# Detecting Similarity of Plane Rational Curves

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**Abstract**

A novel and deterministic algorithm is presented to detect whether two given planar rational curves are related by means of a similarity, which is a central question in Pattern Recognition. As a by-product it finds all such similarities, and the particular case of equal curves yields all symmetries. A complete theoretical description of the method is provided, and the method has been implemented and tested in the Sage system.

**Note of the authors: The final version of this paper was published as "Juan Gerardo Alcázar, Carlos Hermoso, Georg Muntingh: Detecting similarity of rational plane curves. Journal of Computational Applied Mathematics 269: 1-13 (2014)."**

## 1. Introduction

A central problem in Pattern Recognition and Computer Vision is to detect whether a given object corresponds to one of the objects stored in a database. In the case of geometric objects, like the curves considered in this paper, the two objects to be compared need not be in the same position, orientation, or scale. To compare the two objects, it should therefore be checked whether there exists a nontrivial movement, also called *similarity*, transforming one into the other. This problem, known in the Computer Vision literature as *pose estimation*, has been extensively considered using many different techniques, using B-splines [10], Fourier descriptors [18], complex representations [21], statistics [9, 13, 14] in the 3D case, moments [20, 22], geometric invariants [23, 25, 26], Newton-Puiseux parametrizations [15], and differential invariants [3, 4, 24]. The

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[1] Supported by the Spanish "Ministerio de Ciencia e Innovacion" under the Project MTM2011-25816-C02-01. Partially supported by a José Castillejos' grant from the Spanish Ministerio de Educación, Cultura y Deporte. Member of the Research Group ASYNACS (Ref. CCEE2011/R34)

[2] Partially supported by the Giner de los Ríos grant from the Universidad de Alcalá.

interested readers may consult the bibliographies in these papers to find other references on the matter.

However, with exception for the references concerning the B-splines and differential invariants, the above methods use the implicit form of the curves. Moreover, the above methods are either numerical, or only efficient when considered in a numerical setting. The reason for this is that in Pattern Recognition it is often assumed that the inputs are "fuzzy". For instance, in many cases the objects are represented discretely as point clouds. In this situation, an implicit equation is usually first computed for each cloud, and the comparison is performed later. In other cases there may be occluded parts or noise in the input. Finally, in some other cases the input might be exact, but modeling a real object only up to a certain extent. In all of these cases we do not need an perfect matching, so that a numerical comparison is sufficient.

In this paper, we address the problem from a perspective that differs in two ways. First of all, we assume that the curves are given in *exact* arithmetic, so that we can provide a deterministic answer to the question whether these two curves are similar. If so, our algorithm will find the similarities transforming one into the other. A potential application of this is the following. Assume that a database with classical curves is stored in your favourite computer algebra system. Using the algorithm in this paper, the system can recognize a certain curve introduced by the user as one of the curves in the database, and provide that information to the user. A second difference is that we start from rationally *parametrized* curves and carry out all computations in the parameter space. As a consequence, the cost of converting to the implicit form, both in terms of computing time and growth of the coefficients, is avoided. Notice that rational curves remain ubiquitous in Computer-Aided Geometric Design, lying at the very core of (rational) Bézier curves and splines.

We exploit and generalize some ideas used in [1, 2] for the computation of symmetries of rational planar curves, improving the algorithm provided in [2]. Our approach exploits the rational parametrizations to reduce to calculations in the parameter domain, and therefore to operations on univariate polynomials. Thus we proceed symbolically to determine the existence and computation of such similarities, using basic polynomial multiplication, gcd-computations. Additionally, if a numerical approximation is desired, univariate polynomial real-solving must be used as well. We have implemented and tested this algorithm in the Sage system [19]. It is also worth mentioning that, as a by-product, we achieve an algorithm to detect whether a given rational curve is symmetric and to find to find such symmetries. This problem has previously been studied from a deterministic point of view [1, 2, 5, 11], and by many authors from an approximate point of view.

The structure of the paper is the following. Some generalities on similarities and symmetries, to be used throughout the paper, are established in Section 2. The method itself is addressed in Section 3, first for polynomially parametrized curves and then for the general case. Finally, practical details on the implementation, including timings, are provided in Section 4.

## 2. Similarities and symmetries

Throughout the paper, we consider rational plane algebraic curves $\mathcal{C}_1, \mathcal{C}_2 \subset \mathbb{R}^2$ that are neither a line nor a circle. Such curves are irreducible and can be parametrized by rational maps

$$\phi_j : \mathbb{R} \dashrightarrow C_j \subset \mathbb{R}^2, \qquad \phi_j(t) = \big(x_j(t), y_j(t)\big), \qquad j = 1, 2. \tag{1}$$

The components $x_j, y_j$ of $\phi_j$ are rational functions of $t$, and they are defined for all but a finite number of values of $t$. We assume that the parametrizations (1) are *proper*, in the sense that they are injective, except for perhaps finitely many values of $t$. This is no restriction on the curves $\mathcal{C}_1, \mathcal{C}_2$, as any rational curve always admits a proper parametrization. For a proof of this claim and a thorough study on properness, see [17].

Roughly speaking, an *(affine) similarity* of the plane is a linear affine map from the plane to itself that preserves ratios of distances. More precisely, a map $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is a similarity if $f(x) = Ax + b$ for an invertible matrix $A \in \mathbb{R}^{2\times 2}$ and a vector $b \in \mathbb{R}^2$, and there exists an $r > 0$ such that

$$\|f(x) - f(y)\|_2 = r\|x - y\|_2, \qquad x, y \in \mathbb{R}^2,$$

where $\|\cdot\|_2$ denotes the Euclidean norm. We refer to $r$ as the *ratio* of the similarity. Notice that if $r = 1$ then $f$ is an *(affine) isometry*, in the sense that it preserves distances. It is well known, and easy to derive, that a similarity can be decomposed into a translation, an orthogonal transformation, and a uniform scaling by its ratio $r$. The curves $\mathcal{C}_1, \mathcal{C}_2$ are *similar*, if one is the image of the other under a similarity.

For analyzing the similarities of the plane, we have found it useful to identify the Euclidean plane with the complex plane. Through this correspondence $(x, y) \simeq x + iy$, the parametrizations (1) correspond to parametrizations

$$z_j : \mathbb{R} \dashrightarrow C_j \subset \mathbb{C}, \qquad z_j(t) = x_j(t) + iy_j(t), \qquad j = 1, 2. \tag{2}$$

We can distinguish two cases for a similarity of the complex plane. A similarity $f$ is either *orientation preserving*, in which case it takes the form $f(z) = \boldsymbol{a}z + \boldsymbol{b}$, or *orientation reversing*, in which case it takes the form $f(z) = \boldsymbol{a}\overline{z} + \boldsymbol{b}$. In each case, its ratio $r = |\boldsymbol{a}|$.

