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Birational Transformations Preserving Rational Solutions of Algebraic Ordinary Differential Equations

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Abstract

We characterize the set of all rational transformations with the property of preserving the existence of rational solutions of algebraic ordinary differential equations (AODEs). This set is a group under composition and, by its action, partitions the set of AODEs into equivalence classes for which the existence of rational solutions is an invariant property. Moreover, we describe how the rational solutions, if any, of two different AODEs in the same class are related.

Keywords: algebraic differential equation; rational solution; integral birational transformation; integral curve; rational parametrization.

1 Introduction

This paper deals with questions in the symbolic analysis of algebraic differential equations; see e.g. Chapter 9 in [1], [9], [18], [20] for the symbolic treatment, [10] for a numerical treatment, and [17] for the foundations from differential algebra. More precisely, within this area, we describe an algebro-geometric treatment of algebraic ordinary differential equations (AODEs).

1.1 Algebro-geometric treatment of AODEs

Let $\mathbb{K}$ be an algebraically closed field of characteristic zero. Let $'$ be the uniquely defined derivation on $\mathbb{K}(x)$ such that $\ker(') = \mathbb{K}$ and $x' = 1$. An AODE of order n is an ordinary differential equations of the form (see Def. 3.1)

$$F(x, y, y', \ldots, y^{(n)}) = 0 \ , \tag{1}$$

where F is a polynomial with coefficients in $\mathbb{K}$; in practice, we often take $\mathbb{K}$ as the field of complex numbers $\mathbb{C}$. We will assume w.l.o.g. that the defining polynomial F is irreducible over $\mathbb{K}$. If this would not be the case, (1) would split in several AODEs.

Now, we associate to (1) an algebraic variety, namely the irreducible hypersurface $\mathcal{V}(F)$ defined by $F(u_1, \ldots, u_{n+2}) = 0$ in the $(n+2)$-dimensional affine space $\mathbb{K}^{n+2}$, where u_i are new variables. So, a first order AODE corresponds to a surface in $\mathbb{K}^3$, a second order AODE corresponds to a 3-dimensional variety in $\mathbb{K}^4$, etc. The defining polynomial F of an autonomous first order AODE depends only on two variable, namely $\{u_2, u_3\}$, and hence $\mathcal{V}(F)$ is a cylinder in $\mathbb{K}^3$. Therefore, in the case of autonomous first order AODEs we can consider that the associated hypersurface is a plane curve in $\mathbb{K}^2$. Analogously, an autonomous n-order AODE can be associated to a hypersurface in $\mathbb{K}^{n+1}$ instead of in $\mathbb{K}^{n+2}$. A similar treatment can be done with systems of AODEs (see [7], [11], [16]) and for algebraic partial differential equations (see [6]).

In this situation, the strategy consists in achieving information on the solutions of (1) from the algebraic and geometric properties of $\mathcal{V}(F)$. For instance, in [3], [4] the existence and actual computation of rational solutions of autonomous first order AODEs is studied by using rational parametrizations of the associated plane curve. In [5] this analysis is extended to the case of radical solutions by using radical parametrizations of the curve. In [15] the study in [3], [4] is generalized to the case of non-autonomous first order AODEs. For this purpose, the authors, using a rational parametrization of the associated surface, introduce a system of autonomous algebraic ODEs of order 1 and of degree 1, and the problem is treated by means of its invariant algebraic curves in [16]. Other approaches for first order AODEs can be found in [2], and in the chapter by G. Chen, Y. Ma in [20].

1.2 Birational transformations of AODEs and their applications

In the algebro-geometric approach one distinguishes two levels, the differential and the algebraic level, and the idea is to derive information on the differential equation from the associated variety. On the other hand, when working in algebraic geometry it is very usual to perform transformations that preserve the main properties and invariants of the variety, with the aim of reaching a simpler expression or a simpler geometric object from where the final conclusion might be easier to deduce. Typically, one uses birational transformations. Roughly speaking, a birational transformation is an invertible rational map, i.e. an invertible map defined by means of rational functions. Such a rational map may be undefined for special values. If the invertible rational map is between irreducible Zariski open sets, the inverse map is again a rational map. The simplest case of birational transformations are the affine transformations. But, one may have non-linear birational transformation. For instance, the quartic plane curve $-u_2^4 + 2u_1u_2^2 + u_2^3 - u_1^2 = 0$ is transformed, under the birational map

$$\phi : (u_1, u_2) \mapsto \left(\frac{-u_2^2 + u_1}{u_2^2}, \frac{1}{w} \right), \phi^{-1} : (u_1^*, u_2^*) \mapsto \left(\frac{u_1^* + 1}{u_2^{*2}}, \frac{1}{u_2^*} \right),$$

into the parabola $u_2^* = u_1^{*2}$. This implies, for instance, that $\phi^{-1}(t, t^2) = ((t+1)/t^4, 1/t^2)$ is a parametrization of the quartic.

However, birational transformations, although preserving many important algebraic and geometric properties, in general do not preserve the differential properties. For instance, let us consider the homogeneous linear differential equation $y' = 0$. Its associated hypersurface is the plane Π_1 of equation $u_3 = 0$ in $\mathbb{K}^3$. If we consider the affine transformation $\{u_1{}^* = u_1, u_2{}^* = u_2, u_3{}^* = u_3 + u_2\}$, the plane Π_1 is transformed into the plane Π_2 of equation $u_3{}^* - u_2{}^* = 0$, corresponding to the linear differential equation $y' = y$. Nevertheless, while all solutions of the first equation are rational (indeed constant), the second equation does not have any rational solution.

So the natural question, and indeed our problem in this paper, is: what type of birational transformations, if any, on AODEs does preserve the existence of rational solutions? In [13], we consider the simpler case of affine transformations, and we characterize the set of all affine transformations that preserve the rationality of the solutions of first order AODEs. More precisely, these affine transformations are of the type $\{u_1{}^* = u_1, u_2{}^* = \alpha u_2 + \beta u_1 + \gamma, u_3{}^* = \alpha u_3 + \beta\}$, where $\alpha, \beta, \gamma \in \mathbb{K}$, $\alpha \neq 0$, and they form a group under composition. Note that the transformation in the previous paragraph is not of this type and that, applying one of these transformations to Π_1, one gets $u_3{}^* = \beta$ corresponding to $y' = \beta$, all solutions of which are rational.

With the same type of strategy, alternative questions have been treated by other authors analyzing the equivalence, for first order algebraic differential equations, under different criteria such as the preservation of the Painlevé property (see [12]).

Having a description of the birational transformations, preserving rational solutions of AODEs, reinforces the applicability of the existing solving methods. For instance, in Example 4.1 we see how a non-autonomous AODE can be transformed into an autonomous one, and hence algorithms in [3], [4] can be applied to a wider class of equations; see Section 4.

1.3 Main contributions of the paper

In general terms, the main contribution of the paper is the establishment of an equivalence relation in the set of all AODEs such that in any equivalence class the rational solvability is preserved. This leads to a new theoretical and algorithmic framework. More precisely, the main contributions of the paper can be summarized as follows:

- we characterize the set of all birational transformations that, when applied to an AODE of arbitrary order n, preserve the rationality of the solutions (see Theorems 2.1, 2.2). We call them integral birational transformations (see Definition 2.2).

- We prove that this set forms a group (see Proposition 2.1). Moreover, we consider the action of this group on the set of all AODEs of order n, and we prove that the existence of rational solutions is an invariant property for all the differential equations in a given equivalence class (see Theorem 3.1).
- Furthermore, we show how the solutions of two different differential equations, in the same class, are related (see Theorem 3.1). Analogous reasonings can be used for studying invariance of larger classes of solution functions.

1.4 Structure of the paper

The structure of the paper is as follows. In Section 2 we define the notion of a rational integral curve, i.e. a curve having a parametrization of the form $(x, f(x), f'(x), ..., f^{(n)}(x))$, and we characterize the class of birational transformations of $(n+2)$-dimensional mapping rational integral curves into rational integral curves. We call them integral birational transformations. These transformations form a group. In Section 3 we investigate how these transformations act on the set of AODEs. In fact, the orbits w.r.t. to this transformation group form equivalence classes and the property of having regular rational solutions is invariant for elements of a given equivalence class. Also the singular solutions are preserved. In Section 4, we illustrate the potential applicability of the results in the paper. We finish the paper with conclusions and an indication of some open problems.

1.5 Basic notation

Throughout this paper, we use the following notation. $\mathbb{K}$ is an algebraically closed field of characteristic zero, and $'$ is the uniquely defined derivation on $\mathbb{K}(x)$ such that $\ker(') = \mathbb{K}$ and $x' = 1$. y is an indeterminate over a differential extension field of $\mathbb{K}(x)$. The differential equations will depend on $\{x, y, y', \ldots, y^{(n)}\}$. The affine coordinates in $\mathbb{K}^{n+2}$ are denoted by $(u_1, \ldots, u_{n+2})$. So when passing from the differential level to the algebraic level we will replace x by u_1, y by u_2, and $y^{(i)}$ by u_{2+i}.

In addition, let a rational function $f(\overline{u}) = f_1(\overline{u})/f_2(\overline{u})$ in the (set of) variables $\overline{u}$ be represented by coprime polynomials f_1, f_2. Then we call f_1 the numerator of f, $f_1 = \text{numer}(f)$. Observe that the numerator is determined up to constant multiples.

