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Abstract

This paper provides several definitions of efficiency measures in the price space. Economic and scale aspects of inefficiency are considered to give empirical content to the measurement of efficiency when the production technology is described by cost functions models. It shows, in the Hanoch's symmetric duality approach, how the new definitions preserve the ranking of efficiency, are formally dually symmetric to the ones defined in the input space, and both are established with respect to different descriptions of the same technology. In addition, graphical procedures are utilized to make an insight into the achievement of polar technologies from primal ones and into the relationship between Shephard's lemma and Roy's identity.

Keywords: Efficiency measures, Scale measures, Duality theory, Polar forms.
JEL classification: D24

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[1] This paper is a revised version of an old 1982 paper by the author, partially published in Spanish as Muro and Vera (1983) and cited for example in Fare (1985). The usual disclaimer applies.

DUAL FARRELL MEASURES OF ECONOMIC EFFICIENCY

Juan Muro

1. Introduction.

In recent years there has been an increasing concern for the definition and measurement of the efficiency of a productive process. Whatever the reasons behind this concern, it appears that our understanding of such an elusive concept has improved notoriously. See, for instance, Afriat (1972), Fare and Lovell (1978), hereafter FL, Førsund and Hjalmarsson (1974, 1979), hereafter FH[2].

On the other hand, duality theory shows that under certain hypothesis, namely cost minimization and price-taking behaviour, there is more than one way to characterize completely a technology and that under convenient regularity conditions both alternative representations are equivalent. It turns out that cost functions are more and more employed in describing production technologies because, among other reasons, they are especially suitable for empirical analysis[3]. The measurement of efficiency in this type of models must be based on the indirect formulation of the production possibilities, i.e. the representation of the technology in the relative-price space.

Efficiency measures are generally defined in terms of the production function that describes the technology in the input space. They are proper measures in the context of frontier production function models. The empirical content of the different efficiency measures – technical,

[2] These papers, in the line of Debreu, Koopmans and Farrell, are basically devoted to theoretical considerations about the concept and measures of efficiency. For a review of the literature on empirical applications of efficiency measures see the monographic issue of the Journal of Econometrics (1980) on frontier production models, in particular the Førsund-Lovell-Schmidt (1980) paper.

allocative, scale – in the input space is well known too. On the contrary, the empirical development of cost models for the technology has not been associated with a similar theoretical development in the definition and specification of proper efficiency measures in the price space. This lack of adequate specification can lead to erroneous inferences about the structure of efficiency of the industry analyzed and likewise to make meaningless the empirical results of the widely used cost function models.

The purpose of the present paper is to elaborate several definitions of efficiency measures in the price space and to show that such measures are formally dually symmetric to the ones established in the input space. It will follow from duality theory that the new definitions are performed on the same technology as the original ones. Section 2 contains a very brief formal treatment of the alternative representations of a technology. Section 3 describes a graphical procedure for passing from the direct representation to the indirect one and vice versa. The graphical treatment is used in Sections 4 and 5 to give an insight into the concept of polar technologies and the relationship between Roy's identity and Shephard's lemma. Section 6 establishes and explains five definitions of efficiency measures in the price space: two of them do not consider the aspects of inefficiency related with the scale of production and the others are concerned with the overall efficiency in production. Section 7, finally, reveals the duality between the new measures and the well known ones defined in the input space.

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[3] Theoretical arguments can also be given to support the advantage of modelling in special cases the technology by means of cost functions. In Section 3 this reasoning will be briefly outlined.

2. The alternative representations of the technology.

In the present paper direct – in the input space – and indirect – in the price space – descriptions of the technology will be used. If formal treatment of the subject is made, duality theorems of Shephard-Uzawa-McFadden can be utilized.

The following concepts will be convenient in a direct characterization of the technology:

Definition 1. Standard production function. The production function is defined as an application

$$f: \quad R^n_+ \rightarrow R_+; \; y=f(\underline{x}) \quad \forall \, \underline{x} \in R^n_+ \rightarrow y \in R_+.$$

Definition 2. Level sets. For any non-negative output, the production possibilities set $L(y) \subseteq R^n_+$ is defined as

$$L(y)= \{ \underline{x} \mid f(\underline{x}) \geq y\}. \qquad [1]$$

Definition 3. Distance function. The distance function D(1/y, x) of a technology is defined[4] as

$$D(1/y,\underline{x}) = \begin{cases} 0 & \forall \, \underline{x} \notin L(y) \\ \sup\{\alpha \mid 1/\alpha \underline{x} \in L(y); \underline{x} \in R^n{}_+\}, & \underline{x} \in L(y) \end{cases} \qquad [2]$$

The production function (Def. 1) observes the following regularity conditions:

Condition A. $f(\underline{x})$ is defined for all $\underline{x} \in R^n_+$ and is real, single valued, right continuous, non decreasing in $\underline{x}$, quasiconcave, finite for finite $\underline{x}$, with $f(\underline{0})=0$, and unbounded for at least some unbounded sequence $\{\underline{x}^N\}$[5].

[4] It may be thought that the definition of the distance function is in some way affected. This formulation however allows us to introduce a symmetric exposition of the duality relationships, Hanoch (1970, 1978).

The production possibilities set (Def. 2) satisfies:

Condition B. For $y \in R_+$, L(y) is a non-empty, closed, convex set, with free disposal[6]; $L(0)= R^n_+$; if y>0 then $\underline{0} \notin L(y)$; for all $\underline{x}$ there exist a y' >0 such that $\underline{x} \notin L(y')$.

The distance function (Def. 3) satisfies:

Condition C. $D(1/y, \underline{x})$ is positive, real valued, defined and finite for all $\underline{x} \in R^n_+$ finite, 1/y >0; non-decreasing, positive linear homogeneous and concave in $\underline{x}$, for finite 1/y >0[7].

In Figure 1 the concepts above are represented for two inputs one output in the input-input space.

[Insert Figure 1]

The isoquant and the efficient set corresponding to an output level y^0 are

$$\text{Isoq}\ (y^0) = \{\ \underline{x}\ |\ y^0 = f(\underline{x});\ \underline{x} \in R^n_+\}\ ; \qquad [3]$$

$$\text{Eff}\ (y^0) = \{\ \underline{x}\ |\ y^0 = f(\underline{x}),\ y^0 > f(\underline{x}'),\ \surd\ \underline{x}' < \underline{x}\ \}. \qquad [4]$$

---

[5] Condition A may be found in Diewert (1971) and Hanoch (1970, 1978) and guarantees the existence of a unique dual cost function.

[6] Condition B can be found in Hanoch (1970, 1978), where free disposal is defined as

$$\underline{x}' \geq \underline{x} \in L(y) \rightarrow \underline{x}' \in L(y)\ ;\ y' > y \rightarrow L(y') \subseteq L(y).$$

The Shephard-Uzawa-McFadden duality theorems assure the existence of functions and sets in the price-price space dual to the primal stated above. They provide an indirect representation of the technology. In what follows these concepts will be exposed, the regularity conditions satisfied by them outlined and the dual relationships briefly described.

Definition 4. Distance function. The cost function is defined in the standard form

$$C(y, \underline{w}) = \underset{\underline{x}}{\text{Min}} \{ \underline{x} \mid \underline{wx} | \underline{x} \in L(y) \}. \quad [5]$$

This function is uniquely determined by L(y) and is a distance function corresponding to the indirect representation of this technology, i.e. the description of the technology in the price space. In fact, if the price structure is defined in suitable form the definition of the cost function adopts a similar expression to Definition 3,

$$C(y, \underline{p}) = \sup \{ \beta \mid 1/\beta\, \underline{p} \in V(1/y); \underline{p} \in R^n_+ \}, \quad [6]$$

where V(1/y) are level sets in the price space and $\underline{p} = \underline{w}/C$.