A *Möbius transformation* is a rational function

$$\varphi : \mathbb{R} \dashrightarrow \mathbb{R}, \qquad \varphi(t) = \frac{\alpha t + \beta}{\gamma t + \delta}, \qquad \Delta := \alpha\delta - \beta\gamma \neq 0. \tag{3}$$

It is well known that the birational functions on the line are the Möbius transformations [17].

**Theorem 1.** *Let $\mathcal{C}_1, \mathcal{C}_2 \subset \mathbb{C}$ be rational plane curves with proper parametrizations $z_1, z_2 : \mathbb{R} \dashrightarrow \mathbb{C}$. Then $\mathcal{C}_1$, $\mathcal{C}_2$ are similar if and only if there exists a* *similarity $f$ and a Möbius transformation $\varphi$ for which we have a commutative diagram*

$$\begin{tikzcd} \mathbb{C} \arrow[r, "f"] & \mathbb{C} \\ \mathbb{R} \arrow[u, dashed, "z_1"] \arrow[r, dashed, "\varphi"'] & \mathbb{R} \arrow[u, dashed, "z_2"'] \end{tikzcd} \tag{4}$$

*Moreover, if $\mathcal{C}_1, \mathcal{C}_2$ are similar by a similarity $f$, then there exists a unique Möbius transformation $\varphi$ that makes the above diagram commute.*

*Proof.* Suppose that $\mathcal{C}_1$ and $\mathcal{C}_2$ are similar. Then there exists a similarity $f$ of the plane that restricts to a bijection $f : \mathcal{C}_1 \longrightarrow \mathcal{C}_2$. Since $z_1, z_2$ are proper, their inverses $z_1^{-1}, z_2^{-1}$ exist. The composition $z_2^{-1} \circ f \circ z_1$ is a rational function with inverse $z_1^{-1} \circ f^{-1} \circ z_2$, implying that it must be a Möbius transformation $\varphi$. Clearly this choice of $f$ and $\varphi$ makes the diagram (4) commute.

Conversely, suppose we are given a similarity $f(z)$ of the plane and a Möbius transformation $\varphi$ that makes the diagram (4) commute. The similarity $f$ maps any point $z \in \mathcal{C}_1$ to $f(z) = z_2 \circ \varphi \circ z_1^{-1}(z)$, which lies in the image of $z_2$ and therefore on the curve $\mathcal{C}_2$. It follows that $\mathcal{C}_1$ and $\mathcal{C}_2$ must be similar.

For the final claim, suppose that there are two Möbius transformations $\varphi_1, \varphi_2$ making the diagram (4) commute. Then $z_2\big(\varphi_1(t)\big) = f\big(z_1(t)\big) = z_2\big(\varphi_2(t)\big)$. Since the parametrization $z_2$ is proper, we conclude $\varphi_1 = \varphi_2$. $\square$

Although the final claim refers to a Möbius transformation $\varphi$ that is unique, its coefficients, i.e., $\alpha, \beta, \gamma, \delta$ in (3), are only defined up to a common multiple. Since $\varphi$ maps the real line to itself, these coefficients can always be assumed to be real by dividing by a common complex number if necessary.

Depending on whether $f$ preserves or reverses the orientation of the complex plane, the diagram (4) will either take the form

$$z_2\big(\varphi(t)\big) = \boldsymbol{a} z_1(t) + \boldsymbol{b}, \tag{5}$$

or

$$z_2\big(\varphi(t)\big) = \boldsymbol{a}\overline{z_1(t)} + \boldsymbol{b}. \tag{6}$$

Next, let $\mathcal{C}$ be a plane rational curve, neither a line nor a circle, given by a proper rational map

$$\phi : \mathbb{R} \dashrightarrow \mathcal{C} \subset \mathbb{R}^2, \qquad \phi(t) = \big(x(t), y(t)\big).$$

A *self-similarity* $f$ of $\mathcal{C}$ is a similarity of the plane satisfying $f(\mathcal{C}) = \mathcal{C}$.

**Proposition 2.** *Let $\mathcal{C}$ be an algebraic curve that is not a union of (possibly complex) concurrent lines. Any self-similarity $f$ of $\mathcal{C}$ is an isometry.*

*Proof.* Writing $z = x + iy$, the curve $\mathcal{C}$ is the zeroset of a polynomial

$$G(z, \bar{z}) = \sum_{k=0}^{d} G_k(z, \bar{z}),$$

where $G_k(z,\bar{z})$ is homogeneous of degree $k$ and $G_d(z,\bar{z})$ is nonzero. Moreover, since $\mathcal{C}$ is not the union of concurrent lines, at least one other form $G_l(z,\bar{z})$, with $0 \leq l \leq d-1$, must be nonzero.

Let us first consider the case that $f(z) = \boldsymbol{a}z + \boldsymbol{b}$ is an orientation-preserving symmetry. Suppose that $f$ is not an isometry, i.e., $|\boldsymbol{a}| \neq 1$. Then $f$ has a unique fixed point $\boldsymbol{b}/(1-\boldsymbol{a})$, which we can w.l.o.g. assume to be located at the origin by applying an affine change of coordinates if necessary. Hence, in these coordinates $\mathcal{C}$ has the symmetry $f(z) = \boldsymbol{a}z$, so that it is also the zeroset of $G(\boldsymbol{a}z, \bar{\boldsymbol{a}}\bar{z})$, which therefore must be a scalar multiple of $G(z,\bar{z})$. But then $\boldsymbol{a}^i\bar{\boldsymbol{a}}^{l-i} = \boldsymbol{a}^j\bar{\boldsymbol{a}}^{d-j}$ for some $i$ and $j$, implying $|\boldsymbol{a}|^l = |\boldsymbol{a}|^d$ and therefore contradicting our assumption that $|\boldsymbol{a}| \neq 1$.

Next consider the case that $f(z) = \boldsymbol{a}\bar{z} + \boldsymbol{b}$ is an orientation-reversing symmetry. Then it also has the (possibly trivial) orientation-preserving symmetry $f\big(f(z)\big) = \boldsymbol{a}\bar{\boldsymbol{a}}z + \boldsymbol{b} + \boldsymbol{a}\bar{\boldsymbol{b}}$, which must be an isometry by the above paragraph. It follows that $|\boldsymbol{a}|^2 = 1$ and therefore that $f$ must be an isometry. $\square$

The isometries of the plane form a group under composition, which consists of *reflections* $f(z) = \mathrm{e}^{i\theta}\big(\overline{z-\boldsymbol{b}}\big) + \boldsymbol{b}$, which reflect in the line $\Im\big(\mathrm{e}^{-i\theta/2}z - \mathrm{e}^{-i\theta/2}\boldsymbol{b}\big) = 0$, *rotations* $f(z) = \mathrm{e}^{i\theta}(z-\boldsymbol{b}) + \boldsymbol{b}$, which rotate around a point $\boldsymbol{b}$, *translations* $f(z) = z + \boldsymbol{b}$, and *glide reflections*, which are a composition of a reflection and a translation [6].