If f is a non-zero polynomial in the variable x, then by $\text{LC}_x(f)$ we denote the leading coefficient of f w.r.t. x, i.e., the coefficient of $x^{\deg_x(f)}$.

Moreover, when $f(\overline{u}, \overline{v})$ is a polynomial in the (sets of) variables $\overline{u}$ and $\overline{v}$, then by the content of f w.r.t. $\overline{v}$, $\text{cont}_{\overline{v}}(f)$, we denote the greatest common divisor of the coefficients of f w.r.t. $\overline{v}$. By the primitive part of f w.r.t. $\overline{v}$ we mean $\text{pp}_{\overline{v}}(f) = f/\text{cont}_{\overline{v}}(f)$.

For a rational function $f(u_1, \ldots, u_r)$ we denote its derivative w.r.t. u_j by f_j; in particular, for the rational functions ϕ_i appearing in Theorems 2.1 and 2.2 we denote their derivative w.r.t. the j-th variable by $\phi_{i,j}$.

For a polynomial $F(\overline{u})$ we denote by $\mathcal{V}(F)$ the hypersurface defined by F over $\mathbb{K}$.

2 Integral birational transformations

For any rational function $f(x)$ and positive $n \in \mathbb{N}$ the set

$$\{(x, f(x), f'(x), \ldots, f^{(n)}(x)) \mid x \in \mathbb{K}\}$$

is a rational (or parametric) curve in the affine space $\mathbb{K}^{n+2}$. For instance, $f(x) = 5$, and $n = 1$, generates the line, in $\mathbb{C}^3$, parametrized as $(x, 5, 0)$. This motivates the following definition.

Definition 2.1. For a given rational function $f(x)$ and a positive integer n, the parametric space curve

$$\mathcal{C}_f^{(n)} := \{(x, f(x), f'(x), \ldots, f^{(n)}(x)) \mid x \in \mathbb{K}\} \ \subset \mathbb{K}^{n+2}$$

is called a *rational integral curve (RIC) of order* n over $\mathbb{K}$ (cf. [13], p. 198). f is the *defining rational function* of $\mathcal{C}_f^{(n)}$. By

$$\mathcal{RIC}^{(n)} := \{\mathcal{C}_f^{(n)} \mid f \in \mathbb{K}(x)\}$$

we denote the *set of rational integral curves of order* n over $\mathbb{K}$. •

So, the line $\{u_2 = 5, u_3 = 0\}$ is the rational integral curve $\mathcal{C}_5^{(1)} \in \mathcal{RIC}^{(1)}$. A rational integral curve is a set of points in $\mathbb{K}^{n+2}$; but sometimes we find it convenient to consider it as a rational mapping from $\mathbb{K}$ to $\mathbb{K}^{n+2}$:

$$\begin{array}{rccc} \mathcal{C}_f^{(n)}: & \mathbb{K} & \longrightarrow & \mathbb{K}^{n+2} \\ & x & \mapsto & (x, f(x), f'(x), \ldots, f^{(n)}(x)) \end{array}.$$

In this context we would like to describe the birational maps of $\mathbb{K}^{n+2}$ sending rational integral curves into rational integral curves. Let us see an example.

Example 2.1. We consider the rational integral curve

$$\mathcal{C}^2_{\frac{1}{x}} = \left\{ \left(x, \frac{1}{x}, -\frac{1}{x^2}, \frac{2}{x^3} \right) \mid x \in \mathbb{C} \right\}$$

and the birational map (compare to Theorem 2.2)

$$\begin{array}{rccc} \Phi: & \mathbb{C}^4 & \longrightarrow & \mathbb{C}^4 \\ & (u_1, u_2, u_3, u_4) & \mapsto & \left(u_1, \frac{1}{u_2+u_1}, -\frac{1+u_3}{(u_2+u_1)^2}, -\frac{u_1u_4+u_2u_4-2\,u_3{}^2-4\,u_3-2}{(u_2+u_1)^3} \right). \end{array}$$

We observe that Φ maps $\mathcal{C}^2_{\frac{1}{x}}$ into $\mathcal{C}^2_{\frac{x}{x^2+1}}$. However, if you consider the birational map

$$\begin{array}{rccc} \Phi^*: & \mathbb{C}^4 & \longrightarrow & \mathbb{C}^4 \\ & (u_1, u_2, u_3, u_4) & \mapsto & (u_1, u_1 + u_2, u_3, u_4). \end{array}$$

$\mathcal{C}^2_{\frac{1}{x}}$ is mapped into the curve

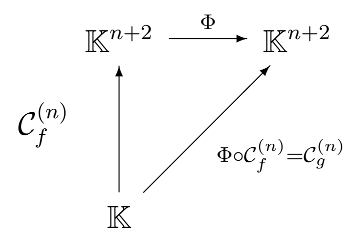
$$\left\{ \left(x, \frac{x^2+1}{x}, -\frac{1}{x^2}, \frac{2}{x^3} \right) \mid x \in \mathbb{C} \right\}$$

which is not a rational integral curve, because $(\frac{x^2+1}{x})' \neq -\frac{1}{x^2}$ •

Our goal in this section is to solve the following problem (see Remark 2.1 (1)):

> *Problem: find all birational transformations* $\Phi : \mathbb{K}^{n+2} \to \mathbb{K}^{n+2}$ *such that the induced map* $\Phi^e : \mathcal{C}_f^{(n)} \mapsto \Phi \circ \mathcal{C}_f^{(n)}$ *is actually a map from a non-empty subset of* $\mathcal{RIC}^{(n)}$ *to* $\mathcal{RIC}^{(n)}$*, i.e., for those* $f \in \mathbb{K}(x)$ *such that* $\Phi^e(\mathcal{C}_f^{(n)}) = \Phi(x, f(x), f'(x), \ldots, f^{(n)}(x))$ *is well defined, there exists a unique rational function* $g \in \mathbb{K}(x)$ *such that*
> $$\Phi^e(\mathcal{C}_f^{(n)}) = \mathcal{C}_g^{(n)}. \tag{2}$$

The following diagram describes the situation in (2):



For a given birational transformation Φ on $\mathbb{K}^{n+2}$, we denote by $\mathcal{RIC}_\Phi^{(n)}$ the subset of $\mathcal{RIC}^{(n)}$ where Φ^e is defined. For instance, if Φ and Φ^* are as in Example 2.1, then $\mathcal{RIC}_\Phi^{(2)} = \mathcal{RIC}^{(2)} \setminus \{\mathcal{C}_{-x}^{(2)}\}$ and $\mathcal{RIC}_{\Phi^*}^{(2)} = \mathcal{RIC}^{(2)}$. In Lemma 2.1 we prove that $\mathcal{RIC}_\Phi^{(n)}$ is never empty.

Lemma 2.1. *Let* $\Phi : \mathbb{K}^{n+2} \to \mathbb{K}^{n+2}$ *be a birational map, then* $\mathcal{RIC}_\Phi^{(n)} \neq \emptyset$*. Furthermore, for almost all polynomials* $f(x)$ *of degree* n*,* $\mathcal{C}_f^{(n)} \in \mathcal{RIC}_\Phi^{(n)}$*.*

Proof. Let $G(u_1, \ldots, u_{n+2})$ be the lcm of all denominators in Φ. Let

$$f = \lambda_2 + \lambda_3(x - \lambda_1) + \cdots + \frac{\lambda_{n+2}}{n!}(x - \lambda_1)^n$$

where λ_i are undetermined coefficients. First we see that $G(\mathcal{C}_f^{(n)}) \neq 0$. Indeed, if $G(\mathcal{C}_f^{(n)}) = G(x, f(x), \ldots, f^{(n)}(x)) = 0$, then $G(\lambda_1, f(\lambda_1), \ldots, f^{(n)}(\lambda_1)) = G(\lambda_1, \lambda_2, \ldots, \lambda_{n+2}) = 0$ which is a contradiction because $G \neq 0$ and $(\lambda_1, \lambda_2, \ldots, \lambda_{n+2})$ is a generic point in $\mathbb{K}^{n+2}$. Now, the set of coefficients of $G(x, f(x), \ldots, f^{(n)}(x))$ w.r.t. x defines an algebraic variety V strictly included in $\mathbb{K}^{n+2}$ such that for $(\lambda_1, \lambda_2, \ldots, \lambda_{n+2}) \in \mathbb{K}^{n+2} \setminus V$, the corresponding polynomial f generates a curve $\mathcal{C}_f^{(n)} \in \mathcal{RIC}_\Phi^{(n)}$. $\square$

In the next theorem, we solve the problem stated in (2) for the case of first order, i.e. for $n = 1$. The basic idea of the proof is to find necessary conditions by checking the map on lines and parabolas to afterwards prove that these conditions are also sufficient.