The cost function satisfies Condition C above changing (1/y, $\underline{x}$) for (y, $\underline{p}$).

Definition 5. Level sets. The unit cost set $V(1/y) \subseteq R^n_+$ is defined as

$$V(1/y) = \{ \underline{p} \mid C(y,\underline{p}) \geq 1 \}. \quad [7]$$

The unit cost set satisfies Condition B above changing y for 1/y.

[7] These are the regularity conditions for the distance function in Hanoch (1970, 1978) where they are proved when

Definition 6. Indirect production function. The indirect production function g($\underline{w}$/C) is defined[8] as

$$g(\underline{w}/C)= \inf \{ y \mid \underline{w} \in V(1/y) \}. \qquad [8]$$

This definition is in the line of the related concepts used in consumer theory and gives the same function defined by Malmquist (1953). However if the above formulation is utilized the symmetry between the direct and indirect alternative representations of the technology is broken. To preserve the symmetry we must define the reciprocal indirect production function, Hanoch (1970, 1978), h ($\underline{w}$/C) as

$$h(\underline{w}/C)= \sup \{ 1/y \mid C(y,\underline{w}) \geq 1 \}, \qquad [9]$$

function that satisfies Condition A above changing x for w/C.

Duality relates f($\underline{x}$), L(y), D(1/y, $\underline{x}$), with h($\underline{w}$/C), V(1/y), C(y, $\underline{p}$), respectively; the duality relationships adopt a symmetric form and the dual concepts are dually symmetric.

3. Price indifference surfaces.

The problem of what can be obtained from a given expenditure for the factors of production is a common one in the theory of production. The firm frequently faces an exogenous price vector for inputs[9], suffering a fixed positive cost rate and tries to make its best in production. The solution to this problem is given in economic-technical terms by an indirect production function. Both public and private sectors are concerned with this type of question. In the case of private sector one

L(y) observes Condition B.

[8] This is the so-called (cost) indirect production function, Shephard (1970, 1974).

may think in small farms behaviour in the agricultural sector of the economy where farmers in most cases operate with given cost and prices; in the public sector it is an obvious problem and its answer is the foundation of cost-benefit analysis.

The subject has been discussed by, among others, Hotelling (1932), Malmquist (1953), Shephard (1953, 1970, 1974) and Hanoch (1970, 1978). Malmquist (1953), quoting Hotelling (1932), establishes the making of price indifference surfaces: starting from a given technology described by a standard production function $y = f(\underline{x})$, suppose the firm faces an input price vector $\underline{w}$, and a fixed cost C^0. The price indifference surfaces are defined by the maximization of the output subject to technology and cost constraints. The (relative) price indifference surface is

$$z^0 = g(\underline{w}/C^0),$$

by varying z^0 we obtain a system of (relative) price indifference surfaces. These price indifference surfaces are the level surfaces of an indirect production function (Def. 6 above) which gives maximum output attainable using prices and cost as arguments. The indirect concept is established in opposition with the standard direct production function in which inputs are used as arguments. Note that arguments in the indirect production function are (relative) normalized prices: observed prices in relation to observed cost. Then, just as the producer is indifferent to different points on a quantity indifference surface he is also indifferent to different (relative) price points on a price indifferent surface.

In Malmquist (1953) a geometrical construction of a price indifference curve for two commodities can be found but the method employed is not a very explicit one; Shephard (1970) also describes a method based in polar constructions; Darrough and Southey (1977) have

[9] For a brief consideration about the nature of prices, market or non-market prices, see Shephard (1974).

developed a pictorial exposition for the consumer utility theory that can be very useful in this analysis. In what follows the concepts defined in the preceding section will be represented for two inputs one output case and a graphical procedure for going from the direct representation of the technology to the indirect one and vice versa will be described.

We will first start from the direct representation to obtain the indirect one. In quadrant 1 of Figure 2 an isoquant corresponding to an output level y^0 is drawn. This isoquant is defined in terms of the production function (3)[10].

The Figure shows three isocost lines AB, CD, EF that represent the minimum cost of producing y^0 in different price conditions. In the input-input space points A, B, C, D, E, F are intersections of isocost lines with input axes. As is well known, the coordinate of these points are the corresponding cost-price ratios. The rational behaviour of the firm leads to input demand vectors (points of tangency) that maximize production subject to the cost constraint.

[Insert Figure 2]

In quadrants 2, 4 of the Figure, unit rectangular hyperbolas are drawn: $x_1p_1=1$, $x_2p_2=1$. In quadrant 2 one can get the points that corresponds to A, C, E through the unit rectangular hyperbola in the price-quantity space; they are A', C', E' whose coordinates are price-cost ratios, i.e. the so-called normalized or accounting prices: observed prices in relation to observed cost. The same method allows us to obtain B', D', F' in quadrant 4. Points in quadrant 3 with

[10] Level sets and the distance function can also be used to define the isoquant

$$Isoq(y^0) = \{\underline{x} \mid \underline{x} \in L(y^0), \forall y' < y^0 \rightarrow \underline{x} \notin L(y'); \underline{x} \in R^n{}_+\};$$
$$Isoq(y^0) = \{\underline{x} \mid D(1/y^0, \underline{x}) = 1; \underline{x} \in R^n{}_+\}.$$

coordinates (A', B'), (C', D'), (E', F') are situated on the price indifference surface level corresponding to y^0 in the price-price space. By iteration of this procedure one can obtain points in price-price space that corresponds to all possible isocost lines tangent to the isoquant y^0 in the input-input-space. The continuum of these points is the so-called "factor-price frontier" or "price-possibility frontier". This frontier may be defined in a variety of ways.

Distance function

$$\text{Isoc}\,(1/y^0) = \{\, \underline{p} \mid C(y^0, \underline{p}) = 1;\ \underline{p} \in R^n_+ \}\ ; \qquad [10]$$

Level sets

$$\text{Isoc}\,(1/y^0) = \{\, \underline{p} \mid \underline{p} \in V(1/y^0),\ \sqrt{}\ 1/y' < 1/y^0 \rightarrow \underline{p} \notin V(1/y') \}\ ; \qquad [11]$$

Indirect production function

$$\text{Isoc}\,(1/y^0) = \{\, \underline{p} \mid h(\underline{p}) = 1/y^0;\ \underline{p} \in R^n_+ \}\ ; \qquad [12]$$

$$\text{Isoc}\,(1/y^0) = \{\, \underline{p} \mid g(\underline{p}) = y^0;\ \underline{p} \in R^n_+ \}\ ; \qquad [13]$$

Equation (10) suggests that the price indifference level may also be called "cost unit frontier", i.e. the location of all price vectors among them the firm is indifferent if seeks to produce y^0 with a fixed cost.

Suppose now that we consider a level of production such as y^1 and we want to draw the corresponding price indifference level. The above procedure can be employed to obtain the curve

$1/y^1$ in quadrant 3 of Figure 2. Thus, the isocost map in price-price space dual to the isoquant map in input-input space can be drawn.