An isometry of the plane leaving $\mathcal{C}$ invariant, is more commonly known as a *symmetry* of $\mathcal{C}$. When $\mathcal{C}$ is different from a line it cannot be invariant under a translation or a glide reflection, as this would imply the existence of a line intersecting the curve in infinitely many points, contradicting Bézout's theorem. The remaining symmetries are therefore the *mirror symmetries* (reflections) and the *rotation symmetries*. The special case of *central symmetries* is of particular interest and corresponds to rotation by $\theta = \pi$. Notice that the identity map is a symmetry of any curve $\mathcal{C}$, called the *trivial symmetry*.

Symmetries of algebraic curves and the similarities between them are related as follows.

**Theorem 3.** *Let $\mathcal{C}_1, \mathcal{C}_2$ be two distinct, irreducible, similar, algebraic curves, neither of them a line or a circle. Then there are at most finitely many similarities mapping $\mathcal{C}_1$ to $\mathcal{C}_2$. Furthermore, there is a unique similarity mapping $\mathcal{C}_1$ to $\mathcal{C}_2$ if and only if $\mathcal{C}_1$ (and therefore also $\mathcal{C}_2$) only has the trivial symmetry.*

*Proof.* If $f, \tilde{f}$ are two similarities of the plane mapping $\mathcal{C}_1$ to $\mathcal{C}_2$, then $g := \tilde{f}^{-1} \circ f$ is a symmetry of $\mathcal{C}_1$. Since an algebraic curve that is neither a line nor a circle has at most finitely many symmetries [12, §5], there can therefore be at most finitely many similarities mapping $\mathcal{C}_1$ to $\mathcal{C}_2$.

For the second claim, if $\mathcal{C}_1$ only has the trivial symmetry, $\tilde{f}^{-1} \circ f$ is the identity and $\tilde{f} = f$ is the unique similarity mapping $\mathcal{C}_1$ to $\mathcal{C}_2$. Conversely, if $\mathcal{C}_1$ has a nontrivial symmetry $\tilde{g}$, then $f \circ \tilde{g}$ is an additional similarity mapping $\mathcal{C}_1$ to $\mathcal{C}_2$. $\square$

## 3. Detecting similarity between rational curves

In this section we derive a procedure for detecting whether the curves $\mathcal{C}_1, \mathcal{C}_2$ are similar by an orientation preserving similarity. In this case Equation (5) holds for some similarity $f(z) = \boldsymbol{a}z + \boldsymbol{b}$ and a Möbius transformation $\varphi$. The case of an orientation reversing similarity is analogous, by replacing $z_2$ by $\overline{z_2}$.

We first attack the simpler case when $z_1(t), z_2(t)$ are polynomials. After that we consider the case when either $z_1(t)$ or $z_2(t)$ is not polynomial, while distinguishing between $\delta \neq 0$ and $\delta = 0$, with $\delta$ as in (3). In each case our strategy is to eliminate $\boldsymbol{a}, \boldsymbol{b}$ and reduce to a simpler problem downstairs in the diagram (4). Once we obtain the possible Möbius transformations downstairs, they can be lifted to corresponding similarities upstairs.

### *3.1. The polynomial case*

A curve is *polynomial* if it admits a polynomial parametrization. From a rational parametrization of a polynomial curve, a polynomial parametrization can be quickly computed [17]. As a consequence, we can assume our polynomial curves to be polynomially parametrized, without loss of generality and without loss of significant computation power. Notice that if $\mathcal{C}_1, \mathcal{C}_2$ are similar and one of them is polynomial, then the other must be polynomial as well.

If both $z_1(t), z_2(t)$ are polynomial and define curves related by an orientation preserving similarity $f$, then comparing degrees in (5) shows that the corresponding Möbius transformation $\varphi$ should be linear affine, i.e., $\varphi(t) = \alpha t + \beta$. In this section we assume that the curves $\mathcal{C}_1, \mathcal{C}_2$ are related by a similarity whose corresponding Möbius transformation is of this form. This includes similar polynomial curves as a special case, but also other rational curves needed for the case treated in Section 3.3.

We require some mild assumptions on our parametrization $z_1$, namely that $z_1$, and therefore also $z_1', z_1''$, are well defined at $t = 0$ and that $z_1', z_1''$ are nonzero at $t = 0$. Notice that there are only finitely many parameters $t$ for which one of these conditions does not hold, so these conditions hold after applying an appropriate, or even random, linear affine change of the parameter $t$.

Evaluating (5) at $t = 0$ yields

$$z_2(\beta) = \boldsymbol{a}z_1(0) + \boldsymbol{b}. \tag{7}$$

To get rid of $\boldsymbol{b}$, we differentiate (5) and evaluate at $t = 0$, which gives

$$z_2'(\beta)\alpha = \boldsymbol{a}z_1'(0). \tag{8}$$

Differentiating (5) twice and evaluating at $t = 0$ yields

$$z_2''(\beta)\alpha^2 = \boldsymbol{a}z_1''(0). \tag{9}$$

The unknowns $\boldsymbol{a}$ and $\alpha$ are nonzero since $f$ and $\varphi$ are invertible, and $z_2'(\beta), z_2''(\beta)$ are not identically zero because $\mathcal{C}_2$ is not a line. Dividing (9) by (8) yields

$$\alpha = \frac{z_1''(0)}{z_1'(0)} \cdot \frac{z_2'(\beta)}{z_2''(\beta)}, \tag{10}$$

which does not involve $\boldsymbol{a}$ or $\boldsymbol{b}$.

Since the Möbius transformation $\varphi(t) = \alpha t + \beta$ maps the real line to itself, both $\alpha$ and $\beta$ are required to be real. That is, their imaginary parts $\Im(\alpha), \Im(\beta)$ are zero. The following lemma shows that this limits $\beta$ to only finitely many candidates.

**Lemma 4.** *If $\Im(\alpha) = 0$ holds for infinitely many values of $\beta$, then $\mathcal{C}_2$ is a line.*

*Proof.* Writing $z_1''(0)/z_1'(0) = a + ib$ and $z_2(\beta) = x(\beta) + iy(\beta)$, the condition $\Im(\alpha) = 0$ is equivalent to

$$b\big(x'x'' + y'y''\big) + a\big(x''y' - x'y''\big) = 0.$$

If $y'$ is identically 0, then $\mathcal{C}_2$ is a line and the result follows. So assume that $y'$ is not identically 0. Then the above condition is equivalent to

$$b\frac{x'x'' + y'y''}{y'^2} + a\left(\frac{x'}{y'}\right)' = 0.$$

Changing to a new variable $z = x'/y'$ and dividing by $z^2 + 1$, we obtain

$$b\frac{zz'}{z^2+1} + a\frac{z'}{z^2+1} = -b\frac{y''}{y'}.$$

Integrating this equation yields

$$\frac{b}{2}\ln\left(x'^2 + y'^2\right) = -a\arctan\left(\frac{x'}{y'}\right) + k, \tag{11}$$

for some constant $k$. Since $z_1''(0)$ is nonzero, $a, b$ cannot both be zero. If $b = 0$, then $x'/y'$ is constant and $\mathcal{C}_2$ is a line. If $b \neq 0$, writing $x' + iy' = re^{i\theta}$, Equation (11) becomes $r = K\mathrm{e}^{-\theta a/b}$ for some nonzero constant $K$. If $a \neq 0$, this curve is a logarithmic spiral, which is a non-algebraic curve and therefore contradicts that $x'$ and $y'$ are rational. If $a = 0$, on the other hand, then the arc length $r = \sqrt{x'^2 + y'^2}$ of $z_2$ is constant, which again implies that $\mathcal{C}_2$ is a line [8]. □