Theorem 2.1. [First order case] *Let* $\Phi : \mathbb{K}^3 \to \mathbb{K}^3$ *be a birational map. If the map* $\Phi^e : \mathcal{C}_f^{(1)} \mapsto \Phi \circ \mathcal{C}_f^{(1)}$ *defines a map from* $\mathcal{RIC}_\Phi^{(1)}$ *to* $\mathcal{RIC}^{(1)}$*, then* Φ *must be of the form*

$$\Phi(u_1, u_2, u_3) = \left(u_1, \quad \frac{au_2 + b}{cu_2 + d}, \quad \frac{\partial}{\partial u_1}\left(\frac{au_2 + b}{cu_2 + d}\right) + \frac{\partial}{\partial u_2}\left(\frac{au_2 + b}{cu_2 + d}\right) \cdot u_3\right), \tag{3}$$

where $a,b,c,d\in\mathbb{K}[u_1]$ *such that* $ad-bc\neq 0$.
Conversely, any map of the form (3) is birational and the induced map $\Phi^e:\mathcal{C}_f^{(1)}\mapsto\Phi\circ\mathcal{C}_f^{(1)}$ *defines a map from* $\mathcal{RIC}_\Phi^{(1)}$ *to* $\mathcal{RIC}^{(1)}$.

Proof. Throughout this proof, for ease of notation we will simply write $\mathcal{C}_f$ instead of $\mathcal{C}_f^{(1)}$, and $\mathcal{RIC}$ instead of $\mathcal{RIC}^{(1)}$.
Let $\Phi(\overline{u})=(\phi_1(\overline{u}),\phi_2(\overline{u}),\phi_3(\overline{u}))$, where $\overline{u}=(u_1,u_2,u_3)$, be a birational map on $\mathbb{K}^3$. Suppose that $\phi_1(\overline{u})=\frac{F_1(\overline{u})}{G_1(\overline{u})}$, where F_1 and G_1 are coprime polynomials in $\mathbb{K}[\overline{u}]$. F_1 and G_1 have to be such that

$$x=\frac{F_1(\mathcal{C}_f)}{G_1(\mathcal{C}_f)}$$

for $f\in\mathbb{K}(x)$ such that $\mathcal{C}_f\in\mathcal{RIC}_\Phi$. Let us consider the polynomial

$$P(\overline{u}):=u_1G_1(\overline{u})-F_1(\overline{u}).$$

We know that for f as above $P(x,f(x),f'(x))=0$. We prove that $P(\overline{u})=0$. Indeed, let $\overline{\lambda}=(\lambda_1,\lambda_2,\lambda_3)\in\mathbb{K}^3$ and consider the line $l(x)=\lambda_2+\lambda_3(x-\lambda_1)$. Then, taking into account Lemma 2.1, for $\overline{\lambda}$ in a non-empty Zariski open subset of $\mathbb{K}^3$, $\mathcal{C}_{l(x)}\in\mathcal{RIC}_\Phi$, and so

$$P(\lambda_1,\lambda_2,\lambda_3)=P(\lambda_1,l(\lambda_1),l'(\lambda_1))=0.$$

Hence, $P(\overline{u})=0$, i.e., $\phi_1(\overline{u})=u_1$.

Now we need to determine the form of $\phi_2(\overline{u})$ and $\phi_3(\overline{u})$ such that $\phi_2(\mathcal{C}_f)'=\phi_3(\mathcal{C}_f)$ for $f\in\mathbb{K}(x)$ such that $\mathcal{C}_f\in\mathcal{RIC}_\Phi$. This is equivalent to

$$\phi_{21}(\mathcal{C}_f)+\phi_{22}(\mathcal{C}_f)f'(x)+\phi_{23}(\mathcal{C}_f)f''(x)=\phi_3(\mathcal{C}_f)\qquad\text{for } f\in\mathbb{K}(x)\text{ such that }\mathcal{C}_f\in\mathcal{RIC}_\Phi.$$

Note that the square-free parts of the denominators of ϕ_{ij} and ϕ_i are equal (here we use the notation introduced just before the Theorem). Let Q be the numerator of

$$\phi_{21}(\overline{u})+\phi_{22}(\overline{u})u_3+\phi_{23}(\overline{u})u_4-\phi_3(\overline{u})$$

We know that for $f\in\mathbb{K}(x)$ such that $\mathcal{C}_f\in\mathcal{RIC}_\Phi$ we have $Q(x,f(x),f'(x),f''(x))=0$. Let $\overline{\lambda}=(\lambda_1,\dots,\lambda_4)\in\mathbb{K}^4$ and consider the parabola $p(x)=\lambda_2+\lambda_3(x-\lambda_1)+\dfrac{\lambda_4}{2}(x-\lambda_1)^2$. Then, for $\overline{\lambda}$ in a non-empty Zariski open subset of $\mathbb{K}^4$, $\mathcal{C}_{p(x)}\in\mathcal{RIC}_\Phi$, and so

$$0=Q(\lambda_1,p(\lambda_1),p'(\lambda_1),p''(\lambda_1))=Q(\lambda_1,\lambda_2,\lambda_3,\lambda_4).$$

Therefore, $Q(u_1,\dots,u_4)=0$. This implies that

$$\phi_3(\overline{u})=\phi_{21}(\overline{u})+\phi_{22}(\overline{u})u_3+\phi_{23}(\overline{u})u_4.$$

Since the left hand side does not contain u_4, $\phi_{23}(\overline{u})$ must be 0. Consequently, $\phi_2(u_1,u_2,u_3)$ does not depend on the third variable. Hence, Φ has the form

$$\Phi(u_1,u_2,u_3)=(u_1,\ \phi_2(u_1,u_2),\ \phi_{21}(u_1,u_2)+\phi_{22}(u_1,u_2)\cdot u_3).$$

Now, it is enough to determine ϕ_2. We observe that ϕ_{22} is not zero, since otherwise Φ would not depend on u_3 and hence could not be birational. Let $\overline{\lambda} = (\lambda_1, \lambda_2, \lambda_3)$ be a generic point in an open subset of $\mathbb{K}^3$. Consider the system of equations

$$\begin{cases} u_1 = \lambda_1 \\ \phi_2(u_1, u_2) = \lambda_2 \\ \phi_{21}(u_1, u_2) + \phi_{22}(u_1, u_2)u_3 = \lambda_3. \end{cases} \tag{4}$$

Let $\phi_2 = \dfrac{F_2}{G_2}$ be in reduced form. Note that $F_2 \neq 0$, because Φ is birational. Consider

$$A(u_2) = F_2(\lambda_1, u_2) - \lambda_2 G_2(\lambda_1, u_2).$$

First we observe that A is of positive degree in u_2. This follows from the fact that $\dfrac{F_2}{G_2}$ does depend on u_2 (because $\phi_{22} \neq 0$). Indeed, $\deg_{u_2}(A) = 1$. If A has two different roots, the fiber will have at least 2 elements. Moreover, let us see that A does not have multiple roots for a generic $\overline{\lambda}$. The polynomial $H(u_1, u_2, w) = F_2(u_1, u_2) - wG_2(u_1, u_2)$ cannot have a multiple factor, since $\deg_w(H) = 1$ (G_2 cannot be 0) and $\gcd(F_2, G_2) = 1$. So $R(u_1, w) = \text{discriminant}_{u_2}(H) \neq 0$. If we take λ_1, λ_2 such that $R(\lambda_1, \lambda_2) \neq 0$, then A will not have multiple roots. This proves that indeed $\deg_{u_2}(A) = 1$. Therefore, $\phi_2(u_1, u_2)$ is of the form

$$\phi_2(u_1, u_2) = \frac{a(u_1)u_2 + b(u_1)}{c(u_1)u_2 + d(u_1)},$$

with $ad - bc \neq 0$.

Conversely, any map of form (3) is birational because its inverse is

$$\Phi^{-1}(u_1, u_2, u_3) = \left(u_1, \ \frac{du_2 - b}{-cu_2 + a}, \ \frac{\partial}{\partial u_2}\left(\frac{du_2 - b}{-cu_2 + a}\right) u_3 + \frac{\partial}{\partial u_1}\left(\frac{du_2 - b}{-cu_2 + a}\right)\right).$$

Moreover, if $\mathcal{C}_f = (x, f(x), f'(x))$, then $\Phi(\mathcal{C}_f) = (x, g(x), g'(x))$, where $g(x) = \dfrac{a(x)f(x) + b(x)}{c(x)f(x) + d(x)}$; of course with the exceptional case in which f is a root of the denominator; namely $f = -d(x)/c(x)$. □

Now we generalize Theorem 2.1 to rational maps on higher dimensional spaces; i.e. we solve the problem in (6) for the general case. The proof is by induction on the order and uses Theorem 2.1 as induction basis.

