A CRITICAL SET OF REVOLUTION SURFACE PARAMETRIZATIONS

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ABSTRACT. Given the classical rational parametrization of a surface of revolution, generated by rotating a rational curve around the z-axis, we determine a superset containing all the possible points of the surface non-reachable by the parametrization; that is a critical set of the parametrization.

1. INTRODUCTION

In some applications of algebraic surfaces, the use of their rational parametric forms to solve certain problems is only meaningful when certain properties are satisfied. For example, in order to intersect two surfaces, one given by its implicit equation and the other by a rational parametrization, substitution of the latter into the former will not provide all the points in the intersection if the parametrization is not surjective (or *normal* as called in this context [2, 3, 5]) and any point outside its image happens to lie in the intersection.

The problem of deciding whether a given surface parametrization is normal can be solved computationally by applying elimination techniques, but this is in general hindered by efficiency issues. On the other hand, for certain types of surfaces it is indeed possible to characterize the set of missing points; in particular, one can compute a *critical set*, that is, a proper closed subset of the surface that contains them. This problem has been analyzed by the authors in [7] and [6] for ruled surfaces and for parametrizations without projective base points. This article solves the problem for surfaces of revolution.

As we show later, the critical set that we calculate is parametrizable (it is a union of lines, circles, and possibly a point), so the algebraic surface can be covered by a union of finitely many parametrizations: the initial two-dimensional and several one-dimensional ones.

In the sequel, let $\mathcal{C}$ be a rational curve (called *rotation curve*) in the (y, z)-plane parametrized by $r(t) = (0, p(t), q(t))$, where $p(t), q(t) \in \mathbb{C}(t)$. Also let $\mathcal{S}$ be the surface of revolution generated by rotating $\mathcal{C}$ around the z-axis. For simplicity in the exposition we will assume that $\mathcal{C}$ is not a line parallel to the y axis, and hence $\mathcal{S}$ is not a plane. The classical parametrization of $\mathcal{S}$, obtained from $r(t)$, is

$$\mathcal{P}(s, t) = \left(\frac{2s}{1 + s^2} \, p(t), \frac{1 - s^2}{1 + s^2} \, p(t), q(t) \right).$$

We denote any critical set of $\mathcal{P}$ by $\mathrm{Crit}(\mathcal{P})$.

A complete version of this extended abstract will be submitted to a journal.

2. MAIN RESULT

Let $\mathcal{P}(s, t)$, $\mathcal{C}$ and $\mathcal{S}$ be as in Section 1. Let $f(y, z)$ be the defining polynomial of the rotation curve $\mathcal{C}$. In [4] a description of the defining polynomial of $\mathcal{S}$ is given in terms of f. Collecting terms of odd and even degree in y we can write $f(y, z)$ as $f(y, z) = A(y^2, z) + yB(y^2, z)$.

The implicit equation of $\mathcal{S}$ depends on whether B is zero or not (i.e. f is symmetric with respect to the z-axis or not):

- $F(x, y, z) = A(x^2 + y^2, z)$ in case that $B = 0$.
- $F(x, y, z) = A^2(x^2 + y^2, z) - (x^2 + y^2)B^2(x^2 + y^2, z)$ in other case.

We consider now the rational curve $\mathcal{C}^s$ (called *mirror curve of* $\mathcal{C}$) defined as

$$\mathcal{C}^s = \{(0, -y, z) \in \mathbb{C}^3 \,/\, (0, y, z) \in \mathcal{C}\}.$$

Its implicit equation is $0 = g(y, z) = f(-y, z)$ and it is parametrized as $\gamma(s) = (0, -p(s), q(s))$. When $\mathcal{C}$ is not symmetric with respect to the z-axis, $\mathcal{C}^s$ is not reachable by $\mathcal{P}$ as we will see.

We represent by $\mathcal{C}_{\alpha,c}$ the circle of radius α in the plane $z = c$ centered at $(0, 0, c)$:

$$\mathcal{C}_{\alpha,c} = \{(x, y, z) \in \mathbb{C}^3 \,/\, x^2 + y^2 = \alpha^2, z = c\}.$$

In the next lemma we study the level curves of $\mathcal{S}$.

Lemma 2.1. *The intersection of $\mathcal{S}$ with the plane $z = c$ is either empty, or a finite union of circles $\mathcal{C}_{\alpha,c}$ with $\alpha \neq 0$, or the pair of lines $\{x \pm iy = 0, \ z = c\}$. Moreover, if $(x_0, y_0, z_0) \in \mathcal{S}$ where $x_0^2 + y_0^2 = \alpha^2 \neq 0$, then $\mathcal{C}_{\alpha,z_0} \subset \mathcal{S}$.*

When a point $P \in \mathcal{C}$ rotates around the z-axis it generates a circle in $\mathcal{S}$ except when P belongs to the rotation axis. In the following lemmas we analyze these cases.