As a consequence, $\Im(\alpha) = 0$ provides a polynomial condition $\xi(\beta) = 0$ on $\beta$. Furthermore, by (10), any real zero $\beta$ of $\xi$ determines the Möbius transformation through

$$\alpha(\beta) = \Re\left(\frac{z_1''(0)}{z_1'(0)} \cdot \frac{z_2'(\beta)}{z_2''(\beta)}\right), \tag{12}$$

as long as $\beta$ is not a zero or pole of $z_2'$ or $z_2''$. Applying this to (8) and (7) yields

$$\boldsymbol{a}(\beta) = \frac{z_2'(\beta)}{z_1'(0)}\alpha(\beta), \qquad \boldsymbol{b}(\beta) = z_2(\beta) - \boldsymbol{a}(\beta)z_1(0), \tag{13}$$

determining the similarity corresponding to $\beta$.

Finally, coming back to (5), we can check whether there exists a real zero $\beta$ of $\xi$, such that

$$z_2\big(\alpha(\beta)t+\beta\big)-\boldsymbol{a}(\beta)z_1(t)-\boldsymbol{b}(\beta)=0 \tag{14}$$

for all $t$. This is a rational function $R(t)=P_1(t)/P_2(t)$, whose coefficients are rational functions of $\beta$. For (14) to hold for all $t$, the coefficients of the numerator $P_1(t)$, which are rational functions of $\beta$, have to be zero. Taking the GCD of the numerators of these coefficients and of $\xi(\beta)$ gives a polynomial $P(\beta)$. Finally let $Q(\beta)$ be the result of taking out from $P(\beta)$ all the factors that make the numerators and denominators of $\alpha(\beta),\boldsymbol{a}(\beta)$ and the denominator of $\boldsymbol{b}(\beta)$ vanish. Analogously, a polynomial $Q(\beta)$ can be found in the case of an orientation reversing similarity.

**Theorem 5.** *The curves $\mathcal{C}_1,\mathcal{C}_2$ are similar if and only if $Q(\beta)$ has at least one real root. More than that, in that case they are symmetric if and only if $Q(\beta)$ has several real roots.*

*Proof.* If $\mathcal{C}_1,\mathcal{C}_2$ are similar, then Theorem 1 implies that there exists a similarity $f$ and a Möbius transformation $\phi$ for which either (5) or (6) holds. By construction, this implies that $Q(\beta)$ has a real zero. Conversely, if $\beta$ is a zero of $Q(\beta)$, then (12) – (13) are well defined, and $f(z)=\boldsymbol{a}(\beta)z+\boldsymbol{b}(\beta)$ is a similarity that satisfies (5) with $\varphi(t)=\alpha(\beta)t+\beta$. Theorem 1 then implies that $\mathcal{C}_1,\mathcal{C}_2$ are similar. The second statement follows from Theorem 3. □

Thus we obtain a recipe, spelled out as Algorithm `SimilarPol`, for detecting whether two rational curves are related by a similarity whose corresponding Möbius transformation is linear affine, which is the case for all similar polynomial curves.

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**Algorithm** `SimilarPol`

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**Require:** Two proper rational parametrizations $z_1,z_2:\mathbb{R}\dashrightarrow\mathbb{C}$ of curves $\mathcal{C}_1,\mathcal{C}_2$ of equal degree greater than one, such that $z_1$ is well defined at $t=0$ and $z_1',z_1''$ are nonzero at $t=0$.

**Ensure:** Returns whether $\mathcal{C}_1,\mathcal{C}_2$ are related by a similarity that corresponds to a linear affine change of the parameter downstairs in (4).

1: *Look for orientation preserving similarities.*
2: Let $\xi(\beta)$ be the numerator of $\Im(\alpha)$, with $\alpha$ from (10).
3: Find $\alpha(\beta)$ from (12).
4: Find $\boldsymbol{a}(\beta)$ and $\boldsymbol{b}(\beta)$ from (13).
5: Let $P(\beta)$ be the GCD of $\xi(\beta)$, and all the polynomial conditions obtained when substituting $\alpha(\beta),\boldsymbol{a}(\beta),\boldsymbol{b}(\beta)$ into (5).
6: Let $Q(\beta)$ be the result of taking out from $P(\beta)$ any factor shared with a denominator of $\alpha(\beta),\boldsymbol{a}(\beta),\boldsymbol{b}(\beta)$ or a numerator of $\alpha(\beta),\boldsymbol{a}(\beta)$.
7: If $Q(\beta)$ has real roots, return `TRUE`.
8: *Look for orientation reversing similarities.*
9: Replace $z_1\leftarrow\bar{z}_1$ and proceed as in lines 1 – 7.
10: If the above step did not return `TRUE`, then return `FALSE`.

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Although we have not made it explicit in the above algorithm, one can compute the similarities mapping $\mathcal{C}_1$ to $\mathcal{C}_2$, either symbolically, i.e., in terms of a real root $\beta$ of $Q(\beta)$, or numerically, at the additional cost of approximating the real roots of $Q(\beta)$ with any desired precision.

*3.2. The rational case I: $\delta \neq 0$*

Next we consider the case that $\mathcal{C}_1, \mathcal{C}_2$ are related by a similarity whose corresponding Möbius transformation $\varphi$ as in (3) satisfies $\delta \neq 0$. After performing a linear affine change of the parameter $t$ if necessary, we may and will assume that $z_1$ is well defined at $t = 0$ and that $z_1', z_1''$ are nonzero at $t = 0$. Furthermore, after dividing the coefficients of $\varphi$ by $\delta$, we can assume that $\delta = 1$ and that $\alpha, \beta, \gamma$ are real.