Note that both alternative representations are dually symmetric in the sense of Hanoch (1970, 1978). Points on the cost unit frontier are "poles" of the corresponding isocost lines in input-input space. Namely, if $\underline{p}^0$, (A', B'), is the point obtained in the Figure from AB, the following equation is always satisfied $\underline{p}^0\underline{x}=1$, $\surd\ x \in$ AB. Even more, the point p^0 and the line AB are polar reciprocal to each other with respect to the unit sphere $\sum x_i^2 = 1$, Shephard (1953). Thus, going, in input-input space, outwards from the origin isoquants with increasing levels of output are found ($y^1 > y^0$), and making the same movement in price-price space isocost curves with increasing levels of output ($1/y^0 < 1/y^1$) are also found.

After moving from the direct representation of the technology to the indirect one, the way of return will be described. Suppose the price indifference map in price-price space, i.e. the indirect production function, is known, knowledge of normalized prices, i.e. observed prices and observed cost, is also given and we try to reconstruct the direct representation of the technology in input-input space. To do that a similar method to the one used earlier will be appropriated.

In Figure 3 the price-indifference map of $g(\underline{p})$ (indirect production function) has been drawn in price-price space. Suppose that a normalized price $\underline{p}^*$ is given, then there will be a price-indifference curve in the map which will contain $\underline{p}^*$; such a curve is in the Figure the one labelled by y^0. To derive the direct representation of the technology it will be first shown that in current conditions the rational behaviour of the firm leads it to demand an input bundle represented by a tangent to the indifference curve y^0 at $\underline{p}^*$.

[Insert Figure 3]

To prove that the firm behaviour is approached by a surrogate problem in the sense of Samuelson (1953): instead of considering prices as the given constraint, the firm is taken as finding, given cost and inputs, the combination of prices under which it would have demanded the known inputs as the optimum. As is well known the solution to this surrogate problem is to minimize the maximum output attainable. Let MN be the cost line that represents different combination of prices given an input bundle (x^0_1, x^0_2) and a cost C, i.e. $p_1 x^0_1 + p_2 x^0_2 = 1$, where p_i are normalized prices. Since g(p) increases as we move towards origin and the value of g(p) represents the maximum output attainable for any particular normalized price, the minimization of the maximum will be reached at the point in which the cost line is tangent to g(p), i.e. p*.

If the indirect map had been represented by the level curves of h(p) (reciprocal indirect production function) to preserve the symmetry, a different explanation could have been given to justify that the firm for a given normalized price demands an input bundle represented by a tangent to the price indifference curve at the given point. In Figure 4 this reasoning is performed. Suppose again that p* is on the $1/y^0$ level curve ($1/y^0 = h(p^*)$); the question in this context can be formulated as: which input bundle will explain that the firm, subject to a given cost constraint, will choose p* as the optimum price combination?[11]

[Insert Figure 4]

Point p* gives us information about observed prices and cost. In presence of definite observed prices infinite combinations of inputs can be demanded by the firm with the sole limitation that the observed cost constraint be satisfied; all these possible bundles are represented in the Figure 4 by lines passing through p*, i.e. $m(p_1-p^*_1)= p_2-p^*_2$. The boundary possibilities are described by the horizontal and vertical lines; the former representing the case in which x_2 is uniquely demanded and the later the exclusive demand of x_1. Assuming rational behaviour once the input combination is selected the firm will maximize its output choosing a price vector represented by the point of tangency between the cost line and a price indifference level. If the input bundle were the one depicted by AB it would produce $1/y^1 > 1/y^0$, selecting p** ≠ p*. Any input bundle different from x* (represented by MN) would induce the firm to produce an output greater than $1/y^0$ (remember that the reciprocal indirect production function increases outwards) taking price vectors different from p*. Thus only MN (i.e. x*) can explain the selection of p* as the optimum price vector[12].

Arguments above allow us to reconstruct the isoquant map. In Figure 5 we have drawn again unit rectangular hyperbolas in quadrants 2 and 4. Points M', N' corresponding points to M, N through the rectangular hyperbolas have been obtained, they are coordinates of a point x* in the input-input space situated on the isoquant labelled by y^0 and represents the input demand for the maximizing behaviour of the firm in given conditions. By iteration of this procedure the isoquant y^0 can be reconstructed starting from the isocost curve $1/ y^0$.

[Insert Figure 5]

[11] The approach based on h(p) considers the firm looking for the input vector which allows it to make its best in current conditions. Note however that in this process the firm moves like pursuing maximum index of product level instead of maximum real product level, i.e. it is viewed using seemingly a specular image as its reference point.

[12] The maximum index of product is reached minimizing the level of h(p), function that includes itself a maximizing process. Observe that the minimization of the reciprocal leads in fact to maximum real product.

4. Getting two for the price of one.

As is well known, the symmetric formulation of the direct and indirect representations of the technology, Hanoch (1970, 1978), has a further advantage what in Hanoch's words is called "getting two for the price of one". Given a technology which may be described directly (primal forms) or indirectly (dual forms) by means of the concepts defined above: f($\underline{x}$), L(y), D(1/y,$\underline{x}$); C(y, $\underline{p}$), V(1/y), g($\underline{w}$/C), h($\underline{w}$/C), there exists an unique technology, which may be called "polar technology", that may be uniquely represented by anyone of the following direct or indirect representations:

$$g(\underline{x}),\ V(y),\ C(1/y,\ \underline{x});\ D\ (y,\ \underline{p}),\ L(1/y),\ f\ (\underline{w}/D),\ (f(\underline{w}/D))^{-1};$$

i.e. the representations of the polar technology adopts the forms of the primitive technology changing primal and dual forms and replacing in this exchange $\underline{p}$ for $\underline{x}$ and 1/y for y and vice versa.

In Figure 6 a graphical exposition of the polar technologies has been made. Isoquants and isocosts may be obtained by a symmetric construction performed between quadrants 1 and 3 of the Figure. If we draw in quadrant 2, 4 the line $x_i = p_i$, points of the polar technology are obtained from dual points of the primitive one reflecting on the bisecting line and vice versa. Then when the symmetric dual forms for a technology are being obtained in fact we are also getting primal forms of a polar technology. Forms through the looking-glass are recuperating life.

[Insert Figure 6]

5. Roy's identity and Shephard's lemma.

The graphical exposition of the symmetric duality between both representations of the technology has the additional gain of giving an insight into the existing relationships between Roy's identity and Shephard's lemma[13]. Going back to Figure 5, consider a firm represented by point p* (w_1/C, w_2/C) in quadrant 1 of the Figure. The firm with observed prices $\underline{w}$ (w_1, w_2) and observed cost C is on the isocost curve labelled by $1/y^0$. Coordinates of point $\underline{x}$* in quadrant 3 which represents the maximizing behaviour input demands have been obtained geometrically through the unit rectangular hyperbolas in quadrants 2 and 4 of the Figure. The coordinate x^*_1 may be calculated by the following reasoning:

$$x^*_1 p_1 = 1;\ \ x^*_1\, OM = 1;\ \ x^*_1 = 1/OM.$$

But in the figure OM= OP+ PM and OP= w_1/C, while PM can be calculated

$$PM = (w_2/C)\ tg\ (90-\alpha). \qquad [14]$$

Expression that allows us to obtain the value sought. The specific form of the above expression depends on the particular form adopted by the slope included in it. In fact, it may be expressed at least in two different ways:

a) Based on the cost function.