Theorem 2.2. [General order case] *Let $\Phi = (\phi_1, \ldots, \phi_{n+2}) : \mathbb{K}^{n+2} \to \mathbb{K}^{n+2}$ be a birational map, where $n > 1$.*
If the map $\Phi^e : \mathcal{C}_f^{(n)} \mapsto \Phi \circ \mathcal{C}_f^{(n)}$ defines a map from $\mathcal{RIC}_\Phi^{(n)}$ to $\mathcal{RIC}^{(n)}$, then $\hat{\Phi} = (\phi_1, \ldots, \phi_{n+1}) : \mathbb{K}^{n+1} \to \mathbb{K}^{n+1}$ is birational, $\hat{\Phi}^e$ maps $\mathcal{RIC}_{\hat{\Phi}}^{(n-1)}$ to $\mathcal{RIC}^{(n-1)}$, and

$$\phi_{n+2}(\overline{u}, u_{n+2}) \ = \ \phi_{n+1,1}(\overline{u}) + \sum_{i=2}^{n} \phi_{n+1,i}(\overline{u}) \cdot u_{i+1} + \phi_{n+1,n+1}(\overline{u}) \cdot u_{n+2} \ , \tag{5}$$

where $\overline{u} = (u_1, \ldots, u_{n+1})$.
So Φ *is triangular in the sense that its* i*-th component depends only on the first* i *variables, and for* $i \geq 3$ *the variable* u_i *is introduced linearly in* ϕ_i.
Conversely, any map of the form (5) *extending a birational map* $\hat{\Phi}$ *from* $\mathbb{K}^{n+1}$ *to* $\mathbb{K}^{n+1}$*, such that* $\hat{\Phi}^e$ *maps* $\mathcal{RIC}_{\hat{\Phi}}^{(n-1)}$ *to* $\mathcal{RIC}^{(n-1)}$*, is birational and the induced map* $\Phi^e : \mathcal{C}_f^{(n)} \mapsto \Phi \circ \mathcal{C}_f^{(n)}$ *defines a map from* $\mathcal{RIC}_{\Phi}^{(n)}$ *to* $\mathcal{RIC}^{(n)}$.

Proof. If $\hat{\Phi}$ would not be birational, clearly Φ could not be birational. And if $\hat{\Phi}$ would not map $\mathcal{RIC}_{\hat{\Phi}}^{(n-1)}$ to $\mathcal{RIC}^{(n-1)}$, then Φ could not map $\mathcal{RIC}_{\Phi}^{(n)}$ to $\mathcal{RIC}^{(n)}$.
We must have $\phi_{n+1}(\mathcal{C}_f^{(n)})' = \phi_{n+2}(\mathcal{C}_f^{(n)})$ for $f \in \mathbb{K}(x)$ such that $\mathcal{C}_f \in \mathcal{RIC}_{\Phi}^{(n)}$. This is equivalent to

$$\phi_{n+1,1}(\mathcal{C}_f^{(n)}) + \sum_{i=2}^{n+2} \phi_{n+1,i}(\mathcal{C}_f^{(n)}) \cdot f^{(i-1)} = \phi_{n+2}(\mathcal{C}_f^{(n)}) \quad \text{for } f \in \mathbb{K}(x) \text{ such that } \mathcal{C}_f \in \mathcal{RIC}_{\Phi}^{(n)}.$$

Let Q be the numerator of

$$\phi_{n+1,1}(\overline{u}) + \sum_{i=2}^{n+2} \phi_{n+1,i}(\overline{u}) \cdot u_{i+1} - \phi_{n+2}(\overline{u}, u_{n+2})$$

As in the proof on Theorem 2.1 we see that $Q(u_1, \ldots, u_{n+3}) = 0$. This implies that

$$\phi_{n+2}(\overline{u}, u_{n+2}) = \phi_{n+1,1}(\overline{u}) + \sum_{i=2}^{n+2} \phi_{n+1,i}(\overline{u}) \cdot u_{i+1}.$$

But ϕ_{n+1} does not depend on u_{n+2}, so $\phi_{n+1,n+2} = 0$. Consequently, ϕ_{n+2} is of the form (5).

Applying the process of shortening the map several times, until finally we arrive at a map on $\mathbb{K}^3$ which is covered by Theorem 2.1, we see that Φ is indeed triangular and the variables $u_3, \ldots, u_{n+2}$ are introduced linearly in the corresponding components of the map.

Conversely, any birational map $\hat{\Phi}$, such that $\hat{\Phi}^e$ maps $\mathcal{RIC}_{\hat{\Phi}^e}^{(n-1)}$ to $\mathcal{RIC}^{(n-1)}$, must be such that $\phi_{n+1,n+1} \neq 0$, since it is triangular and could not be birational otherwise. So the extension Φ of the form (5) is also triangular and birational. And Φ^e maps $\mathcal{RIC}_{\Phi}^{(n)}$ to $\mathcal{RIC}^{(n)}$.
In fact, let us determine the inverse of Φ. Φ^{-1} is also birational and $(\Phi^{-1})^e$ maps $\mathcal{RIC}_{(\Phi^{-1})^e}^{(n)}$ to $\mathcal{RIC}^{(n)}$. So it must also be of the shape (5). That means, it must be generated by a linear function in the 2nd component. If Φ is generated by the linear function

$$L(u_1, u_2) = \frac{a(u_1)u_2 + b(u_1)}{c(u_1)u_2 + d(u_1)}$$

(compare Theorem 2.1), then Φ^{-1} must be generated by the linear function

$$L^{-1}(u_1, u_2) = \frac{d(u_1)u_2 - b(u_1)}{-c(u_1)u_2 + a(u_1)} \; .$$

□

Theorems 2.1 and 2.2 motivate the next definition.

Definition 2.2. An *integral birational transformation of order* n is a rational transformation of the form

$$\Phi(u_1,\dots,u_{n+2}) \;=\; \big(u_1,\phi_2(u_1,u_2),\dots,\phi_{n+2}(u_1,\dots,u_{n+2})\big),$$

where $\phi_2(u_1,u_2)=L(u_1,u_2)$ is an invertible linear function in $\mathbb{K}[u_1](u_2)$, i.e.,

$$L(u_1,u_2)=\frac{a(u_1)u_2+b(u_1)}{c(u_1)u_2+d(u_1)},\quad \text{with}\quad a,b,c,b\in\mathbb{K}[u_1] \text{ and } ad-bc\neq 0,$$

and ϕ_r is derived from ϕ_{r-1} as in (3) and (5), for $3\le r\le n+2$.
We call L the *defining function* of this integral birational transformation Φ, and we write $\Phi=\Phi_L$.
We call $\Delta=a\cdot d-b\cdot c\in\mathbb{K}[u_1]$ the *determinant* of Φ, and we write $\Delta=\Delta_\Phi$.
By

$$\mathcal{G}^{(n)} := \{\Phi_L \mid L\in\mathbb{K}[u_1](u_2) \text{ linear in } u_2 \text{ and invertible }\} \tag{6}$$

we denote the *set of integral birational transformations of order* n. •

With this new notation, we have that Φ in Example 2.1 is $\Phi_{1/(u_1+u_2)}\in\mathcal{G}^{(2)}$.

Remark 2.1. We observe the following.

1. Observe that $\mathcal{G}^{(n)}$ is the answer to the problem stated above.
2. From the proof of Theorem 2.1 and Theorem 2.2 we see that if the integral birational transformation Φ is defined by the function $L=\frac{a\cdot u_2+b}{c\cdot u_2+d}$, then its inverse Φ^{-1} is defined by $\frac{d\cdot u_2-b}{-c\cdot u_2+a}$.
3. If $\Phi\in\mathcal{G}^{(n)}$ is generated by $\frac{a\cdot u_2+b}{c\cdot u_2+d}$, then $\mathcal{RIC}^{(n)}_\Phi=\mathcal{RIC}^{(n)}$ if $c=0$, and $\mathcal{RIC}^{(n)}_\Phi=\mathcal{RIC}^{(n)}\setminus\{\mathcal{C}^{(n)}_{-d/c}\}$ if $c\neq 0$.
4. This type of reasoning in the proofs of the previous results may also be applied to differentiable functions instead of rational functions, in case $\mathbb{K}$ is a differential field; i.e., $\mathcal{RIC}$ may be replaced by the set

   $$\{(x,f(x),f'(x),\dots,f^{(n)}(x)) \mid f \text{ is a differentiable function in } x\}.$$

   For instance, if you consider the integral curve

   $$\mathcal{C}^2_{\mathrm{e}^x\sin(x)}=\{(x,\mathrm{e}^x\sin(x),\mathrm{e}^x\sin(x)+\mathrm{e}^x\cos(x),2\,\mathrm{e}^x\cos(x))\,|\,x\in\mathbb{C}\}$$

   and Φ is the map in Example 2.1, then $\mathcal{C}^2_{\mathrm{e}^x\sin(x)}$ is transformed into the integral curve $\mathcal{C}^2_{\frac{1}{\mathrm{e}^x\sin(x)+x}}$.

In the following, we see that $\mathcal{G}^{(n)}$ is a group, under composition, and we study some of its subgroups.