Evaluating (5) at $t = 0$ again gives (7), but differentiating (5) now yields

$$z_2'(\varphi(t)) \cdot \frac{\Delta}{(\gamma t + 1)^2} = \boldsymbol{a} z_1'(t). \tag{15}$$

Evaluating (15) at $t = 0$, we get

$$z_2'(\beta) \cdot \Delta = \boldsymbol{a} z_1'(0), \tag{16}$$

while differentiating (15) and evaluating at $t = 0$ gives

$$\Delta\big(\Delta z_2''(\beta) - 2\gamma z_2'(\beta)\big) = \boldsymbol{a} z_1''(0). \tag{17}$$

Dividing (17) by (16), using that $z_1'(0), \Delta \neq 0$, and solving for $\gamma$ gives

$$\gamma = -\frac{z_1''(0)}{2z_1'(0)} + \Delta \frac{z_2''(\beta)}{2z_2'(\beta)}. \tag{18}$$

Assuming $\delta = 1$ forces $\gamma$ to be real. Writing

$$-\frac{z_1''(0)}{2z_1'(0)} = A + Bi, \qquad \frac{z_2''(\beta)}{2z_2'(\beta)} = C(\beta) + D(\beta)i,$$

and since $\Delta$ is also real, we have $B + \Delta \cdot D(\beta) = 0$, and therefore

$$\Delta(\beta) = -\frac{B}{D(\beta)}, \quad \gamma(\beta) = A + \Delta(\beta)C(\beta), \quad \alpha(\beta) = \Delta(\beta) + \beta\gamma(\beta). \tag{19}$$

The following lemma proves that the above expressions are well defined.

**Lemma 6.** *If* $\Im\left(\dfrac{z_2''(\beta)}{z_2'(\beta)}\right) = 0$ *holds for infinitely many* $\beta$*, then* $\mathcal{C}_2$ *is a line.*

*Proof.* Writing $z_2(\beta) = x_2(\beta) + iy_2(\beta)$, one observes that

$$\Im\left(\frac{z_2''(\beta)}{z_2'(\beta)}\right) = \frac{x_2'^2(\beta)}{x_2'^2(\beta) + y_2'^2(\beta)} \cdot \frac{\mathrm{d}}{\mathrm{d}\beta}\left(\frac{y_2'(\beta)}{x_2'(\beta)}\right)$$

is identically zero if and only if either $x_2'$ is identical to zero or $y_2'/x_2'$ is identical to a constant. In either case one deduces that $\mathcal{C}_2$ is a line. $\square$

Hence the similarity corresponding to $\varphi$ is found from (16) and (7), i.e.,

$$\boldsymbol{a}(\beta) = \frac{z_2'(\beta)}{z_1'(0)}\Delta(\beta), \qquad \boldsymbol{b}(\beta) = z_2(\beta) - \boldsymbol{a}(\beta)z_1(0). \tag{20}$$

Finally, substituting $\alpha(\beta), \gamma(\beta), \boldsymbol{a}(\beta), \boldsymbol{b}(\beta)$ into (5), and taking the GCD of the numerators as before, we again reach a polynomial condition $P(\beta) = 0$. Let $Q(\beta)$ be the polynomial obtained from $P(\beta)$ by taking out the factors that make a denominator of $\Delta(\beta), \alpha(\beta), \gamma(\beta), \boldsymbol{a}(\beta), \boldsymbol{b}(\beta)$ or a numerator of $\Delta(\beta), \boldsymbol{a}(\beta)$ vanish.

Any real zero $\beta$ of $Q$ corresponds to a Möbius transformation and similarity satisfying (5). Furthermore, by the final claim of Theorem 1, different real roots of $Q$ correspond to different similarities. If $Q$ were identically zero, then there would be infinitely many similarities mapping $\mathcal{C}_1$ to $\mathcal{C}_2$, contradicting Theorem 3. We conclude that Theorem 5 also holds for similarities of this type.

### *3.3. The rational case II: $\delta = 0$*

The remaining case happens when $\mathcal{C}_1, \mathcal{C}_2$ are related by a similarity whose corresponding Möbius transformation (3) satisfies $\delta = 0$. Then $\gamma \neq 0$, and we may and will assume $\gamma = 1$ and therefore that $\alpha, \beta$ are real. Equation (5) becomes

$$z_2(\alpha + \beta/t) = \boldsymbol{a}z_1(t) + \boldsymbol{b}.$$

Making the change of parameter $t \longmapsto 1/t$, and defining $\tilde{z}_1(t) := z_1(1/t)$, we get

$$z_2(\alpha + \beta t) = \boldsymbol{a}\tilde{z}_1(t) + \boldsymbol{b}$$

It follows that $\mathcal{C}_1, \mathcal{C}_2$ are related by a similarity that corresponds to a linear affine change of parameters downstairs, which is the case of Section 3.1, with $\tilde{z}_1$ satisfying the same conditions as $z_1$. We thus arrive at a recipe for detecting whether two rational curves are similar, spelled out as Algorithm `SimilarGen`. As in the case of `SimilarPol`, the corresponding similarities can be computed explicitly.

### *3.4. The similarity type*

Notice that if two curves $\mathcal{C}_1, \mathcal{C}_2$ are identified as similar by Algorithm `SimilarPol` or `SimilarGen`, and if $\beta$ is a real root of $Q$, then $\boldsymbol{a}(\beta)$ and $\boldsymbol{b}(\beta)$ define a similarity transforming $\mathcal{C}_1$ into $\mathcal{C}_2$. The nature of the similarity $f$ can be found as follows by analyzing the fixed points.

In case $f$ is orientation preserving, any fixed point $z_0$ satisfies $(1-\boldsymbol{a})z_0 = \boldsymbol{b}$. If $\boldsymbol{a} = 1$ and $\boldsymbol{b} = 0$ then every $z_0 \in \mathbb{C}$ is a fixed point, and $f$ is the identity. If $\boldsymbol{a} = 1$ and $\boldsymbol{b} \neq 0$ then there are no fixed points, and $f$ is a translation by $\boldsymbol{b}$. If $\boldsymbol{a} \neq 1$, then there is a unique fixed point $z_0 = \boldsymbol{b}/(1-\boldsymbol{a})$. Writing $\boldsymbol{a} = re^{i\theta}$, the similarity is a counter-clockwise rotation by an angle $\theta$ around the origin, followed by a scaling by $r$ and a translation by $\boldsymbol{b}$.

In case $f$ is orientation reversing, any fixed point $z_0$ satisfies $z_0 - \boldsymbol{a}\bar{z}_0 = \boldsymbol{b}$. If $\boldsymbol{a} = 1$ and $\boldsymbol{b} = 0$, then the $x$-axis is invariant and $f$ is a reflection in the

**Algorithm** `SimilarGen`

**Require:** Two proper rational parametrizations $z_1, z_2 : \mathbb{R} \dashrightarrow \mathbb{C}$ of curves $\mathcal{C}_1, \mathcal{C}_2$ of equal degree, none of which is a line or a circle, such that $z_1(t), z_1(1/t)$ are well defined at $t = 0$ and $z_1'(t), z_1''(t), \frac{\mathrm{d}}{\mathrm{d}t} z_1(1/t), \frac{\mathrm{d}^2}{\mathrm{d}t^2} z_1(1/t)$ are nonzero at $t = 0$.

