

# Stochastic Discount Factors (SDFs) and the Equity Premium Puzzle Under a Power Utility Specification

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## Introduction

This paper will present a simple equilibrium pricing model in order to describe the existence of the Stochastic Discount Factor (SDF) as a set of values that correct payoffs of risky assets for risk, dividends, time and the marginal utility of consumption in order to obtain a price for said payoff stream. This is relevant for asset pricing theory insofar as it motivates a rigorous methodology to arrive at stable asset pricing models.

In financial and economic literature the SDF plays a major role both in discrete and continuous time models as it implies both a means to adjust future payoffs and a change of probability measure in order to allow the existence of risk neutral pricing, which is the main tool for valuating complex financial assets such as derivatives.

A second benefit of analyzing the SDF can be found in its deep economic interpretation. Stochastic discount factors are deeply related with performance measures such as the Sharpe Ratio and with economic variables such as the intertemporal marginal rate of substitution which measures both risk aversion and impatience in the agent's preferences in consumption. It is also relevant to note that the various transformations to be performed in the definition of the SDF will allow a very powerful economic interpretation of the way an asset pricing model values risks. In this paper we explain diversification and the price of an excess rate of return through equilibrium theory.

The goal in this paper is twofold: to describe and apply. We want to offer a simple model to describe the existence of an SDF in equilibrium pricing which has also very strong implications for no arbitrage pricing models. Then we will use Colombian market and economic data in order to verify the model and justify the relevance of the study of the Equity Premium Puzzle, a concept that has been discussed frequently in financial and economic literature since the works of Mehra and Prescott in 1985.

When performing the fitting of the model we will find several issues with Colombian data that will add a further layer of difficulty when analyzing information for developing and not for developed economies. Replicating the results of previous papers for a different data set always presents issues that are unique to the new information. This paper is also subject to this caveat since Colombia has recently suffered from a period of high inflation, low economic growth, high unemployment and important fiscal deficits that make its economic data significantly different (and shorter) than that which may be available for developed economies.

This paper is divided in 7 sections. Part 1 will discuss the pricing equation and the existence of the SDF. Part 2 will present the same results as those in Part 1 for asset returns and not for asset prices, it will motivate the existence of portfolio diversification and its effect on asset pricing. Part 3 discusses time varying expected returns and price predictability which will result in further research on the conditions that must be fulfilled by a stochastic discount factor to price assets correctly. Part 4 will develop the model for a power utility function in order to test the model in Part 5 using a Colombian database. Part 6 will present a brief review of the equity premium puzzle and will discuss why it is relevant to take this discussion further for Colombian data. Part 7 presents concluding remarks and further extensions to this cornerstone work.

## Part 1: The pricing equation and the SDF

Let us start with a simple pricing model based on Cochrane (2005). We have a risk averse representative agent in a one period economy with uncertainty about future payoffs. The agent is subject to an additive and separable utility function which is increasing in wealth. Risk aversion implies a decreasing marginal utility of consumption.

$$U(c_t, c_{t+1}) = U(c_t) + \delta E U(c_{t+1}) \quad (1)$$

Where  $\delta$ ,  $[0,1]$  is a subjective discount factor that captures impatience. As  $\delta$  increases the agent will be more patient and thus more willing to trade present consumption for future consumption. Given that there is empirical evidence that shows that real interest rates tend to be positive, there is no need to assume  $\rho$  to be larger than 1.

We denote by  $e_t$  the endowment the agent receives at time  $t$ , by  $e_{t+1}$  the endowment at time  $t+1$  and by  $Z_j$  the amount of asset  $j$  the agent will purchase at time  $t$  as a mechanism to transfer wealth from time  $t$  to time  $t+1$ . The asset  $j$  has a price of  $P_{jt}$  at time  $t$  and a payoff of  $X_{jt+1}$  at time  $t+1$ . Thus we can define consumption constraints at time  $t$  and  $t+1$  as:

$$c_t = e_t - Z_j P_{jt} \quad (2a)$$

$$c_{t+1} = e_{t+1} + Z_j X_{jt+1} \quad (2b)$$

Replacing the constraints in (1) and maximizing utility we have:

$$\max_{Z_j} U(c_t, c_{t+1}) = \max_{Z_j} U(e_t - Z_j P_{jt}) + \delta E U(e_{t+1} + Z_j X_{jt+1}) \quad (3)$$

The resulting first order condition is:

$$P_{jt} U'(c_t) = E \delta U'(c_{t+1}) (X_{jt+1}) \quad (4)$$

Solving (4) for  $P_{jt}$ ,

$$P_{jt} = E_t \delta \frac{U'(c_{t+1})}{U'(c_t)} X_{jt+1} \quad (5)$$

Let us define a stochastic discount factor (SDF)  $m_{t+1}$  as  $\delta \frac{U'(c_{t+1})}{U'(c_t)}$ . Then,

$$P_{jt} = E_t m_{t+1} X_{jt+1} \quad (6)$$

It is important to note that the different subscripts for the expectation operator and for the payoffs of the asset remind us constantly that we are evaluating uncertain payoffs that will happen at time  $t+1$  with the information available at time  $t$ . In order to better understand the role played by  $m_{t+1}$ , we can use (6) to evaluate a risk free bond with a price of  $\beta_t$  at time  $t$  and a payoff of 1 at time  $t+1$ .

$$\beta_t = E_t [m_{t+1}] = \frac{1}{R_{f,t,t+1}} \quad (7a)$$

From (7a) we can see that the price of a risk free bond is both the inverse of the risk free rate paid on an investment from time  $t$  to time  $t+1$  and the expected value of the SDF. In the case where there is no uncertainty we obtain the well-known present value formula. When we have uncertainty as in (6) we must find an asset-specific stochastic discount factor  $m_{t+1}$ .

As stated by Cochrane, “one can incorporate all risk corrections by defining a *single* stochastic discount factor –the same for each asset– and putting it inside the

expectation.  $m_{t+1}$  is *stochastic* or *random* because it is not known with certainty at time  $t$ . The correlation between the random components of the common discount factor  $m$  and the asset-specific payoff  $X$  generate asset-specific risk corrections.” (Cochrane 2005, 7)

In an Arrow-Debreu one period economy with discrete payoffs for  $S$  states of nature in  $t+1$  we can find state prices  $\varphi_i$  such that,

$$\beta_t = \frac{1}{R_{f,t,t+1}} = \sum_{i=1}^S \varphi_i \quad (7b)$$

Interpreting the results from 7a and 7b we obtain,

$$E_t m_{i,t+1} = \sum_{i=1}^S \pi_i m_i = \sum_{i=1}^S \varphi_i \quad (7c)$$

Thus,

$$m_i = \frac{\varphi_i}{\pi_i} \quad (7d)$$

Where  $\pi_i$  are the probabilities for each state under the true probability measure. Thus,  $m$  works as a Radon-Nikodym derivative which performs a change of measure from a measurable space  $(\Omega, F, \mathbb{P})$  to another measurable space  $(\Omega, G, \mathbb{Q})$ . This is a very relevant result since it sets the basis for risk neutral valuation. Still,  $m$ 's economic interpretation is poor if left only at the level of a means to perform a change of measure.