Proposition 2.1. *For a positive integer n, the set of integral birational transformations of order n, $\mathcal{G}^{(n)}$, (cf. (6)) is a subgroup (under composition) of the group of birational transformations of the space $\mathbb{K}^{n+2}$.*

Proof. Since the identity map belongs to $\mathcal{G}^{(n)}$, we have $\mathcal{G}^{(n)} \neq \emptyset$. As we have seen above (Remark 2.1), with Φ the set $\mathcal{G}^{(n)}$ also contains Φ^{-1}.
Let Φ_1, Φ_2 be in $\mathcal{G}^{(n)}$ and let $L_1(u_1, u_2), L_2(v_1, v_2)$ be their defining functions, respectively. Then the composition $\Phi_1 \circ \Phi_2$ is defined by the linear function $L_1(v_1, L_2(v_1, v_2))$, with the determinant being the product of the determinants of L_1 and L_2.
So $\mathcal{G}^{(n)}$ is a group under composition. □

Definition 2.3. We call $\mathcal{G}^{(n)}$ the *group of integral birational transformations of order n.* •

Remark 2.2. In our previous paper [13], we have studied the affine case for order 1, i.e., the case in which $\Phi(u_1, u_2, u_3) := \Phi_{\alpha u_2 + \beta u_1 + \gamma} = (u_1, \alpha u_2 + \beta u_1 + \gamma, \alpha u_3 + \beta)$, where $\alpha, \beta, \gamma \in \mathbb{K}$, $\alpha \neq 0$. The set of all such affine transformations forms a subgroup of the group $\mathcal{G}^{(1)}$ of integral birational transformations on $\mathbb{K}^3$. Now, using Theorem 2.2, the linear subgroup in [13] can be generalized to the n-order case. This leads to the subgroup of $\mathcal{G}^{(n)}$ $(n > 3)$ formed by all integral transformations of $\mathbb{K}^{n+2}$ of the form $\Phi(u_1, \ldots, u_{n+2}) := \Phi_{\alpha u_2 + \beta u_1 + \gamma} = (u_1, \alpha u_2 + \beta u_1 + \gamma, \alpha u_3 + \beta, \alpha u_4, \ldots, \alpha u_{n+2})$; note that $\Delta_\Phi = \alpha$.

Besides the linear subgroup mentioned in Remark 2.2, we introduce three additional subgroups. Consider the following subsets of $\mathcal{G}^{(n)}$:

$$\mathcal{G}^{(n)}_{inv} = \left\{\Phi_{u_2}, \Phi_{1/u_2}\right\}, \mathcal{G}^{(n)}_{mult} = \left\{\Phi_{a(u_1)u_2} \mid a \in \mathbb{K}(u_1), a \neq 0\right\}, \mathcal{G}^{(n)}_{plus} = \left\{\Phi_{u_2 + b(u_1)} \mid b \in \mathbb{K}(u_1)\right\}.$$

Note that $\mathcal{G}^{(n)}_{inv}$ only consists of two elements, while $\mathcal{G}^{(n)}_{mult}$ and $\mathcal{G}^{(n)}_{plus}$ are infinite sets.

Proposition 2.2. *$\mathcal{G}^{(n)}_{inv}$, $\mathcal{G}^{(n)}_{mult}$ and $\mathcal{G}^{(n)}_{plus}$ are subgroups of $\mathcal{G}^{(n)}$.*

Proof. The identity transformation is in $\mathcal{G}^{(n)}_{inv}$, $\mathcal{G}^{(n)}_{mult}$ and $\mathcal{G}^{(n)}_{plus}$.
Clearly, $\mathcal{G}^{(n)}_{inv}$ is a subgroup of $\mathcal{G}^{(n)}$, since the non-identity element in $\mathcal{G}^{(n)}_{inv}$ is its own inverse.
Let $\Phi_1, \Phi_2 \in \mathcal{G}^{(n)}_{mult}$, with defining functions $a_1(u_1)u_2$ and $a_2(u_1)u_2$, respectively. Then $\Phi_1 \circ \Phi_2^{-1}$ is in $\mathcal{G}^{(n)}_{mult}$ with defining function $\frac{a_1(u_1)}{a_2(u_1)} u_2$.
Let $\Phi_1, \Phi_2 \in \mathcal{G}^{(n)}_{plus}$, with defining functions $u_2 + b_1(u_1)$ and $u_2 + b_2(u_1)$, respectively. Then $\Phi_1 \circ \Phi_2^{-1}$ is in $\mathcal{G}^{(n)}_{plus}$ with defining function $u_2 + b_1(u_1) - b_2(u_1)$.
Therefore, $\mathcal{G}^{(n)}_{inv}$, $\mathcal{G}^{(n)}_{mult}$ and $\mathcal{G}^{(n)}_{plus}$ are subgroups of $\mathcal{G}^{(n)}$. □

Proposition 2.3. *Every $\Phi \in \mathcal{G}^{(n)}$ can be decomposed into a product of elements in $\mathcal{G}^{(n)}_{inv}$, $\mathcal{G}^{(n)}_{mult}$ and $\mathcal{G}^{(n)}_{plus}$.*

Proof. Let Φ be an arbitrary element in $\mathcal{G}^{(n)}$, defined by the function

$$L(u_1, u_2) = \frac{a(u_1)u_2 + b(u_1)}{c(u_1)u_2 + d(u_1)}.$$

1. If $c = 0$, then we can write
$$L(u_1, u_2) = \frac{a}{d}u_2 + \frac{b}{d} \ .$$
So $\Phi = \Phi_2 \circ \Phi_1$, where $\Phi_1 \in \mathcal{G}^{(n)}_{mult}$ is defined by $\frac{a}{d}u_2$, and $\Phi_2 \in \mathcal{G}^{(n)}_{plus}$ is defined by $u_2 + \frac{b}{d}$.
2. If $c \neq 0$, then we can write
$$L(u_1, u_2) = \frac{a}{c} + \frac{bc - ad}{c^2\left(u_2 + \frac{d}{c}\right)} \ .$$
So $\Phi = \Phi_4 \circ \Phi_3 \circ \Phi_2 \circ \Phi_1$, where $\Phi_1 \in \mathcal{G}^{(n)}_{plus}$ is defined by $u_2 + \frac{d}{c}$, $\Phi_2 \in \mathcal{G}^{(n)}_{inv}$ is defined by $\frac{1}{u_2}$, $\Phi_3 \in \mathcal{G}^{(n)}_{mult}$ is defined by $\frac{bc-ad}{c^2}u_2$, and $\Phi_4 \in \mathcal{G}^{(n)}_{plus}$ is defined by $u_2 + \frac{a}{c}$.

Therefore, every element in $\mathcal{G}^{(n)}$ can be decomposed into a product of elements in those three subgroups. □

3 The transformation of AODEs

In this section, we study the action of the group $\mathcal{G}^{(n)}$ (see Def. 2.3) on the set of all algebraic ordinary differential equations. In addition, we show that the equivalence classes generated by the action of $\mathcal{G}^{(n)}$ satisfy the expected property, namely, the rational solvability is invariant. We start with the following definition.

Definition 3.1. Let $F(\overline{u}) \in \mathbb{K}[u_1, \ldots, u_{n+2}]$ be such that $\deg_{u_{n+2}}(F) \geq 1$. The *algebraic ordinary differential equation (AODE) of order n* (over $\mathbb{K}$) defined by F is of the form
$$F(x, y, y', \ldots, y^{(n)}) = 0.$$
Let $\mathcal{AODE}^{(n)}$ be the set of all algebraic ODEs of order n over $\mathbb{K}$. •

If the defining polynomial of an AODE can be factored, then the set of solutions is clearly the union of the sets of solutions of the AODEs defined by the factors. So throughout this paper we will assume that the AODE $F(x, y, y', \ldots, y^{(n)}) = 0$ is given by an irreducible polynomial F. We will study the integral birational transformations of the associated hypersurfaces to AODEs. For this purpose, we must first ensure that for any such hypersurface and for every element in $\mathcal{G}^{(n)}$ the map is well-defined. This first step is given in the next proposition.

Proposition 3.1. *For every $F \in \mathcal{AODE}^{(n)}$ and for every $\Phi \in \mathcal{G}^{(n)}$ there exists a non-empty Zariski dense open subset $\Omega \subset \mathcal{V}(F(u_1, \ldots, u_{n+2})) \subset \mathbb{K}^{n+2}$ such that Φ is defined on Ω.*

Proof. The denominator of each rational component of Φ is either a constant or a power of a polynomial in $\mathbb{K}[u_1, u_2]$, and F does depend on at least one variable u_i, $i > 2$. Therefore, since F is irreducible, no denominator of Φ vanishes on $\mathcal{V}(F)$. □

Now, we are ready to introduce the notion of transformed AODE. Later, in Theorem 3.3, we will see how to actually compute the transformed differential equation of a given AODE.

Definition 3.2. Let $F \in \mathbb{K}[\overline{u}]$ be irreducible and non-constant. Let $\Phi \in \mathcal{G}^{(n)}$ be an integral birational transformation on $\mathbb{K}^{n+2}$. Let W be the irreducible image variety of $\mathcal{V}(F)$ under Φ, i.e., the Zariski closure $\Phi(\mathcal{V}(F))^*$ of the set theoretic image. Let G be the defining polynomial of the hypersurface W; i.e. $W = \mathcal{V}(G)$. Then, we say that Φ transforms the AODE $F(x, y, y', \ldots, y^{(n)}) = 0$ into the AODE $G(x, y, y', \ldots, y^{(n)}) = 0$. We denote this relation by $G = \Phi \cdot F$. •

We observe that, because of the triangularized form of the elements in $\mathcal{G}^{(n)}$, integral birational transformations preserve the order of the AODE. So we get the following action of $\mathcal{G}^{(n)}$ on $\mathcal{AODE}^{(n)}$:

$$\begin{array}{rcl} \mathcal{G}^{(n)} \times \mathcal{AODE}^{(n)} & \longrightarrow & \mathcal{AODE}^{(n)} \\ (\Phi, F) & \longmapsto & \Phi \cdot F. \end{array}$$

This group action induces an equivalence relation in $\mathcal{AODE}^{(n)}$, say $\sim_{\mathcal{G}^{(n)}}$, and hence provides a quotient set

$$\overline{\mathcal{AODE}^{(n)}} := \mathcal{AODE}^{(n)} / \sim_{\mathcal{G}^{(n)}}$$

such that if $F, G \in \mathcal{AODE}^{(n)}$ then $F \sim_{\mathcal{G}^{(n)}} G$ if and only if there exists $\Phi \in \mathcal{G}^{(n)}$ such that $\Phi \cdot F = G$. We denote the equivalence class of $F \in \mathcal{AODE}^{(n)}$ as $\overline{F}$.

