A, 2{,}5H_c \; and \; 3{,}0H_D$$

The value of each one is obtained by decomposing the shear force *Q*, or the shear stress (the polygon of forces located in the upper left of Fig. 3), according to the three resultants of the tangential forces:

$$H_A = \frac{0{,}77Q}{9{,}3} = 0{,}083Q,\; H_C = \frac{0{,}64Q}{2{,}5} = 0{,}256Q,\; H_D = \frac{0{,}59Q}{3{,}0} = 0{,}197Q$$

Since the lever arm is $1{,}90m$, then:

$$H = H_A + H_C + H_D = \frac{Q}{1{,}90m} \qquad [5]$$

Similarly, by making cuts only through *Ta* and *Td*, we obtain:

$$H_A = \frac{Q}{1{,}90} \cdot \frac{T_A}{T} = 0{,}097Q$$

$$H_C + H_D = \frac{Q}{1{,}90} \cdot \frac{T_D}{T} = 0{,}430Q$$

With the ratio of these values to those previously determined in the polygon of forces we obtain:

$$H_C = 0{,}243Q \;\; and \;\; H_D = 0{,}187Q$$

Johansen thereby guarantees the real location of the neutral axis, and as has been demonstrated, he does so by means of the equilibrium of forces.

The process would be the same for the cross-sections of the shell for the location of the corresponding negative moment.

As in the case of the cross-section of the shell belonging to the positive moments, the normal compressive forces located in this case at both points *A* and *D*, with a lever arm of $1,80m$ would be calculated as follows:

$$T = \frac{370tm}{1{,}80m} = 205t = N = N_A + N_D \qquad [6]$$

Thereby obtaining a stress for the concrete of 66 kg/cm^2.

In this case, Johansen divides the cross-section of the shell into three parts: $\widehat{AT}$, $\widehat{TD}$ and $\widehat{TCD}$, positioning the tangential forces belonging to each of these three sections and the magnitude of their resultants, after decomposing the force *Q* into a funicular polygon.

Calculation of transverse moments:

By sectioning an element of the shell (Fig. 5), of length *dx* in the direction of the generatrix of the shell and width *ds*, orthogonal to the previous one, we see how the resultant of the shear stresses *H* act upon it, due to the action of the external loads *P*.

As the resultant of the shear stresses, *H*, it is proportional to the shear force, or shear *Q*, meaning that the shear stresses, t, are also proportional to the external loads, i.e.:

$$\frac{\partial Q}{\partial x} = \bar{P} = 7{,}4\, t/m, \qquad [7]$$

where $\bar{P}$ is the oblique resultant of the external forces.

Johansen obtains the various values for the shear stresses t in each section according to the values previously calculated for the tangential forces (Fig. 3):

For the section $\widehat{AB}$:

$$t_A = 0{,}097 \cdot 7{,}4 = 0{,}72\, t/m^2$$

For the section $\widehat{B'''C}$:

$$t_C = 0{,}243 \cdot 7{,}4 = 1{,}80\, t/m^2$$

For the section $\widehat{B''D}$:

$$t_D = 0{,}187 \cdot 7{,}4 = 1{,}38\, t/m^2$$

As *t* does not vary along the length of the shell, i.e. in direction *x*, the tangential moments due to *t* will also be constant along that length.

The transverse moments that Johansen analyses (Johansen 1944) are those due to (Fig. 5).

- Transverse moments due to shear forces (Fig. 6a).
- Transverse moments due to the external load *P* (fig 6b).
- Transverse moments due to reactions in the pillars (Fig 6c).

The following expressions are obtained for all of these:

Moment due to the tangential force *t*, (Fig 6a):

$$m_t = \int_0^{\varphi} \big(r - r\cos(\varphi - \theta)\big) \cdot t \cdot r\partial\theta = \int_0^{\varphi} r^2\, t[1 - \cos(\varphi - \theta)]\partial\theta = (\varphi - \sin\varphi)tr^2 \qquad [8]$$

Moment due to the external load *P* (Fig. 6b):

$$m_p = -pr\varphi z \qquad [9]$$

where:

$$z = r\frac{\sin(1/_2\varphi)}{1/_2\varphi}\sin\big(\alpha - 1/_2\,\varphi\big) - r\sin(\alpha - \varphi)$$

i.e. [15]:

$$m_p = \big(\cos\alpha - \cos(\alpha - \varphi) + \varphi\sin(\alpha - \varphi)\big)pr^2$$

And finally, the moment due to the reaction in the supports (Fig 6c):

$$m_s = S_T r(1 - cos\varphi) - S_N r sin\varphi, \qquad [10]$$

where S_T is the tangential component and S_N is the normal component of the reactions in the pillars.

In short, the resulting moment is given by the sum of the previous three moments, i.e.:

$$m_\varphi = m_t + m_p + m_s, \qquad [11]$$

where, m_t is the transverse moments due to shear forces (*t*); m_p is the transverse moments due to the external load (*P*) and m_s is the transverse moments due to reactions in the pillars.