[13] A similar exposition, for the consumer theory, can be found in Weymark (1978).

$$x^*_1 = \left(\frac{w_1}{C} + \frac{w_2}{C} \frac{\frac{\partial C(y,p)}{\partial p_2}\Big|_{p^*}}{\frac{\partial C(y,p)}{\partial p_1}\Big|_{p^*}} \right)^{-1},$$

let $w_1/C = p_1^*$, then

$$\boldsymbol{x^*_1 = \left(p_1 * + p_2 * \frac{\frac{\partial C(y,p)}{\partial p_2}\Big|_{p^*}}{\frac{\partial C(y,p)}{\partial p_1}\Big|_{p^*}} \right)^{-1} = \frac{\partial C(y,p)}{\partial p_1}\Big|_{p^*} \left(p_1 * \frac{\partial C(y,p)}{\partial p_1}\Big|_{p^*} + p_2 * \frac{\partial C(y,p)}{\partial p_2}\Big|_{p^*} \right)^{-1}},$$

$$x^*_1 = \frac{\partial C(y,p)}{\partial p_1}\Big|_{p^*}. \qquad [15]$$

This result can be extended for all i and is the well known Shephard´s lemma.

b) Based on the indirect production function.

$$\boldsymbol{x^*_1 = \left(\frac{w_1}{C} + \frac{w_2}{C} \frac{\frac{\partial h(p)}{\partial p_2}\Big|_{p^*}}{\frac{\partial h(p)}{\partial p_1}\Big|_{p^*}} \right)^{-1} = \frac{\partial h(p)}{\partial p_1}\Big|_{p^*} \left(p_1 * \frac{\partial h(p)}{\partial p_1}\Big|_{p^*} + p_2 * \frac{\partial h(p)}{\partial p_2}\Big|_{p^*} \right)^{-1}},$$

$$x^*_1 = \frac{h_1}{p_1 * h_1 + p_2 * h_2}. \qquad [16]$$

That can be considered as the Roy`s identity for the case of indirect production functions.

Going back to Figure 2, similar expressions can be obtained by replacing primal variables for the dual one. The firm is now represented by point x* (x_1*, x_2*) with an isocost line such as AB and we try to find the inverse demand (normalized price) in this situation. Following the same method the coordinate p_1* of point p* can be calculated

$$P_1{}^*OA=1;\; p_1{}^*=1/OA=w_1/C.$$

In Figure 2, OA= OX+ XA, and OX= x_1*, while XA can be expressed in terms of the slope of the isoquant

$$XA= x_2{}^*\ tg.(90-\beta) \qquad [17]$$

Expression that can be evaluated in two different ways:

c) Based on the distance function

$$\boldsymbol{p_1}^* = \left(\boldsymbol{x_1}^* + \boldsymbol{x_2}^* \frac{\boldsymbol{D_2}|_{x^*}}{\boldsymbol{D_1}|_{x^*}} \right)^{-1} = \boldsymbol{D_1}|_{x^*} \left(\boldsymbol{x_1}^* \boldsymbol{D_1}|_{x^*} + \boldsymbol{x_2}^* \boldsymbol{D_2}|_{x^*} \right)^{-1},$$

then

$$p_1{}^* = \frac{\partial D(1/y, x)}{\partial x_1}\Big|_{x^*}. \qquad [18]$$

The Shephard´s lemma again.

d) Based on the production function

$$p_1* = \left(x_1 * + x_2 * \frac{\frac{\partial f(x)}{\partial x_2}\Big|_{x*}}{\frac{\partial f(x)}{\partial x_1}\Big|_{x*}} \right)^{-1},$$

and then

$$p_1* = \frac{f_1}{x_1 * f_1 + x_2 f_2}. \qquad [19]$$

That in a broad sense can be considered a form of Roy´s identity.

Therefore the symmetric duality has been extended to the calculation of input demand and inverse demand functions (a)-(c), (b)-(d), respectively, as a manifestation of the structural duality existing between both alternative representations of the technology.

6. Measuring the efficiency of production.

Cost functions are more and more employed in describing production technologies because, among other reasons, they are specially suited to empirical analyses. The measurement of the efficiency of production in this type of models is based on the indirect formulation of the production possibilities described in the preceding sections. However, while the empirical content of the different measures of efficiency - technical, allocative, scale - in the input space (or input coefficient space) is well known, it seems that similar measures in the price space are not so well specified. In this section five measures of efficiency in the price space will be defined to characterize different aspects of the inefficiency problem. In the next section duality relationships between efficiency measures in the input space and "dual" measures will be shown.

At the first stage we will not deal with scale measures. Let the technology be described by a cost function C(y, p) defined by Definition 4 and satisfying Condition C above. Given an observed point of output and normalized prices (y, p) ≡ (y, w/C), the productive inefficiency of a firm can be evaluated by two different measures in the price space.

Definition 7: Output Measure of Economic Efficiency. The measure of economic efficiency according to output is a function E_1: $R_{++}xR^n_+ \rightarrow R_+$ defined for y>0 as

$$E_1(y,p) = \begin{cases} Max\{\delta \mid C(y/\delta, p) \geq 1\}, \forall p \in IntV(1/y) \\ \infty \qquad\qquad\qquad\qquad\quad , \forall p \notin IntV(1/y) \end{cases} \qquad [20]$$

Remember that p are normalized prices[14].

This measure may be called "dual" Farrell measure 1 of economic efficiency because its meaning is implicit in the Farrell's analysis. It has been couched in terms of the cost function but definitions in terms of level sets or the indirect production function are also equivalents[15]. It is an output measure in the sense that provides an evaluation of the non-reached output performance

[14] Int V(1/y)= {p| p∈ R^n_{++}, C(y,p)≤1}.

[15] Preserving the dominion of definition, equivalent definitions in terms of level sets and in terms of the indirect production function are respectively

E_1(y,p)= Max {δ | p ∈ V(δ/y) };

E_1(y, p)= Max {δ | h(p) ≥ δ/y }.

presented by inefficient firms in relation to the efficient one at the same price-cost operating conditions. It can be better understood by considering an alternative expression

$$E_1(y,p) = \frac{y}{\{\alpha \mid C(\alpha,p) = 1\}}, \qquad [21]$$

it shows, for the one output case, the computation of the output measure in terms of observed variables and assuming the knowledge of the unit cost frontier. It gives and index of the maximum amount by which the observed output can be increased operating with best practice technology.

Definition 8: (Normalized) Price measure of economic efficiency. The measure of economic efficiency in relation to normalized prices is a function E_2: R_{++}x $R^n_+ \rightarrow R_+$ defined for y>0 as

$$E_2(y,p) = \begin{cases} Max\{\nu \mid C(y,p/\nu) \geq 1\}, \forall p \in IntV(1/y) \\ \infty \qquad\qquad\qquad\qquad\qquad , \forall p \notin IntV(1/y) \end{cases} \qquad [22]$$

This measure may be called "dual" Farrell measure 2 of economic efficiency and is also in the spirit of Farrell's work. Equivalent definitions in terms of level sets and indirect production functions can also be formulated as in footnote 15. It is a measure of cost reductions that inefficient firms can achieve producing the same output with best practice technology. For one single output and observed variables, E_2 becomes

$$E_2(y,p) = \frac{\| p \|}{\{\| p \| \mid C(y,p^*) = 1\}} = \frac{\| w \| / C}{\| w \| / C^*} = \frac{C^*}{C}, \qquad [23]$$

i.e. in presence of the same price vector, E_2 reduces to a ratio between minimum and observed costs.