From (5) we can infer that the SDF is a marginal rate of substitution. This is the rate at which the investor is willing to exchange consumption at time  $t+1$  for consumption at time  $t$ . From general equilibrium theory, through a maximization process with restrictions, we find that equilibrium prices result from stating the equality between Lagrangian multipliers –marginal utilities of consumption per unit of price– which must be equal in order for Pareto efficient prices and consumption levels to exist. Note that the relevance of  $m_{t+1}$  as a discount factor is independent of the utility function, thus making (6) a general valuation formula of which any single model is a particular case.

Following Marin and Rubio we can perform a transformation on the SDF as defined in the pricing formula in (6). (Marín and Rubio 2001) Recall:

$$Cov(m_{t+1}, X_{t+1}) = E[m_{t+1}X_{t+1}] - E[m_{t+1}]E[X_{t+1}] \quad (8)$$

We can rewrite (6),

$$P_{jt} = E[m_{t+1}]E[X_{t+1}] + Cov(m_{t+1}, X_{t+1}) \quad (9)$$

From (7a) we know,

$$P_{jt} = \frac{E[X_{t+1}]}{R_{f,t,t+1}} + Cov(m_{t+1}, X_{t+1}) \quad (10)$$

Alternatively,

$$P_{jt} = \frac{E[X_{t+1}]}{R_{f,t,t+1}} + \frac{Cov(\rho U(c_{t+1}), X_{t+1})}{U(c_t)} \quad (11)$$

From (10) we can see that the price of an asset can be broken down in two parts. The first term discounts all future cash flows using the risk free rate. This confirms that the marginal rate of substitution works as a mechanism to perform a change of measure from the asset's probability to the risk neutral measure as shown in (7d). The second term performs a risk correction in which the asset price will increase if the covariance between the payoffs and the SDF is positive and the price will decrease if the covariance between the asset's payoffs and the SDF is negative.

This has a very relevant economic interpretation. Since risk in this model can be understood as volatility in consumption, a risk averse agent will want to perform investments that aid him in performing a process of consumption smoothing both across all states of nature and across all future dates. This smoothing process, understood as reducing the volatility in consumption in order to maintain a constant level of utility derived from consumption, is explained by Franco Modigliani in his life-cycle hypothesis of saving (Modigliani 1975). Thus the price of an asset will increase (decrease) if its payoffs increase (decrease) when the investor has

high marginal utility. This is shown when the case covariance between payoffs and marginal utility is positive (negative).

Since marginal utility, by definition, is decreasing in wealth it is important to notice that when payoffs covariate positively with consumption they add volatility to the latter. Assets whose payoffs increase consumption when marginal utility decreases will be hence known as *risky assets* and assets that do the opposite will be hence known as *hedging assets*. Investors will demand higher returns and lower prices from risky assets than from hedging assets that provide insurance.

## Part 2: A return based CCAPM

We can write the pricing equation in (5) in terms of return by dividing by the price of the asset  $j$ ,  $P_{jt}$  on both sides of the equation:

$$1 = E_t \delta \frac{U(c_{t+1})}{U(c_t)} \frac{X_{jt+1}}{P_{jt}} = E_t \delta \frac{U(c_{t+1})}{U(c_t)} R_{jt+1} \quad (12a)$$

Where  $R_{jt+1}$  is the gross return for asset  $j$  from time  $t$  to time  $t+1$ . Since the marginal utility of consumption at time  $t$  is known it can be taken outside of the expectation.

$$U(c_t)(1) = E_t \delta U(c_{t+1}) R_{jt+1} \quad (12b)$$

The intuition behind (12b) implies that the marginal utility of sacrificing one monetary unit of sure [certain] consumption today must be compensated in expectation by the marginal utility of consuming  $R_{jt+1}$  monetary units tomorrow. In shorthand we can express (12a) as,

$$1 = E_t m_{t+1} R_{jt+1} \quad (13)$$

Equation (13) implies that uncertain gross rates of return when corrected by the marginal rate of substitution, or any other variable that acts as an SDF, should be constant. This is the basis for the development of risk neutral pricing which for two payoffs  $Z_u$  and  $Z_d$  could be expressed as:



$$P_{Z,t} = \frac{1}{R_{f,t,t+1}} Z_{u,t+1} \pi_u^* + Z_{d,t+1} \pi_d^* = \frac{1}{R_{f,t,t+1}} E_t^* [Z_{t+1}] \quad (14)$$

Where  $E_t^* [Z_{t+1}]$  is the expectation of future payoffs under the risk neutral measure and risk neutral probabilities are defined as  $\pi_u^* = R_{f,t,t+1} \varphi_u$  and  $\pi_d^* = R_{f,t,t+1} \varphi_d$ . Thus, under the risk neutral measure all asset payoffs should return the risk free rate.

$$R_{f,t,t+1} = \frac{Z_{u,t+1}}{P_{Z,t}} \pi_u^* + \frac{Z_{d,t+1}}{P_{Z,t}} \pi_d^* = R_u \pi_u^* + R_d \pi_d^* \quad (15)$$

Now we can apply the covariance decomposition in (5) to (12a),

$$Cov \delta \frac{U(c_{t+1})}{U(c_t)}, R_{j,t+1} = E_t \delta \frac{U(c_{t+1})}{U(c_t)} R_{j,t+1} - E_t \delta \frac{U(c_{t+1})}{U(c_t)} E_t U(c_{t+1}) \quad (16)$$

We can solve for  $E_t R_{j,t+1}$ ,

$$E_t R_{j,t+1} = R_{f,t,t+1} - \frac{Cov U(c_{t+1}), R_{j,t+1}}{E_t U(c_{t+1})} \quad (17)$$

From (17) we can show that risky assets will have a negative covariance between returns and marginal utility of future consumption while hedging assets have a positive covariance between the two variables. Since marginal utility of consumption is positive by definition we can say that a positive covariance reduces volatility in consumption (risk) while a negative covariance increases volatility in consumption. Thus the return an investor expects from asset  $j$  from time  $t$  to time  $t+1$  will react in consequence.