Lemma 2.2. *Let $P = r(t_0) = (0, p(t_0), q(t_0))$ with $p(t_0) \neq 0$, i.e. $P \in \mathcal{C}$ is reachable by $r(t)$ and is not on the revolution axis. Then $\mathcal{C}_{p(t_0),q(t_0)}$, obtained by rotating P around the z-axis, is reachable by $\mathcal{P}$, except possibly the symmetric point $P_s = (0, -p(t_0), q(t_0)) \in \mathcal{C}^s$.*

Lemma 2.3. *The following statements are equivalent:*

1. $(0, 0, z_0) \in \mathcal{C}$,
2. $\mathcal{S}$ contains the lines $\{x \pm iy = 0, z = z_0\}$.

Moreover,

(i) *If $(0, 0, z_0) \in \mathcal{C}$, the lines $\{x \pm iy = 0, z = z_0\}$ are not reachable by $\mathcal{P}$, except possibly the point $(0, 0, z_0)$.*
(ii) *If $(x_0, y_0, z_0) \in \mathcal{S}$ with $x_0^2 + y_0^2 = 0$, then $(0, 0, z_0) \in \mathcal{C}$.*
(iii) *If $(x_0, y_0, z_0) \in \mathcal{S}$ with $x_0^2 + y_0^2 = 0$, then $\{x \pm iy = 0, z = z_0\} \subset \mathcal{S}$.*

Lemma 2.4. *Let $(x_0, y_0, z_0) \in \mathcal{S}$. Then $P^+ = (0, \sqrt{x_0^2 + y_0^2}, c)$ or $P^- = (0, -\sqrt{x_0^2 + y_0^2}, c)$ belongs to $\mathcal{C}$.*

Lemma 2.5. *Let $\mathcal{C}_{\alpha,c} \subset \mathcal{S}$, with $\alpha \neq 0$, and let $P_1 = (0, \alpha, c)$, $P_2 = (0, -\alpha, c)$. The following statements are equivalent:*

1. $\mathcal{C}_{\alpha,c}$ *contains at least one point reachable by $\mathcal{P}$.*
2. $\mathcal{C}_{\alpha,c}$ *is reachable by $\mathcal{P}$ except, at most, one of the points P_i.*
3. *One of the points P_i is reachable by $r(t)$.*

The next theorem describes a critical set of the surface $\mathcal{S}$. This description is general; nevertheless, as we will see in the proof, in some situations the critical set can be optimized (see Remark 2.7). We speak about the critical point of the curve parametrization $r(t)$ in the sense of [5]: the only point on the Zariski closed curve that might not be reachable by $r(t)$.

Theorem 2.6. *A critical set for $\mathcal{P}$ is*

$$\mathrm{Crit}(\mathcal{P}) = \bigcup_{i=1}^{k} \{x \pm i\, y = 0, z = \lambda_i\} \cup \mathcal{C}^s \cup \mathcal{C}_{\rho,c} \quad \text{where}$$

- $\mathcal{C} \cap \{x = y = 0\} = \{(0, 0, \lambda_i)\}_{i=1,\dots,k}$,
- $P^* = (0, \rho, c)$, *with $\rho \neq 0$, is the critical point of the parametrization $r(t)$.*

Proof. Let $N = (x_0, y_0, z_0) \in \mathcal{S} \setminus \mathcal{P}(\mathbb{C}^2)$. If $x_0^2 + y_0^2 = 0$, by Lemma 2.3, $N \in \{x \pm i\, y = 0, z = z_0\} \subset \mathrm{Crit}(\mathcal{P})$. Let $x_0^2 + y_0^2 = \alpha^2 \neq 0$. By Lemma 2.1, $\mathcal{C}_{\alpha,z_0} \subset \mathcal{S}$, and by Lemma 2.4 $P^+ = (0, \alpha, z_0)$ or $P^- = (0, -\alpha, z_0)$ belongs to $\mathcal{C}$. We now distinguish two cases:

(1) Assume $P^\pm \in \mathcal{C}$. Note that, by [5, Theorem 2], al least one of them is reachable by $r(t)$, and hence by $\mathcal{P}(s, t)$. Assume w.l.o.g. that P^+ is reachable by $r(t)$. By Lemma 2.5, $\mathcal{C}_{\alpha,z_0}$ is reachable by $\mathcal{P}$ with the possible exception of P^-. Since $N \in \mathcal{C}_{\alpha,z_0}$ and it is non-reachable, then $P^- = N$. Moreover, $P^* = P^-$. Thus, $N \in \mathcal{C}^s \subset \mathrm{Crit}(\mathcal{P})$.
(2) Assume either P^+ or P^- belong to $\mathcal{C}$. Assume also w.l.o.g. that $P^+ \in \mathcal{C}$:
  (a) If $P^+ = r(t_0)$, by Lemma 2.2, $\mathcal{C}_{\alpha,z_0}$ is reachable except at $P^- = N$. So, $N \in \mathcal{C}^s \subset \mathrm{Crit}(\mathcal{P})$.
  (b) If P^+ is not reachable by $r(t)$, by Lemma 2.5, $N \in \mathcal{C}_{\alpha,z_0} \subset \mathrm{Crit}(\mathcal{P})$. $\qquad\square$

Remark 2.7. Taking into account the reasoning in the last proof, we derive the following process to optimize the critical set.