With regard to Definitions 7 and 8 several points can be stressed

i) E_1 and E_2 are measures of economic efficiency. They are defined in relation to a unit cost frontier (or price-indifference frontier) which contains full information about the economic aspects of the process of production, so the overall (economic) behaviour of production units can be measured by them. However, technical or allocative components of the observed overall behaviour cannot be derived from the exclusive analysis of E_1, E_2.

ii) Both are radial measures. The sense of radial measures of efficiency has been questioned in the input space, even for homothetic technologies, by FL (1978) because they assume a given input mix. In the price space, under the assumption of exogenous price vector, the measurement of efficiency using radial methods has an obvious economic meaning: all production units face the same price vector and therefore their representative points are situated on the same ray in the space of prices.

iii) They provide, under the regularity conditions exposed in previous sections (Conditions A, B, C above), different quantitative measures of economic efficiency. The different empirical content of both measures can lead to erroneous inferences about the structure of efficiency of an industry if the particular measure used in any empirical analysis is not clearly specified. The choice of a specific measure depends, in the general case, on the goals pursued by the research[16].

[16] The results of section 3 in FL (1978) can also be applied to Definitions 7 and 8. In the next section will be shown the duality between the above Definitions and Definitions 1, 2 in FL (1978).

A graphical exposition of Definitions 7 and 8 is provided, for the two inputs-one output case, by Figure 7.

[Insert Figure 7]

Let A be an inefficient firm which produces an output y^0 with a cost C^0 facing a price vector w^0. The point that represents firm A in the Figure will be on the ray whose slope corresponds to the observed combination of prices: tg $\alpha = w_2^0/w_1^0$. The assumption of inefficiency implies that C^0 is greater than the minimum cost of producing y^0 in presence of w^0; let C^* be this minimum and let B be an efficient firm, i.e. B will produce y^0 with a cost C^*. The point B will be situated on the same ray than A and also on the unit cost frontier corresponding to an output level y^0, i.e. on $C(y^0, p) = 1$. The coordinates of point B in the price space will be $(w_1^0/C^*, w_2^0/C^*)$. Point A has as coordinates $(w_1^0/C^0, w_2^0/C^0)$, where for all i it verifies $w_i^0/C^0 < w_i^0/C^*$, so that the point A is placed inwards the unit cost frontier $C(y^0, p) = 1$. Nevertheless, it will be on a unit cost frontier corresponding to another output level y, i.e. on $C(y, p) = 1$, where $y > y^0$. In the price space inefficient units are situated inwards the frontiers labelled by its output level (remember that cost frontiers are labelled, in the symmetric exposition framework, by level indexes 1/y, reciprocal observed outputs). In the indirect representation of the technology all points are on the same ray, therefore the measure of efficiency as a radial measure has a logical sense. The observed variables are prices, cost and output, and we are interested in comparing the relative performance between points A and B in the Figure; then, it seems obvious to define the efficiency measure as OA/OB. This measure defined in a simple way has not, however, a simple and uniquely determined economic meaning; in fact, it can be interpreted, at least, in two different ways:

a) Output measure. The ratio between the two output levels corresponding to the unit cost frontiers passing through A, B in Figure 7.

$$E_1 = \frac{OA}{OB} = \frac{1/y}{1/y^0} = \frac{y^0}{y}. \qquad [24]$$

b) Price (normalized) measure. The ratio between normalized prices in both points in Figure 7. This ratio is measured along the common ray.

$$E_2 = \frac{OA}{OB} = \frac{\| w^0 \| / C^0}{\| w^0 \| / C*} = \frac{C*}{C^0}. \qquad [25]$$

These two different interpretations of the radial measure defined by means of the graphical exposition of Figure 7 correspond to Definitions 7 and 8 above.

In absence of complementary information about the process of production we cannot know anything about the causes of the observed overall inefficiency, i.e. which part can be accounted for technical inefficiency and which for allocative inefficiency. However, if knowledge of observed input bundles is provided we can measure allocative efficiency by means of the application of Shephard´s lemma in point B. In the indirect representation of the technology technical efficiency can only be measured as a residual measure, the overall inefficiency does not accounted for allocative inefficiency.

After explaining Definitions 7 and 8, scale efficiency measures in the price space will be defined. To obtain a measure of the scale efficiency of a firm, that is, an index of how close it is to the optimal scale of production we need a further hypothesis, namely, that for every vector of market prices there exists an unique level of output, y*, that minimizes the average cost function. In other words, the measurement of scale efficiency in production implies the existence and the definition of a price-efficiency frontier that represents the optimal scale of production in different conditions of normalized prices. As is well known, all points on that price-efficiency frontier are points with unitary elasticity of scale (elasticity of scale couched in terms of the cost function). Under the above condition three different measures of scale efficiency have been defined.

Definition 9: Gross Measure of Scale Efficiency. The measure of overall scale efficiency is a function S_1: R_{++}x R^n_+ $\rightarrow R_+$ defined for y>0 as

$$S_1(y,p) = \begin{cases} Max\{\lambda\mu \mid C(y/\lambda, p/\mu) \geq 1, \varepsilon(y/\lambda, p/\mu) = 1\}, \forall p \in IntV(1/y) \\ \infty \qquad , \forall p \notin IntV(1/y) \end{cases} \quad [26]$$

This measure may be considered "dual" to the FH (1979) measure of scale efficiency. It measures globally the inefficiency of a firm that does not produce with minimizing behaviour in the optimal scale of production. It is a gross measure in the sense that does not discriminate between the effects of scale and economic efficiency in the overall observed behaviour of the firm.

Definition 10: Net (according output) Measure of Scale Efficiency. The net measure of scale efficiency that eliminates the contribution of the output economic inefficiency (the measure of economic efficiency according outputs, E_1) is a function S_2: R_{++}x R^n_+ $\rightarrow R_+$ defined for y>0 as

$$S_2(y,p) = \begin{cases} Max\{\lambda\mu \mid \varepsilon(y^*/\lambda, p/\mu) = 1; C(y^*/p) = 1\}, \forall p \in IntV(1/y) \\ \infty \qquad , \forall p \notin IntV(1/y) \end{cases} \quad [27]$$

This measure is a net one in the sense that quantifies the scale inefficiency by removing previously the aspects of economic inefficiency related with the level of output; i.e. it measures the scale inefficiency from a point placed on the unit cost frontier with the observed cost.

Definition 11: Net (according prices) Measure of Scale Efficiency. The net measure of scale efficiency that eliminates the contribution of the price economic inefficiency (i.e. E_2) is a function S_3: $R_{++}x\ R^n_+ \rightarrow R_+$ defined for y>0 as

$$S_3(y,p) = \begin{cases} Max\{\lambda\mu \mid \varepsilon(y/\lambda, p^*/\mu) = 1; C(y,p^*) = 1\}, \forall p \in IntV(1/y) \\ \infty \qquad ,\forall p \notin IntV(1/y) \end{cases} \qquad [28]$$

This net measure quantifies the scale inefficiency by leaving out the aspects of economic inefficiency related with relative prices; i.e. the scale inefficiency is measured from a point on the unit cost frontier with the observed output.

[Insert Figure 8]

In Figure 8, Definitions 9, 10 and 11 have been graphically described. The Figure shows the evolution in relation to output of the average cost function for a given price vector w*. The average cost curve is U-shaped presenting an optimal scale of production in point D in which the average cost equals the marginal cost, i.e. the elasticity of scale is unity. Points placed to the right of D present (ε>1) increasing returns to scale, while points situated to the left show decreasing returns to scale (ε<1). Let A be an inefficient firm that produces y^0 with an average cost C (so that its total cost is Cy^0). Firm A is economically and scale inefficient: it is placed over the cost average curve, i.e. produces y^0 with greater cost than the minimum or does not produce a greater output with the same observed cost; even more, its production is also greater than the optimal scale of production. Firms B and B' are economically efficient: they are on the average cost curve, the former produces an output greater than firm A ($y>y^0$) with the same cost ($C'=Cy^0/y$), the later produces the same output y^0 with lower cost ($C^0<C$) than firm A. However, B and B' are scale inefficient. In this context the scale inefficiency can be measured by three different methods:

i) Directly by comparing the observed firm A with the economic efficient one producing at the optimal scale of production D. We have called it a gross measure of scale efficiency (Definition 9) because it includes economic inefficiency too.

ii) Indirectly by means of a way through point B, i.e. by comparing points A, D, but eliminating the inefficiency accounted for economic aspects measured according outputs. What we have called a net output measure of scale efficiency (Definition 10).

iii) Indirectly through point B', i.e. eliminating the inefficiency accounted for the price-measure of economic efficiency. What we have called a net price measure of scale efficiency (Definition 11).