Recall (13),

$$\frac{E_t [m_{t+1}]}{E_t [m_{t+1}]} = E_t m_{t+1} R_{j,t+1} \quad (18a)$$

$$E_t m_{t+1} R_{f,t,t+1} = E_t m_{t+1} R_{j,t+1} \quad (18b)$$

$$0 = E_t m_{t+1} (R_{j,t+1} - R_{f,t,t+1}) \quad (18c)$$

From (18c) we can deduce that all excess returns, when corrected for risk have a price of 0. This is a way to prove that the model prices correctly for a certain utility function.

Another topic that has been important in asset pricing theory, since the early 1950s, is the formal relationship between the expected return an asset will provide an investor and the risk that holding such an asset entails. Since Harry Markowitz's seminal work on portfolio selection it has been well known that holding a portfolio of assets may better serve an investor willing to obtain a higher level of expected return while holding constant a level of risk-defined as the variance of the return.

When speaking of risk diversification Markowitz states: "Not only does the E-V (Expected Return-Variance) hypothesis imply diversification, it implies the 'right kind' of diversification for the 'right reason'. (...) In trying to make variance small it is not enough to invest in many securities. It is necessary to avoid investing in securities with high covariances among themselves" (Markowitz 1952, 89)

Even under the assumption that asset returns are fully explained by both their expectation and their variance, which will later progress to become the assumption of normality of asset returns, Markowitz's proposition is a useful one. It can be extended to say that a security's price is linked both to its expected return and to the amount of risk it *adds* to an investor's portfolio and not to its total risk.

Let  $\sigma_p^2$  be the variance of a portfolio of N assets defined as:

$$\sigma_p^2 = \sum_i^N \sum_j^N \omega_i \omega_j \sigma_{ij} \quad (19)$$

For all  $i$  and  $j$ . Where  $\omega_i$  and  $\omega_j$  are the portfolio weights of each asset and  $\sigma_{ij}$  is the covariance between the returns of asset  $i$  and  $j$ . Let us assume the following:

1. All assets are equally weighted in this portfolio such that  $\omega_j = \omega_i = 1/N$ .
2. All assets have a constant variance  $\sigma_i^2 = \sigma_j^2 = \bar{\sigma}$ .
3. The covariance among all pairs of assets are constant  $\sigma_{ij} = \bar{\sigma}_{ij}$ .

Then (19) can be transformed by defining two cases.

Case 1:  $i = j$

$$\sigma_p^2 = \sum_i^N \frac{1}{N^2} \bar{\sigma}^2 = \frac{1}{N} \bar{\sigma}^2 \quad (20a)$$

Case 2:  $i \uparrow j$

$$\sigma_p^2 = \sum_i^N \sum_j^N \frac{1}{N^2} \bar{\sigma}_{ij} = \frac{N-1}{N} \bar{\sigma}_{ij} \quad (20b)$$

We then add both cases and take limits for  $N \rightarrow \infty$  to reflect an increasing number of assets in a portfolio.

$$\lim_{N \rightarrow \infty} \frac{1}{N} \bar{\sigma}^2 + \frac{N-1}{N} \bar{\sigma}_{ij} = \bar{\sigma}_{ij} \quad (20c)$$

The result of (20a) is known as *asset specific risk* while the result of (20c) is referred to as *market risk*. Thus, as Markowitz concludes, asset specific risk can be diversified away while market risk remains no matter how many securities are included in a portfolio. Thus a risk averse investor will prefer assets with low correlations rather than those with high correlations.

As previously stated the price of an asset depends on how much risk it adds to a diversified portfolio or, as stated by the CAPM, how strongly it covaries with the market portfolio which is held by all investors. This can be extrapolated to the pricing equation in (10). Suppose that  $Cov(m_{t+1}, X_{t+1}) = 0$ . Then,

$$P_{jt} = \frac{E[X_{t+1}]}{R_{f,t,t+1}} \quad (21)$$

Thus “if the payoff is uncorrelated with the discount factor  $m$ , the asset receives *no* risk correction to its price, and pays a return equal to the risk free rate. (...) This prediction holds even if the payoff  $X$  is highly volatile and investors are highly risk averse.” (Cochrane 2005, 15) The same is to say that the well known CAPM is

just a general case of this pricing equation that recognizes that only *systematic risk* should be priced because *idiosyncratic risk* can be diversified away.

### Part 3: Time varying expected returns: revisiting the random walk hypothesis

One of the fundamental applications of asset pricing theory is the prediction of the behavior of asset returns. Recall that the behavior of an asset over time can be expressed using (5) and the definition  $X_{jt+1} = P_{jt+1} + D_{jt+1}$ . Where  $D_{jt+1}$  is the dividend paid by the asset at time  $t+1$ . Then,

$$P_{jt} = E_t \delta \frac{U(c_{t+1})}{U(c_t)} (P_{jt+1} + D_{jt+1}) \quad (22)$$

Under the assumption that agents live in a risk neutral world where assets pay no dividends we can state then that prices behave as martingales during short periods of time. This is because there is no change in marginal utility from time  $t$  to time  $t+1$  and because  $\rho=1$ . Thus,

$$P_{jt} = E_t P_{jt+1} \quad (23)$$

The previous equation can be expressed as an additive martingale as follows,

$$P_{jt+1} = P_{jt} + \epsilon_{jt+1} \quad (24)$$

Assuming homoskedasticity in the noise term we can say that prices behave like random walks. This is extensive to returns where, under certain characterizations of the shock, different levels of complexity and rigor of the random walk hypothesis can be tested (Forero 2011). According to (24) prices should not be predictable:

1. During *short* periods of time.
2. *After* adjusting for dividends.
3. *After* scaling for marginal utility.

In this sense financial economics, as a science, has tried to provide models that are useful mimicking reality as well as predicting the behavior of asset prices and returns. The idea behind these models is that, with an ample time horizon before any corrections, the random walk hypothesis can be disregarded and thus both asset prices and returns tend to be predictable.

To formalize the origin of the predictability of asset returns let us recall (17) and substitute the covariance by its expression in terms of the correlation coefficient,

$$E_t R_{j,t+1} - R_{f,t,t+1} = - \frac{\sigma_t(m_{t+1})}{E_t[m_{t+1}]} \sigma_t(R_{j,t+1}) \rho(m_{t+1}, R_{j,t+1}) \quad (25)$$

Where  $\sigma_t(m_{t+1})$  is the volatility of the stochastic discount factor,  $\sigma_t(R_{j,t+1})$  is the volatility in the returns of asset j and  $\rho(m_{t+1}, R_{j,t+1})$  is the correlation coefficient between the asset's returns and the stochastic discount factor. First it can be said that marginal utility of agents  $\frac{\sigma_t(m_{t+1})}{E_t[m_{t+1}]}$  changes over time since preferences, levels of wealth and the economic cycle all affect the way agents consume over time. Secondly, the measure of risk is the way an asset's returns vary with respect to the stochastic discount factor  $\rho(m_{t+1}, R_{j,t+1})$ , and this correlation coefficient cannot be assumed constant along the business cycle.