**Ensure:** Returns whether $\mathcal{C}_1, \mathcal{C}_2$ are related by a similarity.

1: *Look for orientation preserving similarities with $\delta \neq 0$.*
2: Find $\Delta(\beta), \gamma(\beta), \alpha(\beta)$ from (19).
3: Find $\boldsymbol{a}(\beta), \boldsymbol{b}(\beta)$ from (20).
4: Let $P(\beta)$ be the GCD of all the polynomial conditions obtained when substituting $\alpha(\beta), \gamma(\beta), \boldsymbol{a}(\beta), \boldsymbol{b}(\beta)$ in (5).
5: Let $Q(\beta)$ the result of taking out from $P(\beta)$ any factor shared with a denominator of $\Delta(\beta), \alpha(\beta), \gamma(\beta), \boldsymbol{a}(\beta), \boldsymbol{b}(\beta)$ or a numerator of $\Delta(\beta), \boldsymbol{a}(\beta)$.
6: If $Q(\beta)$ has a real root, return `TRUE`.
7: *Look for orientation reversing similarities with $\delta \neq 0$.*
8: Replace $z_1 \leftarrow \bar{z}_1$ and proceed as in lines $1 - 6$.
9: *Look for the remaining similarities with $\delta = 0$.*
10: Replace $(\alpha, \beta) \leftarrow (\beta, \alpha)$ and $z_1(t) \leftarrow z_1(1/t)$.
11: Return the result of `SimilarPol`.

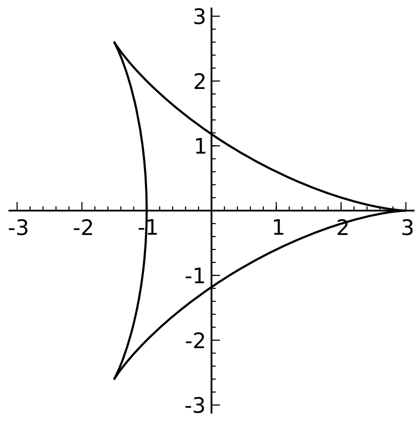
$x$-axis. If $\boldsymbol{a} = 1$ and $\boldsymbol{b} \neq 0$ is real, then the $x$-axis is invariant and $f$ is a glide reflection, which first reflects in the $x$-axis and then translates by $\boldsymbol{b}$. If $\boldsymbol{a} = 1$ and $\Im(\boldsymbol{b}) \neq 0$, then there are no fixed points and $f$ is a reflection in the $x$-axis, followed by a translation by $\boldsymbol{b}$. If $\boldsymbol{a} \neq 1$, write $\boldsymbol{a} = r\mathrm{e}^{i\theta}$. If $r = 1$ then there is a line of fixed points and $f$ is an isometry that first reflects in the $x$-axis, then rotates counter-clockwise by $\theta$, and then translates by $\boldsymbol{b}$. If $r \neq 1$ then there is a unique fixed point $z_0 = (\boldsymbol{a}\overline{\boldsymbol{b}} + \boldsymbol{b})/(1 - |\boldsymbol{a}|^2)$, and $f$ first reflects in the $x$-axis, rotates counter-clockwise by $\theta$ around the origin, scales by $r$, and finally translates by $b$.

### *3.5. Computing symmetries*

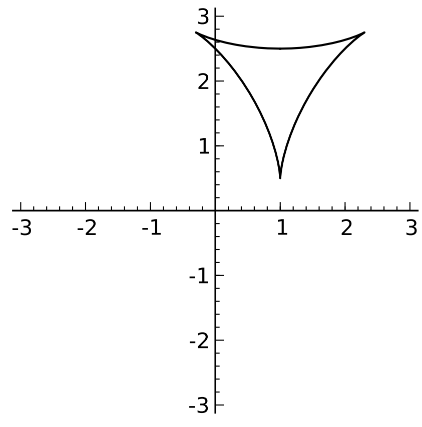
From Proposition 2, we can find the symmetries of $\mathcal{C}$ by applying Algorithms `SimilarPol` and `SimilarGen` with both arguments the parametrization $z_1 = z_2$ of $\mathcal{C}$. Furthermore, the nature of the symmetry can be deduced from the set of fixed points, as observed in Section 3.4. Since an irreducible algebraic curve different from a line cannot have a nontrivial translation symmetry or glide-reflection symmetry, we are left with the cases mentioned in Section 2.

## 4. Implementation and experimentation

Algorithms `SimilarPol` and `SimilarGen` were implemented in Sage [19], using Singular [7] as a back-end. The corresponding worksheet "Detecting Similarity of Plane Rational Curves" is available online for viewing at the third author's website [16] and at https://cloud.sagemath.org, where it can be tried online once SageMathCloud supports public worksheets.



(a)



(b)

Figure 1: The deltoid given as the image of (21) (left) and a related deltoid obtained by a similarity (right).

### *4.1. An example: computing the symmetries of the deltoid*

Let $\mathcal{C}_1 \subset \mathbb{R}^2$ be the *deltoid*, defined parametrically as the image of the rational map

$$\phi = (\phi_1, \phi_2) : \mathbb{R} \longrightarrow \mathbb{R}^2, \qquad t \longmapsto \left( \frac{-t^4 - 6t^2 + 3}{(t^2+1)^2}, \frac{8t^3}{(t^2+1)^2} \right), \tag{21}$$

or implicitly as the zero locus of the equation

$$(x^2 + y^2)^2 - 8(x^3 - 3xy^2) + 18(x^2 + y^2) - 27 = 0.$$

As can be seen in Figure 1a, this curve is invariant under the symmetric group $S_3$ with six elements, which is generated by a 120°-rotation around the origin and the reflection in the horizontal axis.

Since $\phi'(0) = (0, 0)$, this curve needs to be reparametrized before we can apply the algorithm. One checks that $z_1 = \phi_1(t-1) + \phi_2(t-1)i$ satisfies the conditions of the algorithm. To detect the symmetries of $\mathcal{C}_1$, we let $z_2 := z_1$ and apply the algorithm. We find $\Delta = \frac{1}{2}(\beta^2 - 2\beta + 2)$,

$$\gamma(\beta) = -\frac{\beta(2\beta^3 - 3\beta^2 - 18\beta + 22)}{4(\beta - 1)(\beta^2 - 2\beta - 2)},$$

$$\alpha(\beta) = -\frac{7\beta^4 - 34\beta^3 + 30\beta^2 + 8\beta - 8}{4(\beta - 1)(\beta^2 - 2\beta - 2)},$$

$$\boldsymbol{a}(\beta) = \frac{(\beta - 1)(\beta^2 - 2\beta - 2)}{(\beta^2 - 2\beta + 2)^2}\big(2 - \beta - \beta i\big), \tag{22}$$

$$\boldsymbol{b}(\beta) = 3\frac{\beta(\beta^2-6\beta+6)}{(\beta^2-2\beta+2)^2}\big(1-(\beta-1)i\big) \tag{23}$$