[Insert Figure 9]

The exposition above can be better understood if we pass from Figure 8 to Figure 9 where the price-efficiency frontier has been drawn. Note that like in the direct representation the efficiency frontier must be represented in the input-coefficient space, in the indirect representation of the technology the price-efficiency frontier must be represented in an observed prices-observed average cost coefficient space; namely, in Figure 9 points are characterized by observed prices-observed average cost ratios, i.e.

$$\frac{w}{\frac{C}{y}} = y\frac{w}{C}.$$

Products of normalized prices by observed output.

In the Figure the ray through the origin has a slope determined by the observed price vector w*. All firms will lie on that ray; firms with lower average costs will be placed outwards and the efficient point D on the efficiency frontier (the geometric locus of all points where the elasticity of scale is unity). Then a descriptive view of the above measures is given in the Figure. They are radial measures that explain the aspects of inefficiency in which we are interested:

i) A in relation to D, the overall inefficiency: scale and economic inefficiency.
ii) B in relation to D, the aspect of inefficiency does not account for economic inefficiency according to outputs.
iii) B' in relation to D, the aspect of inefficiency does not account for economic inefficiency according to prices.

7. The Duality exhibited.

Once the relevant measures of efficiency in terms of the cost function have been defined, it remains to reveal the duality between such measures and the one defined by FL (1978) and FH

(1979). Firstly, efficiency measures in the input space will be briefly outlined and after that a graphical treatment will be employed to show that Definitions 7-11 are dually symmetric to the input space efficiency measures. In order to do that a reelaboration of Figure 6 will be needed.

FH (1979) use the efficiency frontier to obtain an unambiguous representation of efficiency measures in terms of production function. This method assumes that the direct production function f(x) is of the "regular ultra-passum" type, Frisch (1975). This requires that:

i) Along any expansion path there is a unique level of output, y*, that satisfies

$$\varepsilon(y^*) = \sum_{i \in I} f_i \frac{x_i}{y^*} = \sum_{i \in I} f_i \theta_i^* = 1. \qquad [29]$$

ii) As we move in a nor-east direction along that expansion path the scale elasticity function is decreasing (strictly).

In other words, the "regular ultra-passum" production function implies that, for any market price vector there is a unique technical optimal contour, i.e. the locus of all production plans (x, y) that satisfy (29).

The transposition of the technical optimal contour from the input space to the input-coefficient space gives us the efficient frontier, i.e. the locus where the input coefficients θ=x/y obtain their minimum values along the expansion paths. The representation of the efficiency frontier for the two inputs-one output case can be found in FH (1979).

In this framework, to show duality between both types of efficiency measures their definitions must be expressed in a similar way. The known efficiency measures in the input space have been properly formulated in Table 1, where they have been set besides Definitions 7-11 above. Definitions $E_1(y, x)$, $E_2(y, x)$ can be found in FL (1978). Scale efficiency measures in the input space can also be found in FH (1979) where they are defined by means of a geometrical procedure; $E_3(y, x)$, $E_4(y, x)$, $E_5(y, x)$ have been defined in a formal way but in the sense of the FH measures[17].

[Insert Figure 10]

The graphical procedure of revealing duality can be performed as follows. In quadrant 1 of Figure 10 (the input-coefficient space) the efficiency frontier and the iso-average-cost line have been represented

$$\frac{C}{y} = \sum_i \frac{x_i}{y} w_i = \sum_i \theta_i w_i,$$

or alternatively,

$$1 = \sum_i \theta_i \frac{w_i}{C} y = \sum_i \theta_i p_i y. \qquad [30]$$

Obviously, the slope of the iso-average-cost line is solely determined by observed prices, i.e.

[17] E_3-E_5 are, of course, a particular interpretation of the F-H measures. The possible errors are our own

$$tg\alpha = tg(180-\alpha) = -tg\beta = \frac{-OA}{OB} = \frac{-p_1 y}{p_2 y} = \frac{-w_1}{w_2}. \quad [31]$$

The point where the iso-average-cost line supports the efficiency frontier gives us the optimal input coefficients θ_1^*, θ_2^*: the input coefficients that, given observed prices, allow the firm to operate at the technical optimal scale. The problem can be stated as

$$\begin{aligned} &Min \sum_i \theta_i w_i = \sum_i x_i \frac{w_i}{y}, \\ &F(\theta y) = y. \end{aligned} \quad [32]$$

The Lagrangian is

$$L(\lambda, \theta) = \sum_i \theta_i w_i - \lambda \{F(\theta y) - y\}, \quad [33]$$

The first order conditions are

$$\begin{aligned} &w_i = \lambda y F_i(\theta y), \qquad \forall i = 1,....,n. \\ &y = F(\theta y), \end{aligned} \quad [34]$$

and, dividing any of the first n equations by the first, we have

$$\frac{w_i}{w_1} = \frac{F_i(.)}{F_1(.)}, \quad [35]$$

---

responsibility.

which guarantees that the solution to the average cost minimization also satisfies the conditions of total cost minimization.

In quadrant 3 of the Figure, the price efficiency frontier or the indirect efficiency frontier, i.e. the locus of the normalized price vectors that allow the firm to operate in the technical optimal scale, will be constructed.

Consider the iso-average-cost line KL. Point L represents the maximum θ_1 that the firm could have used OL= θ_1^{max}. From (30) we have that

$$1 = \theta_1^{max} p_1 y$$
$$\theta_1^{max} = \frac{1}{p_1 y} = \frac{1}{\frac{w_1 y}{C}} = \frac{1}{\xi_1}, \qquad [36]$$

Where ξ_1 is the normalized (average cost) price of input x_1[18].

In quadrant 2, the unit rectangular hyperbola 1= $\theta_1\xi_1$ has been constructed. So, given any θ_1^{max} we can find the corresponding level of normalized (average cost) price of input x_1 as the reciprocal of θ_1^{max}.

Similarly, in quadrant 4, the unit rectangular hyperbola 1= θ_2 ξ_2 has been constructed, that obviously relates θ_2 with ξ_2.

[18] Although the relationship is between input coefficients and normalized (average cost) prices, the axis in Figure 10 are labelled so to make duality apparent. So, instead of θ_i and ξ_j they are labelled x_i/y and $p_i y$ respectively. This, we believe, make more transparent previous results.

Thus, given the iso-average-cost line KL, the point (θ_1*, θ_2*) is related to θ_1^{max} and θ_2^{max} which, in turn, are associated with the point (ξ_1, ξ_2) in the normalized price space. By iteration of this procedure for all iso-average-cost lines that support the efficiency frontier the price or indirect efficiency frontier can be obtained.

The attainment of the efficiency frontier from the indirect efficiency frontier follows the reasoning previously exposed (Section 3) and is not reproduced here.