From (25) we can obtain certain restrictions on the plausible set of stochastic discount factors that would work correctly for the given pricing equation. Let  $R_{j,t+1} - R_{f,t,t+1} = R_{j,t+1}^*$ , and recall that  $E m_{t+1} R_{j,t+1}^* = 0$ . Since,

$$Cov(m_{t+1}, R_{j,t+1}^*) = E_t m_{t+1} R_{j,t+1}^* - E_t[m_{t+1}] E_t R_{j,t+1}^* \quad (26)$$

Then,

$$Cov(m_{t+1}, R_{j,t+1}^*) + E_t[m_{t+1}] E_t R_{j,t+1}^* = 0 \quad (27a)$$

Replacing the definition of covariance,

$$\rho(m_{t+1}, R_{j,t+1}^*) \sigma(m_{t+1}) \sigma(R_{j,t+1}^*) + E_t[m_{t+1}] E_t R_{j,t+1}^* = 0 \quad (27b)$$

And applying the fact that  $|\rho| < 1$ ,

$$\sigma(m_{t+1})\sigma(R_{j,t+1}^*) \geq \left| E_t[m_{t+1}]E_t[R_{j,t+1}^*] \right| \quad (27c)$$

Recall standard deviations are strictly positive numbers. Solving for a restriction to the SDF,

$$\frac{\sigma(m_{t+1})}{E_t[m_{t+1}]} \geq \frac{E_t[R_{j,t+1}^*]}{\sigma(R_{j,t+1}^*)} \quad (27d)$$

Simplifying,

$$\sigma(m_{t+1})R_{f,t,t+1} \geq S_{j,t+1} \quad (27e)$$

This equation is a restriction on the set of possible SDFs that can correctly price a set of returns given a Sharpe ratio  $S_{j,t+1}$ . Since the risk free rate is quite stable over time, in order to explain the volatility of risky returns we will require a sufficiently volatile SDF.

## Part 4: The CCAPM under a power utility specification

The general framework that has been discussed this far can be given an empirical application by choosing a utility function. The purpose of this section is to formalize the pricing model under a power utility function. The next section will test the data to evaluate its performance. This will motivate a later discussion of the equity premium puzzle.

Let us assume the representative agent has an additive and separable utility function of the form:

$$U(C) = \frac{C^{1-\gamma}}{1-\gamma}; \forall \gamma > 0; \gamma \neq 1 \quad (28a)$$

$$U(C) = \ln(C); \gamma = 1 \quad (28b)$$

If we follow Arrow and Pratt the Absolute risk aversion coefficient is,

$$ARA = -\frac{U''(C)}{U'(C)} = \frac{\gamma}{C} \quad (29a)$$

This means the agent has a DARA utility function which shows that his risk aversion is decreasing in wealth. According to Lengwiler there is ample empirical evidence to show this is true for most investors. As an investor becomes wealthier the nominal amount of her investments in risky assets increases (Lengwiler 2006, 81-90).

However the Relative Risk Aversion Coefficient is constant,

$$RRA = -\frac{U''(C)}{U'(C)}C = \gamma \quad (29b)$$

A CRRA utility function assumes agents are willing to pay the same price to avoid risk across time and across states of nature. This contradicts empirical evidence, which shows that risk aversion is mostly countercyclical – agents are less risk averse during economic expansions and more risk averse during recessions (Lengwiler 2006, 81-90).

Following (5) we can define the SDF for this utility specification as,

$$m_{t+1} = \delta \frac{U'(c_{t+1})}{U'(c_t)} = \delta \frac{C_{t+1}^{-\gamma}}{C_t^{-\gamma}} \quad (30)$$

Thus the stochastic discount factor is determined by the growth rate in consumption, the relative risk aversion and the impatience parameter. If future consumption is higher than present consumption the risk averse investor will have no incentive to avoid consumption today in order to invest in financial assets. In this sense the SDF will decrease and so will the actual price of a certain set of constant future payoffs.

Recall equation (12a) and define the pricing equation for returns under the power utility specification,

$$1 = E_t \left[ m_{t+1} R_{j,t+1} \right] = E_t \left[ \delta \frac{C_{t+1}}{C_t} \right]^{-\gamma} R_{j,t+1} \quad (31)$$

From here onwards we will work under the assumption that prices behave as lognormal variables and thus returns behave as normal variables. This assumption of normality has been a standard recourse in recent financial literature although criticisms to its applications abound (Forero 2011).

Based on this assumption we linearize the model using the moment generating function for a normal variable. Recall that if  $X$  is normally distributed then  $e^X$  is lognormally distributed. Also,  $E(e^X) = e^{E(X) + \frac{1}{2}Var(X)}$ . Taking natural logarithms on both sides we obtain,

$$\ln(E(e^X)) = E(X) + \frac{1}{2}Var(X) = E(\ln e^X) + \frac{1}{2}Var(\ln e^X) \quad (32)$$

Replacing (32) in (31),

$$\ln(1) = \ln E_t \left[ \delta \frac{C_{t+1}}{C_t} \right]^{-\gamma} R_{f,t+1} \quad (33a)$$

Recall  $R_{j,t+1}$  is known. Thus we can transform (33a),

$$\ln(R_{f,t+1}) = -\ln E_t \left[ \delta \frac{C_{t+1}}{C_t} \right]^{-\gamma} \quad (33b)$$

$$\ln(R_{f,t+1}) = -E_t \left[ \ln \delta \frac{C_{t+1}}{C_t} \right]^{-\gamma} - \frac{1}{2}Var \left[ \ln \delta \frac{C_{t+1}}{C_t} \right]^{-\gamma} \quad (33c)$$

$$\ln(R_{f,t+1}) = -\ln(\delta) + \gamma E_t \left[ \ln \frac{C_{t+1}}{C_t} \right] - \frac{\gamma^2}{2}Var_t \left[ \ln \frac{C_{t+1}}{C_t} \right] \quad (33d)$$



Lowercase letters indicate logarithms and  $\sigma_c^2 = \text{Var}_t \ln \frac{C_{t+1}}{C_t}$

$$r_{f,t+1} = -\ln(\delta) + \gamma E_t(\Delta c_{t+1}) - \frac{\gamma^2}{2} \sigma_c^2 \quad (33e)$$

From (33e) we can infer several dynamics that occur concurrently in the model and that help determine the risk free rate:

1. When people are more impatient  $-\delta \rightarrow 0$ –the rate of return of a risk free asset increases as they are less willing to sacrifice present consumption for current consumption.
2. When expected consumption growth increases people refrain from investing as they expect future income to increase. This is corrected for risk aversion since in the process of consumption smoothing there is always a level of precautionary saving. (Lengwiler 2006)
3. When volatility in consumption –the functional definition of risk in this paper– increases, the demand for the risk free asset increases thus driving its rate of return downward.