as functions of $\beta$. Substituting these expressions into

$$z_2\left(\frac{\alpha(\beta)t+\beta}{\gamma(\beta)t+1}\right) - \boldsymbol{a}(\beta)z_1(t) + \boldsymbol{b}(\beta)$$

yields a rational function in $t$ whose coefficients are rational functions in $\beta$. Its numerator is a polynomial in $t$, and taking the GCD of the numerators of its coefficients, we find

$$\beta(\beta-1)(\beta^2-6\beta+6)(\beta^2-2\beta-2).$$

We are only interested in the $\beta$ for which the expressions for $\alpha, \gamma, \Delta, \boldsymbol{a}$ and $\boldsymbol{b}$ are well-defined and for which the Möbius transformation and the similarity are invertible, so we discard the factors corresponding to the poles of $\alpha, \gamma, \Delta, \boldsymbol{a}, \boldsymbol{b}$ and zeros of $\Delta, \boldsymbol{a}$, leaving

$$\beta(\beta^2-6\beta+6).$$

Substituting the three zeros of this equation into Equations (22) and (23), we find three orientation preserving symmetries $f : z \longmapsto \boldsymbol{a}z + \boldsymbol{b}$, with $\boldsymbol{a} = 1, \mathrm{e}^{2\pi i/3}, \mathrm{e}^{4\pi i/3}$ and $\boldsymbol{b} = 0$, corresponding to rotations by $0°$, $120°$, and $240°$. Repeating the procedure with $z_1 \mapsto \overline{z}_1$, we find the remaining three orientation reversing symmetries $f : z \longmapsto \boldsymbol{a}\overline{z} + \boldsymbol{b}$ in $S_3$ with again $\boldsymbol{b} = 0$ and $\boldsymbol{a} = 1, \mathrm{e}^{2\pi i/3}, \mathrm{e}^{4\pi i/3}$ and $\boldsymbol{b} = 0$.

### *4.2. An example: detecting similarities between two deltoids*

Next, we let $z_1$ be as in Section 4.1 and define

$$z_2 : \mathbb{R} \longrightarrow \mathbb{C}, \qquad t \longmapsto \frac{t^4+4t^3+2t^2+1}{(t^2+1)^2} + \frac{5t^4+14t^2+1}{2(t^2+1)^2}i.$$

The corresponding curves $\mathcal{C}_1$ and $\mathcal{C}_2$ are shown in Figure 1. Applying the Algorithm `SimilarGen`, we obtain three orientation preserving similarities $f(z) = \boldsymbol{a}z + \boldsymbol{b}$ and three orientation reserving similarities $f(z) = \boldsymbol{a}\overline{z} + \boldsymbol{b}$, each with $\boldsymbol{b} = 1 + 2i$ and $\boldsymbol{a} \in \big\{-\frac{1}{2}i, \frac{1}{2}\mathrm{e}^{\pi i/6}, \frac{1}{2}\mathrm{e}^{5\pi i/6}\big\}$. A direct computation yields

$$z_2(t-1) = -\frac{1}{2}i\big(\phi_1(t-1) + \phi_2(t-1)i\big) + 1 + 2i = -\frac{1}{2}iz_1(t) + 1 + 2i,$$

confirming that (5) holds with $\boldsymbol{a} = -i/2, \boldsymbol{b} = 1 + 2i$ and $\varphi(t) = t - 1$.

### *4.3. Performance*

Let the *bitsize* $\tau := \lceil \log_2 k \rceil + 1$ of an integer $k$ be the number of bits needed to represent it. For various $d$ and $\tau$, we test the performance of Algorithms `SimilarPol` and `SimilarGen` by applying them to a pair of parametrized curves of equal degree $d$ and bitsizes of their coefficients bounded by $\tau$. For every pair $(d, \tau)$ this is done a number of times, and the average execution time $t$ (CPU

| $t_{\text{pol}}$ | $\tau = 1$ | $\tau = 8$ | $\tau = 16$ | $\tau = 32$ | $\tau = 64$ | $\tau = 128$ |
|---|---|---|---|---|---|---|
| $d = 3$ | 0.024 | 0.022 | 0.024 | 0.020 | 0.030 | 0.038 |
| $d = 6$ | 0.076 | 0.198 | 0.248 | 0.336 | 0.650 | 1.464 |
| $d = 9$ | 0.764 | 2.316 | 3.220 | 6.626 | 15.151 | 42.753 |
| $d = 12$ | 3.188 | 14.589 | 24.902 | 61.318 | 159.452 | 448.476 |
| $d = 15$ | 11.885 | 90.340 | 206.469 | 587.249 | 1244.176 | 4439.909 |

Table 1: Average CPU time $t_{\text{pol}}$ (seconds) of `SimilarPol` applied to random polynomial parametrizations of given degree $d$ and with integer coefficients with bitsizes bounded by $\tau$.

| $t_{\text{gen}}$ | $\tau = 1$ | $\tau = 2$ | $\tau = 4$ | $\tau = 8$ | $\tau = 16$ | $\tau = 32$ |
|---|---|---|---|---|---|---|
| $d = 2$ | 0.33 | 0.43 | 0.64 | 1.40 | 3.26 | 9.76 |
| $d = 3$ | 1.50 | 3.90 | 8.53 | 21.94 | 85.00 | 387.69 |
| $d = 4$ | 8.93 | 19.60 | 61.93 | 325.01 | 724.95 | 3881.44 |
| $d = 5$ | 41.66 | 96.12 | 380.02 | 3956.86 | 22608.96 | |
| $d = 6$ | 99.05 | 452.70 | 1028.22 | 17606.67 | | |
| $d = 7$ | 250.39 | 2670.24 | 19478.14 | | | |
| $d = 8$ | 1805.01 | 21610.26 | | | | |

Table 2: Average CPU time $t_{\text{gen}}$ (seconds) of `SimilarGen` applied to random rational parametrizations of given degree $d$ and with integer coefficients with bitsizes bounded by $\tau$.

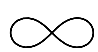
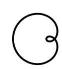
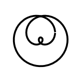
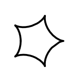
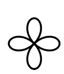
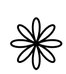
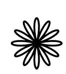
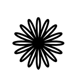
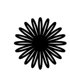
| curve | Descartes' folium | Bernoulli's lemniscate | epitrochoid | cardioid offset | hypocycloid |
|---|---|---|---|---|---|
| |  |  |  |  |  |
| degree | 3 | 4 | 4 | 8 | 8 |
| CPU time | 0.11 | 0.32 | 0.11 | 0.77 | 3.59 |
| curve | 4-leaf rose | 8-leaf rose | 12-leaf rose | 16-leaf rose | 20-leaf rose |
| |  |  |  |  |  |
| degree | 6 | 10 | 14 | 18 | 22 |
| CPU time | 0.24 | 3.50 | 24.83 | 118.74 | 703.12 |

Table 3: Average CPU time (seconds) of `SimilarGen` for well-known curves.

time) is listed in Tables 1 and 2 for a Dell XPS 15 laptop, with 2.4 GHz i5-2430M processor and 6 GB RAM.