Passing now to the efficiency measures, points A, B, B' and D in the input coefficient space in Figure 10 represent firms with different situations of inefficiency. As can be seen they correspond with the firms represented in Figures 7 and 9 above. Firm A is characterized by

i) being neither on the production nor on the average cost curve, which means that it does not minimize the cost of producing output level y^0,

ii) producing an output level y^0 that does not corresponds with the technical optimal scale.

So, the efficiency measure OA/OD is a measure of the gross scale inefficiency type (Definition 9).

Let us see how this firm's position in input space can be "transferred" to the normalized (average cost) price space. As firm A faces the observed prices, it should be on the ray OW. But since it suffers, as Figure 10 shows, the highest average costs, the resulting normalized (average cost) prices of this firm are the smallest of all others considered here. So, as the firm A is situated further outwards that the other firms with respect to the efficiency frontier, it should be placed further inwards with respect to the price efficiency frontier. This can be clearly seen in Figure 10.

The same reasoning can be applied to firms B and B' obtaining in this way the same ranking of efficiency for the firms.

Turning back now to Table 1, the symmetric duality between both types of efficiency measures is exhibited by the symmetric formulation of the expression of the measures. Changing y for 1/y and x for p, respectively, and vice versa we can obtain efficiency measures in the input space from efficiency measures in the price space and vice versa.

8. Concluding remarks.

We provide in the paper several definitions of efficiency measures in the price space. Economic and scale aspects of inefficiency are considered to give empirical content to the measurement of efficiency when the production technology is described by cost functions models. We show, in the Hanoch's symmetric duality approach, that the new measures preserve the ranking of efficiency, are formally dually symmetric to the measures defined in the input space, and both are established with respect to different descriptions of the same technology. Additionally, graphical procedures are utilized to make an insight into the achievement of polar technologies from primal ones and into the relationship between Shephard's lemma and Roy's identity.

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Figure 1. Production function, level set and distance function.

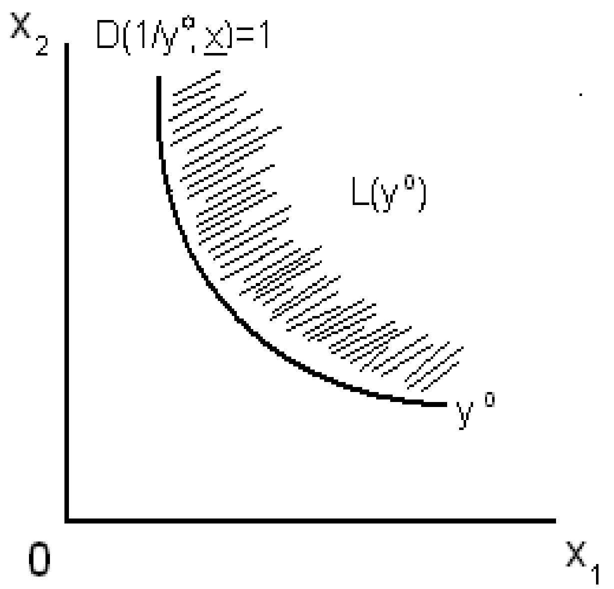


Figure 2. Direct and indirect representations of a technology.

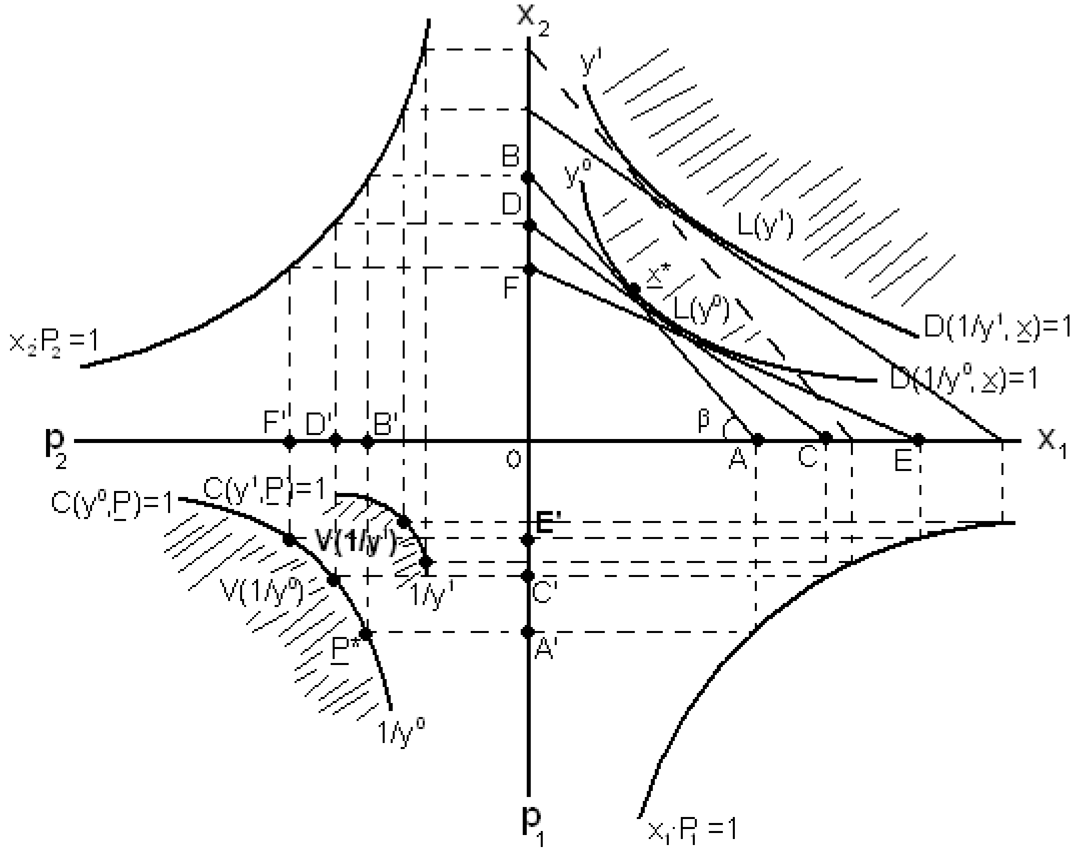


Figure 3. Price-indifference map of the indirect production function.

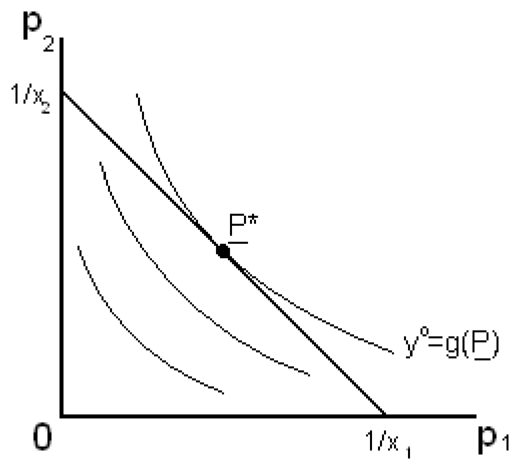


Figure 4. Input bundle for the optimum price combination choice.

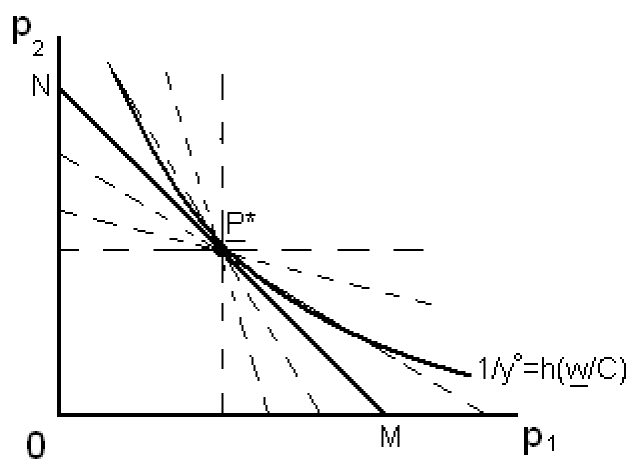


Figure 5. Reconstruction of the isoquant map.