Using (33e) for any risky asset we obtain,

$$E(r_{j,t+1}) = -\ln(\delta) + \gamma E_t(\Delta c_{t+1}) - \frac{1}{2}(\gamma^2 \sigma_c^2 + \sigma_j^2 - 2\gamma \sigma_{jc}) \quad (33f)$$

Where  $\sigma_j^2 = \text{Var}(r_{j,t+1})$  and  $\sigma_{jc} = \text{Cov}(r_{j,t+1}, \Delta c_{t+1})$  We can subtract (33d) from (33f) to obtain the formula for a risk premium.

$$E(r_{j,t+1}) - r_{f,t+1} + \frac{1}{2} \sigma_j^2 = \gamma \sigma_{jc} \quad (33g)$$

The term  $\frac{1}{2} \sigma_j^2$  on the left hand side is a Jensen's inequality correction –resulting from (32)– that can be eliminated by rewriting (33g) as,

$$\ln E_t \frac{R_{j,t+1}}{R_{f,t+1}} = \gamma \sigma_{jc} \quad (33h)$$

$$E(r_{j,t+1}) - r_{f,t+1} \approx \gamma \sigma_{jc} \quad (33i)$$

The intuition behind (33i) is clear: the risk premium increases linearly with the covariance between consumption growth and the asset returns. This means that if an asset increases the volatility of future consumption it should pay a higher risk premium. The risk premium will increase linearly in risk aversion as well since more risk averse agents will be less willing to invest in risky assets.

## Part 5: An application for Colombian data

In this section of the article we will test the model described previously with Colombian data in order to understand if it prices assets correctly. We want to infer information both about the demanded volatility of the risk free rate and about the risk aversion coefficient of the agents. The idea is to check for inconsistencies in the model that will motivate an analysis of the equity premium puzzle.

The necessary variables in the database are:

- *Consumption growth*: Quarterly Data from the National Statistics Bureau and the National Planning Department. (Departamento Administrativo Nacional de Estadística – DANE and Departamento Nacional de Planeación – DNP). The data is presented as annualized compound returns.
- *Inflation rate*: Quarterly geometric mean of monthly CPI obtained from DANE. The data is presented as annualized compounded returns.
- *Real risky rate of return*: We use two nominal risky rates of return. The first is the average lending rate asked for by financial institutions obtained from DNP and the Colombian Central Bank (Banco de la República-BanRep). The data is presented as annualized compounded returns. The second rate is the average stock exchange return for each quarter. Information prior to July 2001 is the compound return of the Bogotá Stock Exchange and the Medellín Stock Exchange. Information from July 2001 onward is the return of the General Index of the Colombia Stock Exchange (IGBC). Information is annualized. Both datasets are corrected for inflation in order to present real and not nominal rates. (BanRep) We will compare results on both rates.

- *Real risk free rate of return:* We use the overnight open markets rate determined by BanRep. We take the geometric mean of the end of month rates for the three months in each quarter.

The database starts in 1995 –I and ends in 2009– IV. The full database has a total of 60 observations per time series. Refer to Annex 1 to see a graphical presentation of the data as well as the main statistics for each time series.

The data shows that during the time period inflation decreased significantly from a maximum value of 40.43% for 1995-IV to a one digit value since 2006. This constant decrease in inflation during the 1990s is typical of developing countries, especially Latin American ones, which during the 1980s suffered from hyperinflation, low growth, high fiscal deficits and over-indebtedness.

The way to cope with these meager economic conditions was to increase the overnight lending rate which, in annualized nominal rates, reached a maximum of 45% during the first two quarters of 1995. As economic conditions stabilized during the later part of the 1990s and the first part of the 2000s the overnight lending rate fell continuously reaching minimum levels during the financial crisis between 2007-2009.

A relevant piece of data that has to be discussed is the performance of the two proxies for the risky rate of return. The composed average lending rate for the economy is very strongly correlated with the behavior of the risk free rate (correlation coefficient = 0.928). This is obvious since the risk free rate is one of the main instruments a central bank can use as part of its monetary policy, therefore it permeates the entire economy through the financial system, and is a main component of the monetary channel. This strong correlation however, will motivate abnormal results which will lead us to discard this proxy as a mimetic variable of the behavior of the returns of risky assets.

Another argument against the lending rate as a proxy for the rate of return of risky assets is its very low correlation with the behavior of stock returns. The behavior of the stock market has been used as a proxy for the risky assets since the seminal works of celebrated authors such as Harry Markowitz, Eugene Fama and Rajnish Mehra, as it reflects in a timely and frequent fashion both market sentiment and market expectations. The lending rate, on the other hand, is a more rigid measure that cannot be averaged or composed in an economy-wide indicator since it is contingent both on the lender and the borrower.

A final caveat that can be found when using the lending rate as a proxy for the risky asset return is the similarity between the volatility of this variable (8.54%) and

that of the risk free rate (8.62%). It is nonsensical to assume that the risky asset is less volatile or, statistically as volatile as the risk free asset. It is understood that the assumption in the model is that the volatility of the risk free rate is 0, still from the real data we can see that the two variables are extremely similar and thus cannot be used to represent two very distinct phenomena.

We will use (33g) to solve for  $\gamma$ —the risk aversion parameter—implied in the full database. Since we have two definitions of the risky asset we will have two possibilities for  $\gamma$ . When performing the evaluation with the full database we obtain negative values for  $\gamma$ . If we use the lending rate as a proxy for the risky asset the value of  $\gamma$  is -107.8893. On the other hand, using the stock market real return the value of  $\gamma$  is -94.6156. These results are highly counterintuitive since according to (29b)  $\gamma$  is a risk aversion parameter that, although constant, should be positive for an economy since the assumption underlying the model, and most microeconomic theory, is that agents are risk averse.

A negative risk aversion coefficient states that an agent will have a risk loving utility function regardless of the level of consumption or wealth. Since the second derivative of the utility function presented in Part 4 of this paper is negative it is contradictory that the result in the model would result in a negative risk aversion coefficient. This would mean, for an economy wide database, that agents pay a risk premium in order to expose themselves to additional risks without paying attention to their level of wealth.