(1) Let $\mathrm{Crit}(\mathcal{P}) = \emptyset$.
(2) Compute the intersections of $\mathcal{C}$ with the z-axis. If $\mathcal{C} \cap \{x = y = 0\} = \{(0, 0, \lambda_i)\}_{i=1,\dots,k}$ then include $\bigcup_{i=1}^{k} \{x \pm i\, y = 0, z = \lambda_i\}$ in $\mathrm{Crit}(\mathcal{P})$.
(3) Check whether $\mathcal{C}$ is symmetric with respect to the z-axis.
  (a) If $\mathcal{C}$ is symmetric check whether $r(t)$ is normal. If $r(t)$ is normal then RETURN $\mathrm{Crit}(\mathcal{P})$, else include the missing point of $r(t)$ in $\mathrm{Crit}(\mathcal{P})$ and RETURN $\mathrm{Crit}(\mathcal{P})$.
  (b) If $\mathcal{C}$ is not symmetric then include $\mathcal{C}^s$ in $\mathrm{Crit}(\mathcal{P})$ and check whether $r(t)$ is normal. If $r(t)$ is normal then RETURN $\mathrm{Crit}(\mathcal{P})$, else
    (i) compute the missing point $(0, a, b)$ of $r(t)$ and if $a = 0$ then RETURN $\mathrm{Crit}(\mathcal{P})$,
    (ii) if $(0, -a, b) \in \mathcal{C}$ then RETURN $\mathrm{Crit}(\mathcal{P})$, else include $\mathcal{C}_{a,b}$ in $\mathrm{Crit}(\mathcal{P})$ and RETURN $\mathrm{Crit}(\mathcal{P})$.

Remark 2.8. To check the normality of $r(t)$ one can apply Theorem 2 in [5]. To check symmetry one can use the ideas in [1] or compute the implicit equation $f(y, z)$ of $\mathcal{C}$ and check whether $f(-y, z) = f(y, z)$.

3. EXAMPLES

We illustrate our results with two examples.

Example 3.1. The *pear–shaped curve* $\mathcal{C}$ given by $x = z^2 - y^3(1 - y) = 0$ is parametrized as

$$r(t) = (0, p(t), q(t)) = \left(0, \frac{16t^2 + 8t + 1}{17t^2 + 32t + 145}, -\frac{64t^4 + 816t^3 + 588t^2 + 145t + 12}{289t^4 + 1088t^3 + 5954t^2 + 9280t + 21025}\right).$$

It holds that $\mathcal{C} \cap \{x = y = 0\} = \{(0,0,0)\}$, $\mathcal{C}$ is not symmetric with respect to the z-axis, and $r(t)$ is not normal (the point $\left(0, \frac{16}{17}, -\frac{64}{289}\right)$ is not reachable). Taking into account Remark 2.7, a critical set for the usual parametrization of the revolution surface generated by $r(t)$ is

$$\mathrm{Crit}(\mathcal{P}) = \{x \pm i\,y = 0, z = 0\} \cup \mathcal{C}^s \cup \mathcal{C}_{\frac{16}{17}, -\frac{64}{284}}.$$

On the other hand we can rotate $\mathcal{C}$ around the y-axis; for simplicity, consider the curve obtained by rotating $\mathcal{C}$ an angle $\pi/2$ in the yz-plane. The new curve, of shape similar to that of $\mathcal{C}$, has equations $\{x = 0, y^2 = z^3(1-z)\}$, and can be parametrized as $r(t) = (0, q(t), p(t))$. In this case $\mathcal{C} \cap \{x = y = 0\} = \{(0,0,0), (0,0,1)\}$, $\mathcal{C}$ is symmetric with respect to the z-axis, and $r(t)$ is not normal, being $\left(0, -\frac{64}{289}, \frac{16}{17}\right)$ its non-reachable point. Then

$$\mathrm{Crit}(\mathcal{P}) = \{x \pm i\,y = 0, z = 0\} \cup \{x \pm i\,y = 0, z = 1\} \cup \left\{\left(0, -\frac{64}{289}, \frac{16}{17}\right)\right\}.$$

Example 3.2. *Kulp's quartic* $\mathcal{C}$ has equations $\{x = 0, y^2(1+z^2) = 1\}$ and parametrization

$$r(t) = \left(0, -\frac{3t^2 + 4t + 1}{5t^2 + 4t + 1}, \frac{2(2t+1)t}{3t^2 + 4t + 1}\right).$$

In this case $\mathcal{C} \cap \{x = y = 0\} = \{\emptyset\}$, $\mathcal{C}$ is symmetric with respect to the z-axis, and $r(t)$ is not normal (the point $\left(0, -\frac{3}{5}, \frac{4}{3}\right)$ is not reachable). Then $\mathrm{Crit}(\mathcal{P}) = \left\{\left(0, -\frac{3}{5}, \frac{4}{3}\right)\right\}$.

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