Double logarithmic plots of the CPU times against the degrees and against the coefficient bitsizes are presented in Figures 2 and 3. In these plots the CPU times lie on a curve that seems to asymptotically approach a straight line, suggesting an underlying power law. The least-squares estimates of these asymptotes gives the power laws

$$t_{\mathrm{pol}} = 2.17 \cdot 10^{-11} d^{9.28} \tau^{1.59}, \qquad t_{\mathrm{gen}} = 6.2 \cdot 10^{-06} d^{9.91} \tau^{1.91} \tag{24}$$

for the CPU times $t_{\mathrm{pol}}$ of `SimilarPol` and $t_{\mathrm{gen}}$ of `SimilarGen`.

We should emphasize that these timings are for *dense* polynomials. The performance is much better in practice than Tables 1, 2 and Figures 2, 3 suggest. To illustrate this, we find the symmetries of various well-known curves $z_1 : \mathbb{R} \longrightarrow \mathbb{C}$, namely the *folium of Descartes*,

$$t \longmapsto 3t \frac{1 + ti}{1 + t^3},$$

the *lemniscate of Bernoulli*,

$$t \longmapsto \frac{(3t^4 + 2t^3 - 2t - 3) + (t^4 + 6t^3 - 6t - 1)i}{5t^4 + 12t^3 + 30t^2 + 12t + 5},$$

an *epitrochoid*,

$$t \longmapsto \frac{(-7t^4 + 288t^2 + 256) + (-80t^3 + 256t)i}{t^4 + 32t^2 + 256},$$

an *offset curve to a cardioid*,

$$\begin{aligned} t \longmapsto & \frac{6t^8 - 756t^6 + 3456t^5 - 31104t^3 + 61236t^2 - 39366}{t^8 + 36t^6 + 486t^4 + 2916t^2 + 6561} \\ & \qquad - \frac{18t(6t^6 - 16t^5 - 126t^4 + 864t^3 - 1134t^2 - 1296t + 4374)}{t^8 + 36t^6 + 486t^4 + 2916t^2 + 6561} i, \end{aligned}$$

a *hypocycloid* of degree 8,

$$\begin{aligned} t \longmapsto & \frac{-3t^8 - 24t^7 - 120t^6 - 384t^5 - 680t^4 - 608t^3 - 224t^2 + 16}{t^8 + 8t^7 + 32t^6 + 80t^5 + 136t^4 + 160t^3 + 128t^2 + 64t + 16} \\ & \qquad + \frac{16t^7 + 112t^6 + 304t^5 + 400t^4 + 320t^3 + 256t^2 + 192t + 64}{t^8 + 8t^7 + 32t^6 + 80t^5 + 136t^4 + 160t^3 + 128t^2 + 64t + 16} i, \end{aligned}$$

and *Rose curves*

$$t \longmapsto \frac{2t + (1 - t^2)i}{(1 + t^2)^{n+1}} \sum_{k=0}^{n} \binom{2n}{2k} (-1)^k t^{2k}, \qquad n = 2, 4, 6, 8, 10,$$

of degrees 6, 10, 14, 18, and 22. In each case the average CPU time used by `SimilarGen` is listed in Table 3. Observe that the degrees and coefficients

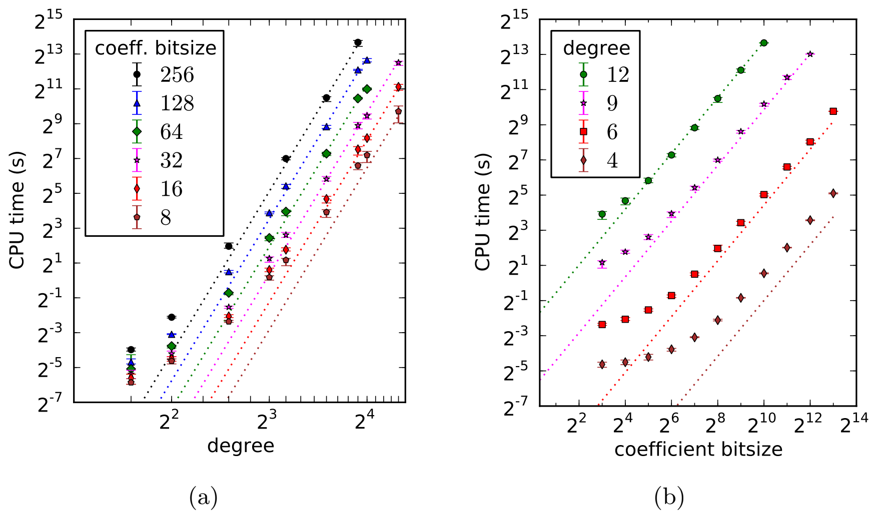



Figure 2: Double logarithmic plots of the average CPU time of `SimilarPol` versus degree (left) and versus the bitsize of the coefficients (right). The error bars show the range of CPU times found for the various random polynomials. The dotted line represents the fitted power law for $t_{\mathrm{pol}}$ in (24).

of these examples are far from trivial. For more details we refer to the Sage worksheet [16].

To conclude, notice that the order of $z_1$ and $z_2$ is important for the performance of `SimilarGen`, so that one should let $z_2$ be the parametrization whose coefficients have the smallest bitsize.

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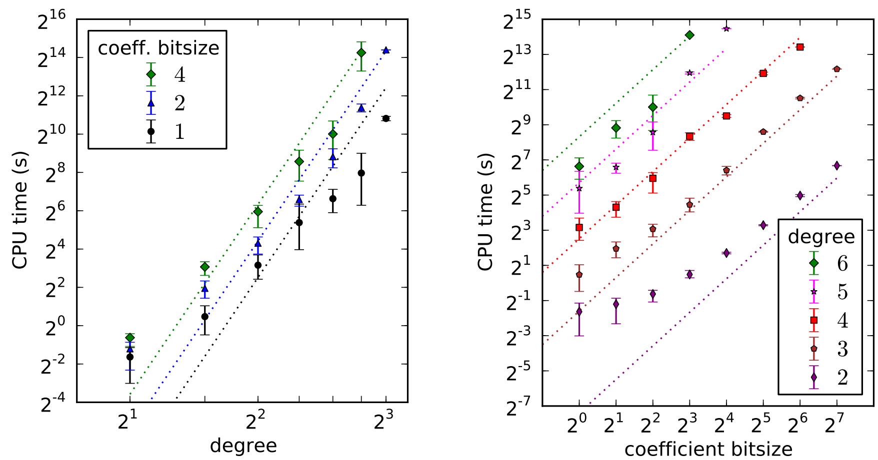


(a) (b)

Figure 3: Double logarithmic plots of the average CPU time of `SimilarGen` versus degree (left) and versus the bitsize of the coefficients (right). The error bars show the range of CPU times found for the various random polynomials. The dotted line represents the fitted power law for $t_{\text{gen}}$ in (24).

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