The explanation for the negative values of  $\gamma$  is twofold and can be found in the data. In the first case, for the model where the risky rate is modeled with the average lending rate,  $\gamma$  is negative because the covariance between consumption growth and the risky rate of return is negative. On the other hand, for the model where the risky asset is modeled with the stock market return,  $\gamma$  is negative because the risk premium is negative.

This contradicts the results in (Mehra and Prescott 1985), (Mehra and Prescott 1988), (Mehra y Prescott 2003) and (Mehra and Prescott 2008). They state that  $\gamma$  is positive since agents are risk averse. This is explained by the positive relation between consumption growth and the risky rate of return. An economic interpretation is that risk averse agents, which are evidenced in several empirical studies cited by Lengwiller, will increase their rate of investment in risky assets, and thus increase the price of the assets, only when their level of consumption increases. On the other hand it is difficult to sustain a long run negative real rate of return for risky assets. The Colombian case is a unique one given the time frame that was chosen for the data base. As explained above, the high level of the risk free

rate of return –due to inflationary pressures– rendered the real risky rate of return negative. This is explained both by the central bank's monetary policy and by a crowding-out phenomenon generated by high yield government bonds issued in order to finance fiscal deficits.

When evaluating the database we discovered that the economic situation is an atypical one and quite different from the one described in the different articles published by Mehra and Prescott. In that sense it seems sensible to adjust the database in order to have data for a regular economic cycle correcting for all the biases discussed in the first part of this section. For that reason we select the data available from 2000 –I until 2009– IV since we believe it reflects better a typical economic cycle in which we can evidence a stable rate of inflation and the complete economic growth cycle from a recession to economic recovery.

The graphical and statistical information for this cropped database can be found in Annex 2. From this annex we can infer that the lending rate is still subject to all the caveats previously discussed and thus is a bad proxy for the real risky rate of return. However we find positive covariances between consumption growth and the proxies for the return of the risky asset as well as positive risk premiums.

Following (33g) we recalculate  $\gamma$  for this new database and obtain two values. In the first case, when the proxy for the risky rate of return is the average lending rate, we obtain a  $\gamma$  of 694.4929. When we use the real stock market returns we obtain a  $\gamma$  of 36.9859. This new database offers positive risk aversion coefficients that are more in line with what would be expected of an economic model. This data moves also in the same direction as works by other researchers with information from developed economies. However, a single value of  $\gamma$  provides little to no information about the database and it shall be used in the model in order to check for consistency.

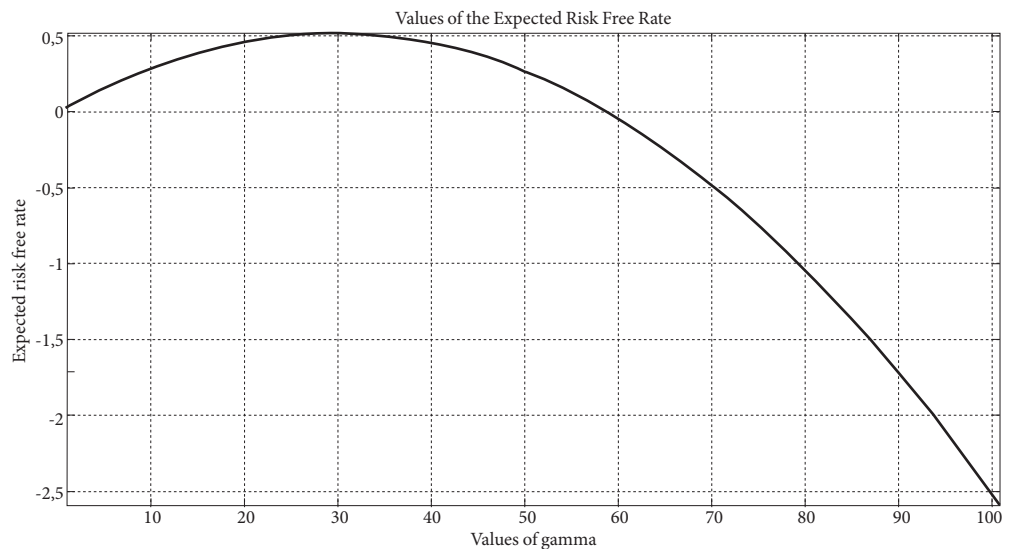
## Part 6: The Equity Premium Puzzle

In order to confirm the usefulness of the values of  $\gamma$  found in the previous section it is important to use the linear version of the power utility CCAPM described in (33e). This equation solves for the expected risk free rate as a function of consumption growth, the volatility of consumption growth and the agent's level of risk aversion.

Following (Marín and Rubio 2001) we will use the inverse of the risk free rate as a subjective discount factor since it is a useful measure to reflect the agent's level of impatience. In future extensions of this work it can be interesting to determine the sensitivity of the result to different values of this parameter.

When using the complete database and the negative values of  $\gamma$  we obtain negative values of the expected risk free rate for both the case where the return of the risky asset is mimicked by the average lending rate and the case when real stock returns serve as a proxy for the risky asset. In the first case we obtain an expected annual risk free rate of -1,245.07% and in the second case the expected annual risk free rate has a value of -994.40%. These results are consistent with a risk loving agent but are not consistent with reality. According to the level of risk aversion this agent would be willing to lose money if he were to shift his portfolio from the risky asset towards a risk free investment. Still, a first approach to the equity premium puzzle can be done from these results since the value of the risk free rate is unusually high given the size of the risk aversion coefficient. This clearly shows that the model, as defined in Part 4 of this paper, does not explain the relation between the risk free rate and the risk aversion parameter.

Figure 1: Expected risk free rate to different values of  $\gamma$ .



In Figure 1 we can find the relation between  $\gamma$  and expected value of the risk free rate. It is noteworthy that the expected risk free rate is positive only for positive values of  $\gamma$  lower than 60. For the case in which we use the reduced database –from 2000– I to 2009 –IV– we obtain an expected risk free rate of -27,078.35% when the average lending rate is used as the risky rate of return. However, when we use the

real stock return as the risky rate of return we obtain a  $\gamma$  of 36.98 and an expected risk free rate of 48.65%. These results are consistent with findings in other papers.

The equity premium puzzle is described as a phenomenon in which the data from the economy, when fitted into an equilibrium model demands an extremely high and volatile risk free rate given the agent's implied risk aversion (Mehra and Prescott 1985). In this sense it is clear that a simple CCAPM model under power utility is unable to explain financial market data. A very positive value of the risk aversion parameter implies that agents are unwilling to perform intertemporal substitution in consumption and that in order to explain the small variations in consumption they would demand extremely high interest rates.