Let $F \in \mathcal{AODE}^{(n)}$, $G \in \overline{F}$ and $\Phi \in \mathcal{G}^{(n)}$ such that $\Phi \cdot F = G$. Let us assume that the AODE $F(x, y, y', \ldots, y^{(n)}) = 0$ of order n has rational solutions (a similar reasoning could be done for other types of solutions as those introduced in Remark 2.1(4)). Because of Remark 2.1, for every rational solution $y = f(x)$, with maybe one exception, $\mathcal{C}_f^{(n)} \in \mathcal{RIC}_\Phi^{(n)}$. Therefore, by Theorems 2.1 and 2.2, there exists a unique $g(x) \in \mathbb{K}(x)$ such that $\Phi^e(\mathcal{C}_f^{(n)}) = \mathcal{C}_g^{(n)}$. So $\Phi(x, f(x), f'(x), \ldots, f^{(n)}(x)) = (x, g(x), g'(x), \ldots, g^{(n)}(x))$. Thus,

$$G(x, g, g', \ldots, g^{(n)}) = F(\Phi^{-1}(\Phi(x, f, f', \ldots, f^{(n)}))) = F(x, f, f', \ldots, f^{(n)}) = 0.$$

We have deduced the following theorem.

Theorem 3.1. [Invariance of the rational solvability] *The existence of rational solutions is an invariant property for the elements in each equivalence class of* $\overline{\mathcal{AODE}^{(n)}}$*. Furthermore, if* $y = f(x)$ *is a general rational solution of* $F(x, y, y', \ldots, y^{(n)}) = 0$ *and* $\Phi \cdot F = G$*, then* $y = g(x)$*, where* $\Phi^e(\mathcal{C}_f^{(n)}) = \mathcal{C}_g^{(n)}$*, is a general rational solution of* $G(x, y, y', \ldots, y^{(n)}) = 0$.

Theorem 3.1 establishes the invariance of the existence of rational solvability of AODEs. Nevertheless, for actually computing the rational solutions of a parametrizable AODE the central tool, developed in [15, 16], is the study of the so called associated system. In the sequel, we show that not only the existence of rational solution but also the tool for computing them is preserved.

Definition 3.3. Let $F(u_1 \ldots, u_{n+2})$ be the irreducible polynomial defining the AODE

$$F(x, y, y', \ldots, y^{(n)}) = 0.$$

We say that $F(x, y, y', \ldots, y^{(n)}) = 0$ is a *parametric ordinary differential equation* (PODE) if the hypersurface defined by $F(u_1, \ldots, u_{n+2})$ is rational; i.e. it can be rationally and properly parametrized over $\mathbb{K}$.
Let $\mathcal{PODE}^{(n)}$ be the set of all PODEs in $\mathcal{AODE}^{(n)}$. •

Remark 3.1. We observe that there exists an important difference between the cases $n = 1$ and $n > 1$. For $n = 1$ (i.e. surfaces in $\mathbb{K}^3$), Castelnouvo's Theorem (see e.g. [19]) ensures that every surface rationally parametrized can be properly and rationally parametrized (note that $\mathbb{K}$ is algebraically closed of characteristic zero). However, for $n > 1$, i.e. for hypersurface in $\mathbb{K}^{n+2}$, the equivalence is not true in general. For the case $n = 1$, we refer to [13].

Clearly, $\mathcal{PODE}^{(n)} \subset \mathcal{AODE}^{(n)}$. Moreover, since the elements in $\mathcal{G}^{(n)}$ are birational transformations, the rationality of the associated algebraic hypersurface is preserved when applying to the differential equation and element in $\mathcal{G}^{(n)}$. Furthermore, if $\mathcal{P}(t_1, \ldots, t_{n+1})$ is a proper rational parametrization of $F \in \mathcal{PODE}^{(n)}$ and $\Phi \in \mathcal{G}^{(n)}$ then $\Phi(\mathcal{P}(t_1, \ldots, t_{n+1}))$ is a proper rational parametrization of $\Phi \cdot F$. So, $\Phi \cdot F \in \mathcal{PODE}^{(n)}$. Therefore, $\mathcal{G}^{(n)}$ also acts on $\mathcal{PODE}^{(n)}$. Similarly, we use the notation $\overline{\mathcal{PODE}^{(n)}}$ and $\overline{F}$ for $F \in \mathcal{PODE}^{(n)}$.

Let us study the equivalence classes in $\overline{\mathcal{PODE}^{(n)}}$. From [15], [16], [8] we know that every element in $\mathcal{PODE}^{(n)}$ is associated to a system of autonomous ODEs in the parameters. More precisely, let $\mathcal{P}(\bar{t}) = (\chi_1(\bar{t}), \chi_2(\bar{t}), \ldots, \chi_{n+2}(\bar{t}))$ be a proper rational parametrization of the solution hypersurface $F(u_1 \ldots, u_{n+2}) = 0$, where we assume that the Jacobian g of $\mathcal{P}(\bar{t})$ is regular; see below the role of g. Then the associated system to $F(x, y, y', \ldots, y^{(n)}) = 0$ w.r.t. $\mathcal{P}$ is

$$\left\{ t_1' = \frac{f_1}{g},\ t_2' = \frac{f_2}{g}, \ldots, t_{n+1}' = \frac{f_{n+1}}{g} \right\}$$

where

$$f_1 = \begin{vmatrix} 1 & \chi_{12} & \cdots & \chi_{1,n+1} \\ \chi_3 & \chi_{22} & \cdots & \chi_{2,n+1} \\ \vdots & \vdots & \vdots & \vdots \\ \chi_{n+2} & \chi_{n+1,2} & \cdots & \chi_{n+1,n+1} \end{vmatrix}, \quad f_2 = \begin{vmatrix} \chi_{11} & 1 & \cdots & \chi_{1,n+1} \\ \chi_{21} & \chi_3 & \cdots & \chi_{2,n+1} \\ \vdots & \vdots & \vdots & \vdots \\ \chi_{n+1,1} & \chi_{n+2} & \cdots & \chi_{n+1,n+1} \end{vmatrix}, \ldots,$$

$$f_{n+1} = \begin{vmatrix} \chi_{11} & \chi_{12} & \cdots & 1 \\ \chi_{21} & \chi_{22} & \cdots & \chi_3 \\ \vdots & \vdots & \vdots & \vdots \\ \chi_{n+1,1} & \chi_{n+1,2} & \cdots & \chi_{n+2} \end{vmatrix}, \quad g = \begin{vmatrix} \chi_{11} & \chi_{12} & \cdots & \chi_{1,n+1} \\ \chi_{21} & \chi_{22} & \cdots & \chi_{2,n+1} \\ \vdots & \vdots & \vdots & \vdots \\ \chi_{n+1,1} & \chi_{n+1,2} & \cdots & \chi_{n+1,n+1} \end{vmatrix}.$$

We now prove that the associated system is invariant under this group action.

Theorem 3.2. [Invariance of the associated system] *Let $F \in \mathcal{PODE}^{(n)}$ and $\Phi \in \mathcal{G}^{(n)}$. Let $\mathcal{P}(\bar{t}) = (\chi_1(\bar{t}), \chi_2(\bar{t}), \ldots, \chi_{n+2}(\bar{t}))$ be a proper rational parametrization of the solution surface $F(u_1 \ldots, u_{n+2}) = 0$ with $\det(J(\mathcal{P}(\bar{t})) \neq 0$. Then the associated system of $F(x, y, y', \ldots, y^{(n)}) = 0$ w.r.t. $\mathcal{P}$ and the associated system of $(\Phi \cdot F)(x, y, y', \ldots, y^{(n)}) = 0$ w.r.t. $\Phi \circ \mathcal{P}$ are equal.*

Proof. The associated system of $F(x, y, y', \ldots, y^{(n)}) = 0$ w.r.t $\mathcal{P}(\bar{t})$ is

$$\left\{ t_1' = \frac{f_1}{g},\ t_2' = \frac{f_2}{g}, \ldots, t_{n+1}' = \frac{f_{n+1}}{g} \right\}$$

where f_i and g are as above. We have

$$(\Phi \circ \mathcal{P})(\bar{t}) = \left(\chi_1,\ \phi_2(\chi_1,\chi_2),\ \phi_{21}+\phi_{22}\chi_3,\ldots,\phi_{n+1,1}+\sum_{i=2}^{n+1}\phi_{n+1,i}\chi_{i+1}\right).$$