Or to put it another way:

$$m_\varphi = pr^2 \left[-\frac{S_N}{pr} sin\varphi + \left(cos\alpha + \frac{S_T}{pr} \right)(1 - cos\varphi) - \varphi sin\varphi cos\alpha - (sin\varphi - \varphi cos\varphi) sin\alpha + (\varphi - sin\varphi)\frac{t}{p} \right] \qquad [12]$$

The value of this expression [14] becomes a minimum when $S_N = 0$, meaning that the first term of the formula is cancelled out. The second-order terms therefore also disappear from the formula when the trigonometric functions are developed in a series according to the angle. In this case, by including series of up to the fifth order, Johansen obtains the following expression:

$$m_\varphi = pr^2 \left[-\left(\frac{\varphi^3}{3} - \frac{\varphi^5}{30} \right) sin\alpha + \left(\frac{\varphi^3}{6} - \frac{\varphi^5}{120} \right) \frac{t}{p} + \frac{\varphi^4}{12} cos\alpha \right] \qquad [13]$$

If we therefore apply this expression to the section $\widehat{AB}$ (Fig. 3), in the cross-section of the shell, where the value of the shear stress is $t = 0{,}72\, t/m^2$ and the value of the external load $P = (340 + 100)\, kg/m^2 = 440\, kg/m^2$:

$$\frac{t}{P} = \frac{0{,}72\, t/m^2}{0{,}440\, t/m^2} = 1{,}63$$

$$\alpha = 0.85 rad(48.62^o), \;\; sin\alpha = 0.75, \;\; cos\alpha = 0.661, \;\; r = 9.28m$$

According to the expression [15], the value of the tangential moment obtained by Johansen would be:

$$m_\varphi = 440 \cdot 9{,}28^2 \left[-\frac{\varphi^3}{3}\left(1 - \frac{\varphi^2}{10}\right) 0{,}75 + \frac{\varphi^3}{6}\left(1 - \frac{\varphi^2}{20}\right) 1{,}63 + \frac{\varphi^4}{12} 0{,}661 \right]$$

or in other words:

$$m_{\varphi} = 815(1 + 2{,}5\varphi + 0{,}5\varphi^2)\varphi^3$$

Johansen adds a correction factor to the value of the reaction in the S pillars, due to the real position of the shell, which would affect its two components - both the normal, S_N, and the tangential S_T. Johansen determines the following values by trial and error:

$$\Delta m = 360kg, \ \ \Delta S_N = -0{,}055\, t/m, \ \ \Delta S_T = -0{,}87\, t/m,$$

This means that the value of the resulting moment, depending on the angle φ, is obtained from the sum of the tangential moments, m_{φ}, a moment of abutment, Δm, and the relative contribution to the correction of the value of the reaction in the supports, $S_N\ and\ S_T$.

Depending on the angle φ, these values are refined in table 1.

Likewise, for the section $\widehat{B'C}$ (Fig. 3), the values adopted were:

$$t_C = 1{,}80\, t/m^2\,, \ \ P = 440\, kg/m^2\,, \ \ \frac{t}{P} = 4{,}09$$

$$\alpha = 0{,}472rad(27{,}04^o), \ \ sin\alpha = 0{,}455, \ \ cos\alpha = 0{,}890, \ \ r = 14{,}06m,$$

and since the value of S is null, the tangential moment would be obtained as follows [12], [13]:

$$m_{\varphi} = 440 \cdot 14{,}06^2[(1 - cos\varphi - \varphi sin\varphi)0{,}890 - (sin\varphi - \varphi cos\varphi)0{,}455 + (\varphi - sin\varphi)4{,}09]$$

i.e.:

$$m_{\varphi} \sim 440 \cdot 14{,}06^2\left[-\frac{\varphi^2}{2}\left(1 - \frac{\varphi^2}{4}\right)0{,}890 - \frac{\varphi^3}{3}0{,}455 + \frac{\varphi^3}{6}4{,}09\right]$$

As in point *B'*, there is a value of $\varphi = 0{,}185$:

$$m_{B'} = -1020kg$$

Finally, for the section $\widehat{B''D}$ (Fig. 3):

$$t_D = 1{,}38\, t/m^2\,, \ \ P = 340\, kg/m^2\,, \ \ \frac{t}{P} = 4{,}06$$

$$\alpha = 0{,}775rad, \ \ sin\alpha = 0{,}70; \ \ \ cos\alpha = 0{,}715; \ \ \ r = 9{,}28m,$$

and since the value of S is null, the tangential moment would be obtained as follows [12]:

$$m_\varphi \sim 340 \cdot 9{,}28^2 \left[-\frac{\varphi^2}{2}\left(1 - \frac{\varphi^2}{4}\right) 0{,}715 - \frac{\varphi^3}{3} 0{,}70 + \frac{\varphi^3}{6} 4{,}06 \right],$$

and as in point *B''*, there is a value of $\varphi = 0{,}33$:

$$m_{B''} = -636kg$$

Thus,

$$m_{B'} + m_{B''} = -1656kg$$

In addition, when making the calculations, Johansen also takes into account the moments related to the sections $\widehat{CD}$ and $\widehat{DE}$, as follows:

In the section $\widehat{CD}$, the tangential force would be obtained from the expression:

$$t_c \cdot 1{,}0m = 1{,}8\, t/m$$

Since on the one hand, the weight of the section $\widehat{CD}$ is 0.34 t/m and that of the section *DE*, with an external load acting on it, is 0.44 t/m, the value of the moment at *B'B''* will therefore be:

$$(1{,}80\, t/m - 0{,}34\, t/m) \cdot 2{,}5m - 0{,}44\, t/m \cdot 3m = 2{,}33t = 2330kg$$

The resulting moment in the section *B'B''* in the cross-section of the shell will consequently be

$$m_{B'B''} = 2330kg - 1656kg \sim 680kg$$

In short, the transverse analysis of the shell is resolved by Johansen in an extremely simple manner, as is the longitudinal calculation, outlined as follows:

1. Calculate the expressions of the transverse forces and their location in the cross-section of the shell.

2. Calculate the transverse moments due to these tangential stresses on the outer load acting on the shell, and finally, those due to the reactions on the supports.

With all this information, the necessary reinforcement to the shell can be determined immediately.

Johansen concludes by referring to the use of the theory of plasticity, which by means of an appropriate selection of both the moment and the shear force, or shear, in the section *CD*, with a plastic approach to the equilibrium of forces, can obtain resultant moments of equal magnitude at B' and B'', similarly to the procedure with a beam.

For the purposes of buckling, Johansen compares the behaviour of the section AB with that of a cylinder subjected to a load of $P = 440\ kg/m^2$ and an axial compression equivalent to the compressive stress in that section, namely $43\ kg/m^2$. After making the relevant calculations, Johansen emphasises that the real situation of the shell is much more favourable than that of a cylinder, for two reasons. First, the section AB of the cross-section of the shell rests on pillars, and second, the remainder is a closed cross-section, where the resistance to buckling effects is considerably high. For these reasons, the shell is sufficiently safe against buckling problems.

As set out above, we have tried to show that while the theory of elasticity was unable to give even an approximate image of the structural behaviour of long cylindrical roof shells, the method described by Johansen is the most suitable for the structural calculation of these types, as it is based solely on considerations of equilibrium, and does not take into account those related to the shell's compatibility and deformation.

CONCLUSIONS

The structural calculation of long cylindrical roof shells originated in Germany in the early 1920s. Although engineers at that time based their calculations on the theory of elasticity, the most appropriate framework for the structural calculation of these types is breakage analysis. Although the "real" state of the structure could not be determined, its strength could be calculated accurately; it is also very insensitive to supposed defects in the manufacture or execution, and to small variations in the conditions of the surroundings.

The Danish engineer K.W. Johansen was the first to apply limit analysis with a balance approach to long cylindrical roof shells in 1944, following the publication of the fundamental theorems of plasticity in 1938, written by the Russian A.A. Gvozdev in 1936.

The plastic method developed by K.W. Johansen in 1944 is a simple and secure method of structural calculation for long cylindrical roof shells, since:

1. The beam method provides a solution of equilibrium which if the shell is made of a ductile material, and in the absence of instability problems, proves to be a safe solution, provided that the reinforced concrete's yield condition is satisfied.
2. The steady state in the shell is achieved by transferring stresses from the areas most subjected to those that are least. This all depends on the transverse geometry of the shell, the location of the neutral axis and the various provisions made for the reinforcement. The state of equilibrium thus obtained is therefore one solution to the problem, but not the only one. Any state of the structure in which the equilibrium of forces occurs can be studied, meaning that the calculating engineer could focus on studying the safety of the shell in each one.
3. When giving up the search for the "only" solution for the long cylindrical shell, the conclusion is that the essential aspect of limit analysis is the application of the "equilibrium approach," the main corollary of the Fundamental Theorem of Safety. This avoids the need to consider the shell's compatibility and deformation.

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Fig. 1. Modelling and geometrical definition of the cross-section of the long cylindrical shell the roof of the General Broadcasting Corporation building.

Fig. 2. Diagram of similarity between a reinforced concrete beam and a long cylindrical Shell.

Fig. 3. Representation of external loads and internal forces in the cross-section of the cylindrical shell for the positive moment and polygons of forces.

Fig. 4. Location of the tangential line of the arc.

Fig. 5. Representation of external loads and internal forces in the cross-section of the shell.

Fig. 6. Obtaining tangential moments.

Table 1. Parameter values depending on the value of the angle φ.