Possible solutions to this issue have been thoroughly discussed in financial and economic literature. Firstly a wide array of utility functions have been discussed in which elements such as dependence of intertemporal consumption, habit formation, and time varying risk aversion are topics of discussion. Another discussion has been directed towards the definition of the consumption variable which does not need to be consumption growth but some other variable such as gross investment growth or other relevant economic and market information. Finally it has been discussed that there might be nonlinear relations between consumption and the risky rate of return.

## Part 7: Concluding Remarks

In this paper we have found that in a simple equilibrium model the stochastic discount factor –defined as the intertemporal rate of substitution– should be able to explain the changes in financial market data for a simple data base. We also found, at the end of Part 3, that the SDF if not unique, must satisfy certain restrictions in order to be a valid SDF that will price assets correctly. Further extensions of this work may be directed into verifying these conditions, and verifying if such a measure can be used to predict future economic conditions. From said model, under a power utility specification we inferred a risk aversion parameter and associated it with an expected risk free rate of return.

When analyzing information for the Colombian economy we found several issues that made the process difficult. First the database fell short of expectations as it covered a time period where the risk free rate was extremely volatile, consumption growth was meager, and inflation was high. This atypical and unsustainable economic situation was not explained by the model which showed that agents

under these economic conditions would have to be risk lovers in order to explain the changes in consumption.

When adjusting the database for it to better resemble a developed economy –insofar as it manifests stable levels of inflation, low risk free rates and positive growth in consumption– we find that the average lending rate for the economy is not a good proxy variable for the real return of the risky asset since it is very strongly correlated with the risk free rate and negatively correlated with consumption growth.

Finally, when the adjusted database was fitted into the model using the average real stock market return as the risk free rate we identified that the equity premium puzzle as described by Mehra and Prescott, does exist in Colombia and that the existence of a high and positive risk aversion parameter would imply the existence of an equally high risk free rate of return.

This paper is the basis of a research project that will perform several tests on Colombian economic data looking for a solution to the Equity Premium Puzzle. It serves the simple purpose of presenting the theoretical background of the puzzle and proves that there is evidence of its existence in Colombia. Future extensions of this paper include a literature review of the main works done by researchers worldwide on the topic as well as employing different specifications of utility functions in order to better grasp the origin of the puzzle.



# Annex 1: Graphical and statistical remarks in the data 1995-2009

Figure 2: Annualized Quarterly CPI

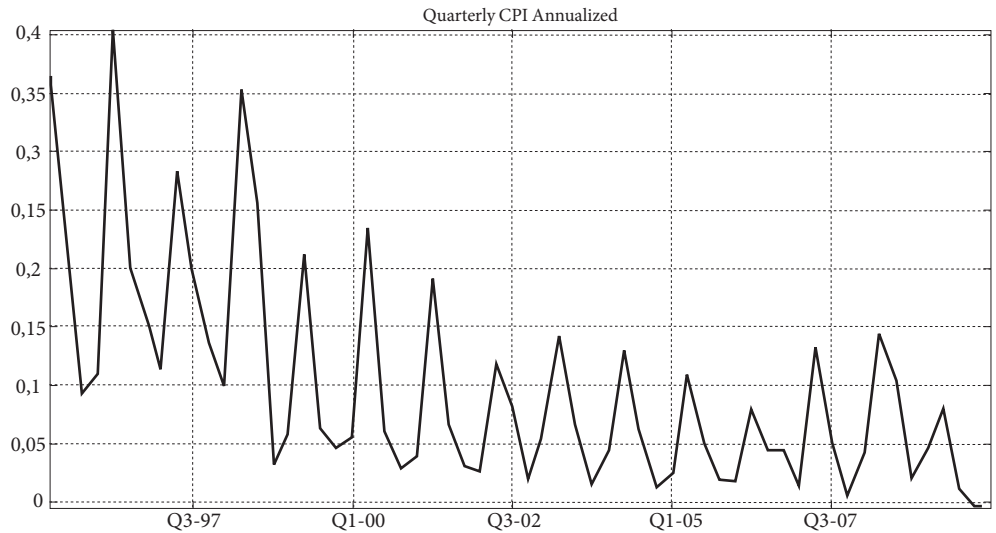


Figure 3: Consumption Growth and Real Rates of Return – Full Sample.

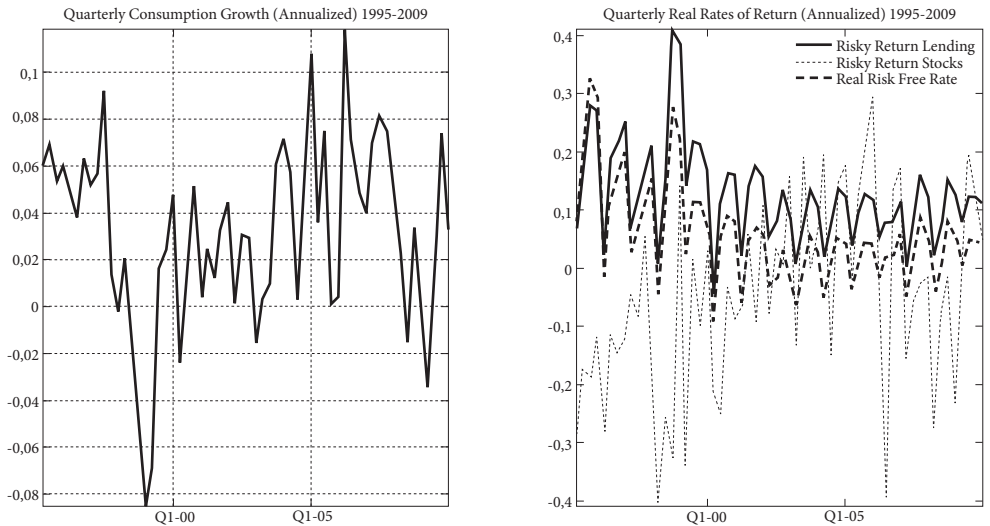


Table 1: Statistical Description Full Sample

	<b>Real Consumption Growth</b>	<b>Risk Rate (Lending Rate)</b>	<b>Risky Rate (Stock indexes)</b>	<b>Real Risk Free Rate</b>	<b>CPI</b>
Minimum	-8,51%	-4,93%	-40,91%	-9,22%	-0,48%
Maximum	11,86%	41,11%	29,70%	32,62%	40,43%
Mean	3,07%	12,52%	-4,72%	5,49%	9,86%
Standard Deviation	3,95%	8,54%	16,60%	8,62%	9,44%