Moreover, $(\Phi \circ \mathcal{P})(\bar{t})$ is a proper rational parametrization of the hypersurface $\Phi \cdot F = 0$. Therefore, the associated system of $(\Phi \cdot F)(x, y, y', \ldots, y^{(n)}) = 0$ w.r.t. $(\Phi \circ \mathcal{P})$ is

$$\left\{t_1' = \frac{\tilde{f}_1}{\tilde{g}},\ t_2' = \frac{\tilde{f}_2}{\tilde{g}},\ldots,\ t_{n+1}' = \frac{\tilde{f}_{n+1}}{\tilde{g}}\right\}$$

where

$$\begin{aligned}\tilde{f}_1 =& \begin{vmatrix} 1 & \chi_{12} & \cdots & \chi_{1,n+1} \\ \phi_{21}+\phi_{22}\chi_3 & \phi_{21}\chi_{12}+\phi_{22}\chi_{22} & \cdots & \phi_{21}\chi_{1,n+1}+\phi_{22}\chi_{2,n+1} \\ \vdots & \vdots & \vdots & \vdots \\ \phi_{n+1,1}+\sum_{i=2}^{n+1}\phi_{n+1,i}\chi_{i+1} & \sum_{i=1}^{n+1}\phi_{n+1,i}\chi_{i,2} & \cdots & \sum_{i=1}^{n+1}\phi_{n+1,i}\chi_{i,n+1} \end{vmatrix} \\ =&\phi_{22}\phi_{33}\ldots\phi_{n+1,n+1}f_1,\end{aligned}$$

and

$$\begin{aligned}\tilde{g} =& \begin{vmatrix} \chi_{11} & \chi_{12} & \cdots & \chi_{1,n+1} \\ \phi_{21}\chi_{11}+\phi_{22}\chi_{2,1} & \phi_{21}\chi_{12}+\phi_{22}\chi_{22} & \cdots & \phi_{21}\chi_{1,n+1}+\phi_{22}\phi_{2,n+1} \\ \vdots & \vdots & \vdots & \vdots \\ \sum_{i=1}^{n+1}\phi_{n+1,i}\chi_{i,1} & \sum_{i=1}^{n+1}\phi_{n+1,i}\chi_{i,2} & \cdots & \sum_{i=1}^{n+1}\phi_{n+1,i}\chi_{i,n+1} \end{vmatrix} \\ =&\phi_{22}\phi_{33}\ldots\phi_{n+1,n+1}g.\end{aligned}$$

Similarly, we can express $\tilde{f}_i$ in terms of f_i with the same factor $\phi_{22}\phi_{33}\ldots\phi_{n+1,n+1}$. Note that

$$\phi_{22}\phi_{33}\ldots\phi_{n+1,n+1} \neq 0,$$

which implies that the jacobian condition satisfies also for $\Phi \circ \mathcal{P}$. It implies that the associated system of $F(x, y, y', \ldots, y^{(n)}) = 0$ w.r.t. $\mathcal{P}$ and the associated system of $(\Phi \cdot F)(x, y, y', \ldots, y^{(n)}) = 0$ w.r.t. $\Phi \circ \mathcal{P}$ are equal. □

Remark 3.2. So, with the notation of the theorem, two equivalent equations F and G in $\mathcal{PODE}^{(n)}$ have the same class of associated systems w.r.t. the various parametrizations of the corresponding algebraic hypersurface. Therefore one might search for an element in the equivalence class having a geometrically simpler solution hypersurface and therefore leading to a simpler parametrization. For instance, if the equation is equivalent to an autonomous one, we can transform the solution hypersurface to a cylinder, thus greatly simplifying the parametrization problem. This is why the algorithm of Feng and Gao is much simpler.

Corollary 3.1. *Integral birational transformations send regular rational solution to regular rational solutions and singular rational solution to singular rational solutions.*

Proof. By Theorem 3.1 rational solutions are mapped to rational solutions. By Theorem 3.2 the general solutions are preserved. So, regular rational solutions are sent to regular rational solutions and, consequently, singular rational solutions must be mapped into singular rational solutions. □

In the last part of this section, let us take a closer look at the transform $G = \Phi \cdot F$, where $F \in \mathcal{AODE}^{(n)}$ and $\Phi \in \mathcal{G}^{(n)}$. It is clear, that G can be easily computed by means of elimination techniques as Gröbner basis, etc. Nevertheless, for further theoretical reasons, we want to have a precise description of G. For this purpose, we start with the following lemma.

Lemma 3.1. *Let Φ be a birational map from $\mathbb{K}^m$ on $\mathbb{K}^m$, and Φ^{-1} its inverse. Let $V = \mathcal{V}(P)$, with $P \in \mathbb{K}[\overline{x}]$ irreducible and non-constant such that Φ is defined on a non-empty Zariski open subset of V. Let W be the irreducible image variety of V under Φ, i.e., the Zariski closure $\Phi(V)^*$ of the set theoretic image. Then Φ^{-1} is defined on*

$$\left(\mathcal{V}(A) \setminus \mathcal{V}\left(\prod_{i=1}^{m} C_i\right)\right) ,$$

where A is the numerator of $P(\Phi^{-1}(\overline{x}))$ and C_i is the numerator of $N_i(\Phi^{-1}(\overline{x}))$, $N_i(\overline{x})$ being the denominators of the rational functions defining $\Phi(\overline{x})$. Moreover,

$$\left(\mathcal{V}(A) \setminus \mathcal{V}\left(\prod_{i=1}^{m} C_i\right)\right)^* = W .$$

Proof. Let Φ and Φ^{-1} be written as

$$\Phi(\overline{x}) = \left(\frac{M_1(\overline{x})}{N_1(\overline{x})}, \cdots, \frac{M_m(\overline{x})}{N_m(\overline{x})}\right), \ \Phi^{-1}(\overline{x'}) = \left(\frac{M_1'(\overline{x'})}{N_1'(\overline{x'})}, \cdots, \frac{M_m'(\overline{x'})}{N_m'(\overline{x'})}\right) .$$

We introduce the algebraic set

$$\mathcal{B} = \left\{ (\overline{x}, \overline{x'}, z) \in \mathbb{K}^m \times \mathbb{K}^m \times \mathbb{K} \ \middle| \ \begin{array}{l} P(\overline{x}) = 0 \\ x_i' N_i(\overline{x}) = M_i(\overline{x}), \ i = 1, \ldots, m \\ zK(\overline{x}, \overline{x'}) = 1 \end{array} \right\}$$

where $K(\overline{x}, \overline{x'}) = N_1(\overline{x}) \cdots N_3(\overline{x}) N_1'(\overline{x'}) \cdots N_3'(\overline{x'})$. Also, we consider the projection $\pi_{\overline{x'}} : \mathbb{K}^m \times \mathbb{K}^m \times \mathbb{K} \to \mathbb{K}^m$ such that $(\overline{x}, \overline{x'}, z) \mapsto \overline{x'}$.

Let us see that $W = \pi_{\overline{x'}}(\mathcal{B})^*$. Indeed, let Ω be the non-empty Zarsiki open subset of $\Phi(V)$ where Φ^{-1} is defined; note that, since W is irreducible because V is, then Ω is dense in W. Let $q \in \Omega$. Then there exists $p \in V$ such that $\Phi(p) = q$. Thus, $(p, q, 1/K(p, q)) \in \mathcal{B}$. So, $q \in \pi_{\overline{x'}}(\mathcal{B})$. Therefore, $\Omega \subset \pi_{\overline{x'}}(\mathcal{B})$, and hence $W = \Omega^* \subset \pi_{\overline{x'}}(\mathcal{B})^*$. Conversely, let $q \in \pi_{\overline{x'}}(\mathcal{B})$ then there exist $p \in \mathbb{K}^n$ and $\alpha \in \mathbb{K}$ such that $(p, q, \alpha) \in \mathcal{B}$, from where one deduces that $p \in V$, $q = \Phi(p)$. So, $q \in \Phi(V)$. Therefore, $\pi_{\overline{x'}}(\mathcal{B}) \subset \Phi(V)$ and hence $\pi_{\overline{x'}}(\mathcal{B})^* \subset W$.