Table 2: Variance – Covariance Matrix

<b>Variance-Covariance Matrix</b>	<b>Real Consumption Growth</b>	<b>Risky Rate (Lending Rate)</b>	<b>Risky Rate (Stock indexes)</b>	<b>Real Risk Free Rate</b>	<b>CPI</b>
Real Consumption Growth	0,00156	-0,00069	0,00093	-0,00009	-0,00009
Risky Rate (Lending Rate)	-0,00069	0,00729	0,00034	0,00683	-0,00257
Risky Rate (Stock indexes)	0,00093	0,00034	0,02754	-0,00115	-0,00905
Real Risk Free Rate	-0,00009	0,00683	-0,00115	0,00744	-0,00140
CPI	-0,00009	-0,00257	-0,00905	-0,00140	0,00890

Table 3: Correlation Coefficient Matrix

<b>Variance-Coefficient Matrix</b>	<b>Real Consumption Growth</b>	<b>Risky Rate (Lending Rate)</b>	<b>Risky Rate (Stock indexes)</b>	<b>Real Risk Free Rate</b>	<b>CPI</b>
Real Consumption Growth	1,000	-0,203	0,142	-0,025	-0,024
Risky Rate (Lending Rate)	-0,203	1,000	0,024	0,928	-0,320
Risky Rate (Stock indexes)	0,142	0,024	1,000	-0,080	-0,578
Real Risk Free Rate	-0,025	0,928	-0,080	1,000	-0,172
CPI	-0,024	-0,320	-0,578	-0,172	1,000

## Annex 2: Graphical and statistical remarks in the data 2000-2009

Figure 4: Consumption Growth and Real Rates of Return

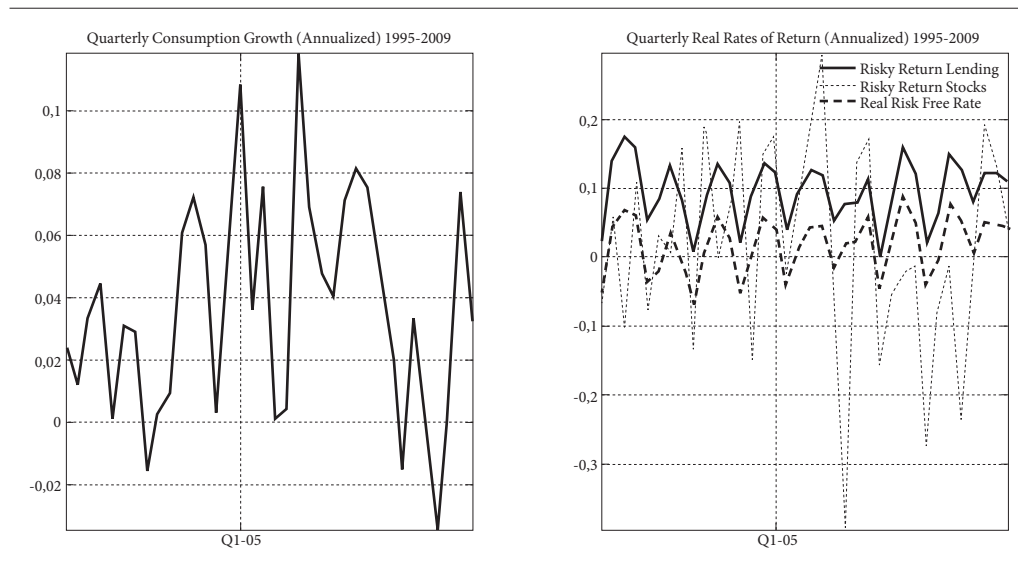


Table 4: Statistical Description of the Database

	Real Consumption Growth	Risk Rate (Lending Rate)	Risky Rate (Stock indexes)	Real Risk Free Rate	CPI
Minimum	-3,47%	0,06%	-39,68%	-6,93%	-0,48%
Maximum	11,86%	17,49%	29,70%	8,73%	19,18%
Mean	3,63%	9,44%	1,83%	1,78%	5,80%
Standard Deviation	3,50%	4,61%	15,15%	4,24%	4,85%

Table 5: Variance – Covariance Matrix

Variance-Covariance Matrix	Real Consumption Growth	Risky Rate (Lending Rate)	Risky Rate (Stock indexes)	Real Risk Free Rate	CPI
Real Consumption Growth	0,00123	0,00011	0,00032	0,00023	-0,00035
Risky Rate (Lending Rate)	0,00011	0,00212	0,00213	0,00187	-0,00196
Risky Rate (Stock indexes)	0,00032	0,00213	0,02296	0,00172	-0,00293
Real Risk Free Rate	0,00023	0,00187	0,00172	0,00180	-0,00188
CPI	-0,00035	-0,00196	-0,00293	-0,00188	0,00235

Table 6: Correlation Coefficient Matrix

Variance-Coefficient Matrix	Real Consumption Growth	Risky Rate (Lending Rate)	Risky Rate (Stock indexes)	Real Risk Free Rate	CPI
Real Consumption Growth	1,000	0,069	0,061	0,156	-0,024
Risky Rate (Lending Rate)	0,069	1,000	0,305	0,958	-0,876
Risky Rate (Stock indexes)	0,061	0,305	1,000	0,268	-0,398
Real Risk Free Rate	0,156	0,958	-0,268	1,000	-0,914
CPI	-0,204	-0,876	-0,398	-0,914	1,000

### Annex 3: MATLAB® Code

```
data=importdata('ColombianDataBase.txt');
dates=data(:,1)+693960;
data=data(:,2:end);
names=['Consumption Growth ','Risky Return Lending','Risky Return Stocks','Real Risk Free Rate','Quarterly CPI Annual'];
```

*Part 1: Full sample*

```
StandardDeviation=std(data);
MeanDatabase=mean(data);
CovarianceMatrix=cov(data);
CorrelationMatrix=corr(data);
GammaActive=(MeanDatabase(2)-MeanDatabase(4)+(0.5*(CovarianceMatrix(2,2)))/(CovarianceMatrix(1,2)));
GammaStocks=(MeanDatabase(3)-MeanDatabase(4)+(0.5*(CovarianceMatrix(3,3)))/(CovarianceMatrix(1,3)));
RFActive=log(1/(1+(MeanDatabase(4))))+(GammaActive*MeanDatabase(1))-(0.5*(GammaActive^2)*CovarianceMatrix(1,1));
RFStocks=log(1/(1+(MeanDatabase(4))))+(GammaStocks*MeanDatabase(1))-(0.5*(GammaStocks^2)*CovarianceMatrix(1,1));
```

*Part 2: 2000-2009*

```
dates2=dates(25:end,1);
data2=data(25:end,:);
```

*The rest is the same as Part 1.*

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