Finally, we prove that $\pi_{\overline{x'}}(\mathcal{B})^* = (\mathcal{V}(A) \setminus \mathcal{V}(\prod_{i=1}^{m} C_i))^*$. Indeed, let $q \in \pi_{\overline{x'}}(\mathcal{B})$. Then, there exits $p \in \mathbb{K}^m, \alpha \in \mathbb{K}$ such that $(p, q, \alpha) \in \mathcal{B}$. Thus, $p \in V$, $K(p, q) \neq 0$, from here

$q = \Phi(p)$ and $\Phi^{-1}(q)$ is well defined. Moreover, $0 \neq N_i(p) = N_i(\Phi^{-1}(q))$. So, $q \notin \mathcal{V}\left(\prod_{i=1}^m C_i\right)$. On the other hand, $P(\Phi^{-1}(q)) = P(p) = 0$. Thus, $q \in \mathcal{V}(A)$. Therefore, taking closures, we get that $\pi_{\overline{x'}}(\mathcal{B})^* \subset (\mathcal{V}(A) \setminus \mathcal{V}\left(\prod_{i=1}^m C_i\right))^*$. Conversely, let B be the denominator of $P(\Phi(\overline{x'}))$; observe that B is a power of the $\mathrm{lcm}(N_1', \ldots, N_m')$. In $\mathcal{V}(A) \setminus \mathcal{V}\left(\prod_{i=1}^m C_i\right)$ we consider the open set $\Sigma = \mathcal{V}(A) \setminus (\mathcal{V}\left(\prod_{i=1}^m C_i\right) \cup \mathcal{V}(B))$. Since $\gcd(A, B) = 1$, $\Sigma \neq \emptyset$ and $\Sigma^* = (\mathcal{V}(A) \setminus \mathcal{V}\left(\prod_{i=1}^m C_i\right))^*$. Let $q \in \Sigma$. Since $B(q) \neq 0$, then $\Phi^{-1}(q)$ is well defined, say $p = \Phi^{-1}(q)$. So, since $q \in \mathcal{V}(A)$, $F(p) = F(\Phi^{-1}(q)) = A(q)/B(q) = 0$. Furthermore, since $q \notin \mathcal{V}\left(\prod_{i=1}^m C_i\right)$, then $0 \neq N_i(\Phi^{-1}(q)) = N_i(p)$. So, $N_i(p) \neq 0$ and $N_i'(q) \neq 0$. Therefore there exists α such that $(p, q, \alpha) \in \mathcal{B}$. Thus $Q \in \pi_{\overline{x'}}(\mathcal{B})$. Now taking closure we get the other inclusion. □

In the next theorems we show how to compute the transformed AODE corresponding to a given AODE.

Theorem 3.3. [Computation of the transformed AODE] *Let $F(u_1, \ldots, u_{n+2})$ be the irreducible polynomial of an AODE of order n, and let $\Phi \in \mathcal{G}^{(n)}$. Then the transformed AODE is defined by the primitive part w.r.t. $\{u_2, \ldots, u_{n+2}\}$ of the numerator $A(u_1, \ldots, u_{n+2})$ of the rational function $F(\Phi^{-1}(u_1, \ldots, u_{n+2}))$. Moreover, the content of A w.r.t. $\{u_2, \ldots, u_{n+2}\}$ is of the form Δ_Φ^r, for some non-negative integer number r.*

Proof. Let $\Phi := \Phi_L$ when $L = L_1/L_2$ and $L_1 = a(u_1)u_2 + b(u_1), L_2 = c(u_1)u_2 + d(u_1)$ are co-prime polynomials. Let $G = \Phi \cdot F$. Because of Proposition 3.1, and taking into account that $\mathcal{G}^{(n)}$ is a group, we have that $\mathcal{V}(F)$ and $\Phi(\mathcal{V}(F))^* = \mathcal{V}(G)$ satisfy the conditions of Lemma 3.1. So, we analyze for our particular case the value of C_i (with the terminology of Lemma 3.1). By Theorem 2.1 and Theorem 2.2, we know that the denominators N_i in Φ are powers of L_2. On the other hand, $\Phi^{-1} = \Phi_{L^{-1}}$ and $L^{-1} = (u_2 d - b)/(-cu_2 + a)$. So, the numerator of $N_i(\Phi^{-1})$ is a power of Δ_Φ. Therefore, G is the irreducible factor of A not depending only on u_1. Thus, $G = \mathrm{PP}_{\{u_2,\ldots,u_{n+2}\}}(A)$ and its content is Δ_Φ^r with $r \geq 0$. □

Remark 3.3. In our paper [13], Section 4, we have considered three special classes of of parametrizable AODEs, namely

- (a) equations solvable for y': $y' = G(x, y)$,
- (b) equations solvable for y: $y = G(x, y')$,
- (c) equations solvable for x: $x = G(y, y')$,

where G is a rational function. The action of the group of linear transformations leaves the first two classes invariant, whereas it leads out of the third class. So if an equation can be transformed by the group of linear transformations to the form (c), we can apply our solution method.

Now the action of the group of birational transformations $\mathcal{G}^{(n)}$ leaves only the first class (solvable for y') invariant, whereas it leads out of the second and third class. For instance, the equation $y = G(x, y')$ is transformed into

$$\frac{ay + b}{cy + d} = G(x,\ A(x, y)y' + B(x, y))\ .$$

The degree of y in this equation is no longer 1, because the y appears in G; this does not happen with the group of linear transformations, because in this case A and B are constant.

4 Potential applicability to solving AODEs

In this section, we illustrate the potential applicability of the previous results to the problem of solving AODEs. We have observed that all differential equations in an equivalence class have the same order, say n. However, the variables $\{x, y', \ldots, y^{(n-1)}\}$ appearing in the equation can be different. So, it may happen that x does not appear, and hence an autonomous equation may be equivalent to a non-autonomous one (see Example 4.1). Also, we may have equivalent equations where the number of involved derivatives is different (see Example 4.2). Nevertheless, the systematic theoretical-algorithm study is an open problem. So, the contribution of this paper must be seen as the establishment of a theoretical frame for developing new algorithm techniques.

Example 4.1. Consider the first order algebraic ODE

$$F(x, y, y') = 25x^2y'^2 - 50xyy' + 25y^2 + 12y^4 - 76xy^3 + 168x^2y^2 - 144x^3y + 32x^4 = 0.$$

Using the transformation

$$\Phi(u_1, u_2, u_3) = \left(u_1, \frac{3u_2 - u_1}{-u_2 + 2u_1}, \frac{5u_1}{(-u_2 + 2u_1)^2}u_3 + \frac{-5u_2}{(-u_2 + 2u_1)^2}\right) \in \mathcal{G}^{(1)}$$

we get that $\Phi \cdot F$ is the primitive part w.r.t. $\{u_2, u_3\}$ of the numerator of $F(\Phi^{-1}(u_1, u_2, u_3))$; i.e.,

$$\Phi \cdot F = u_3^2 - 4u_2.$$

Therefore, $F(x, y, y') = 0$ is transformed into the autonomous AODE

$$G(x, y, y') = {y'}^2 - 4y = 0.$$

In addition, we observe that F cannot be transformed into an autonomous AODE by affine transformations as considered in [13].

Since Φ is birational, the rational general solution of $F(x, y, y') = 0$ is transformed into the rational general solution of $G(x, y, y') = 0$ and vice versa. It is clear that $y = (x + c)^2$ is the rational general solution of $y'^2 - 4y = 0$. Therefore,

$$y = \frac{x(2(x + c)^2 + 1)}{(x + c)^2 + 3},$$

where c is any constant, is the rational general solution of $F(x, y, y') = 0$. The autonomous equation G has the singular solution $y = 0$ which, by the inverse of our transformation, is mapped into the singular solution $y = \frac{1}{3}x$ of F (compare to Corollary 3.1). •

The previous example shows that the property of being autonomous is not invariant w.r.t. the action of the group. This can be helpful in solving AODEs if one can decide the existence of an autonomous equation in a given class. For the case of affine transformations such a decision algorithm is given in [13]. For the more general case, partial results can be found in [14].

Example 4.2. Consider the second order algebraic ODE

$$F(x,y,y',y'') = x^6 + 3\,x^5 u_2 + 3\,x^4 y^2 + x^3 y^3 - x^2 u_4 - x u_4 y + 2xy'^2 + 2xy' - 2y'y - 2y = 0.$$

Using the transformation

$$\Phi(\overline{u}) = \left(u_1, \frac{u_2}{u_2+u_1}, \frac{u_1u_3-u_2}{(u_2+u_1)^2}, \frac{{u_1}^2u_4 + u_1u_4u_2 - 2\,u_1{u_3}^2 - 2\,u_1u_3 + 2\,u_3u_2 + 2\,u_2}{(u_2+u_1)^3} \right) \in \mathcal{G}^{(2)}$$

we get that $\Phi \cdot F$ is the primitive part w.r.t. $\{u_2, u_3, u_4\}$ of the numerator of $F(\Phi^{-1}(u_1,u_2,u_3,u_4))$; i.e.,

$$\Phi \cdot F = {u_1}^3 - u_4.$$

Therefore, $F(x,y,y',y'') = 0$ is transformed into the AODE

$$G(x,y,y',y'') = x^3 - y'' = 0.$$

Since Φ is birational, the rational general solution of $F(x,y,y',y'') = 0$ is transformed into the rational general solution of $G(x,y,y',y'') = 0$ and vice versa. $y = (1/20)x^5 + c_1x + c_2$ is the rational general solution of $G(x,y,y',y'') = 0$. Therefore, applying the inverse map we that

$$y = -\frac{(\frac{1}{20}x^5 + c_1x + c_2)x}{\frac{1}{20}x^5 + c_1x + c_2 - 1},$$

is the rational general solution of $F(x,y,y',y'') = 0$. •

5 Conclusion and future directions of research

We have characterized the birational transformations preserving rational solvability of AODEs. The set of all these birational transformations forms a group under composition. The action of this group partitions the set of all AODEs into equivalence classes for which the rational solvability is invariant. The rational general solutions of two elements in the same class, if they exist, can be transformed into each other by such transformations. The same holds for the singular solutions.

This opens a new spectrum of open questions, such as deciding the equivalence between two given AODEs or the transformability of a given AODE into an equivalent autonomous one. Decision algorithms for both problems are known for the case of affine transformations.

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