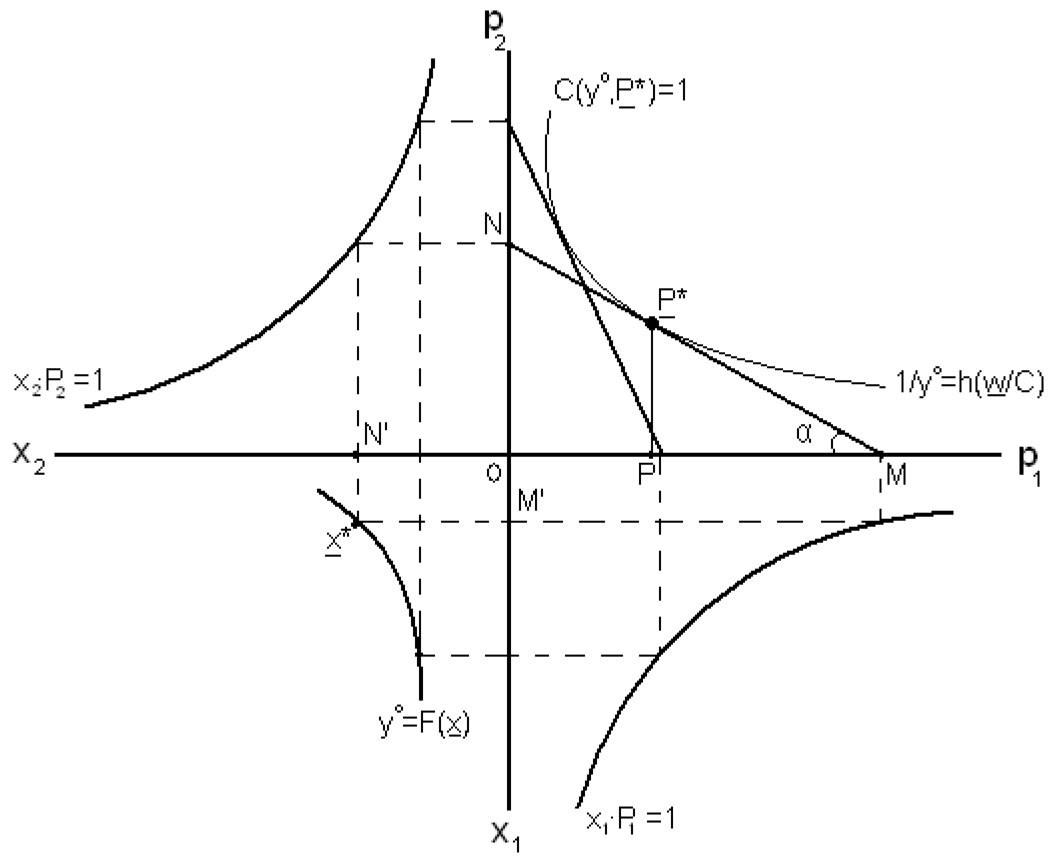


Figure 6. Polar technologies.

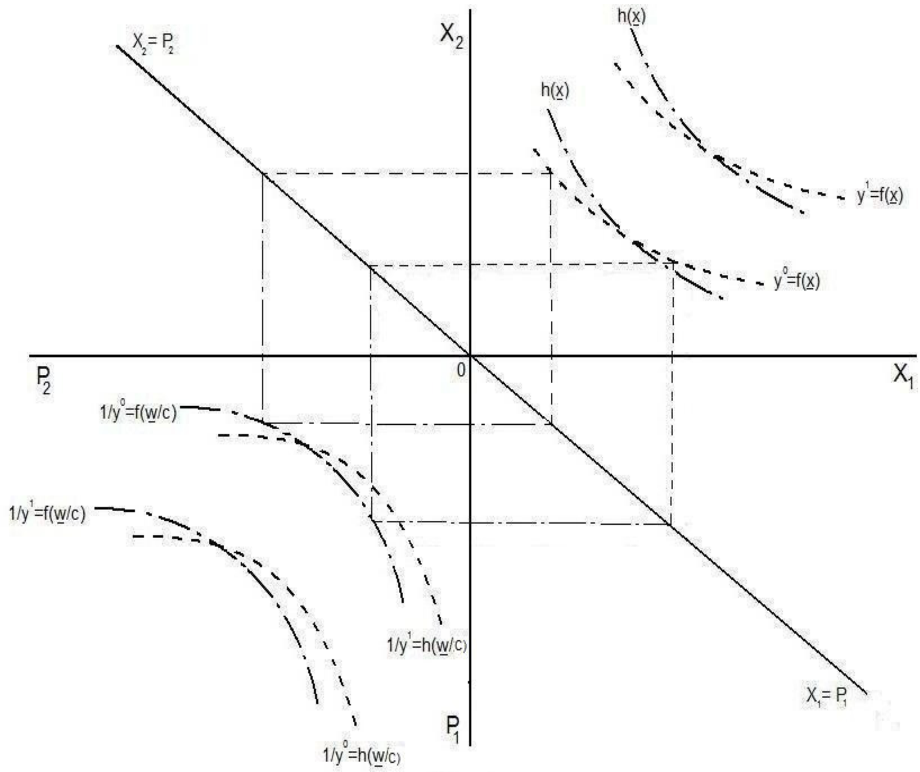


Figure 7. Output and (normalized) price measures of economic efficiency.

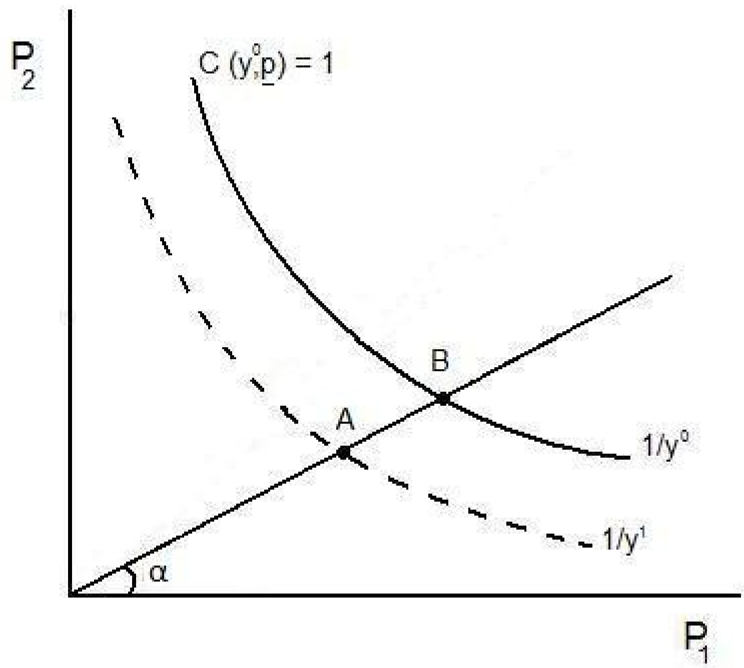


Figure 8. Gross, net (according output) and net (according prices) measures of scale efficiency.

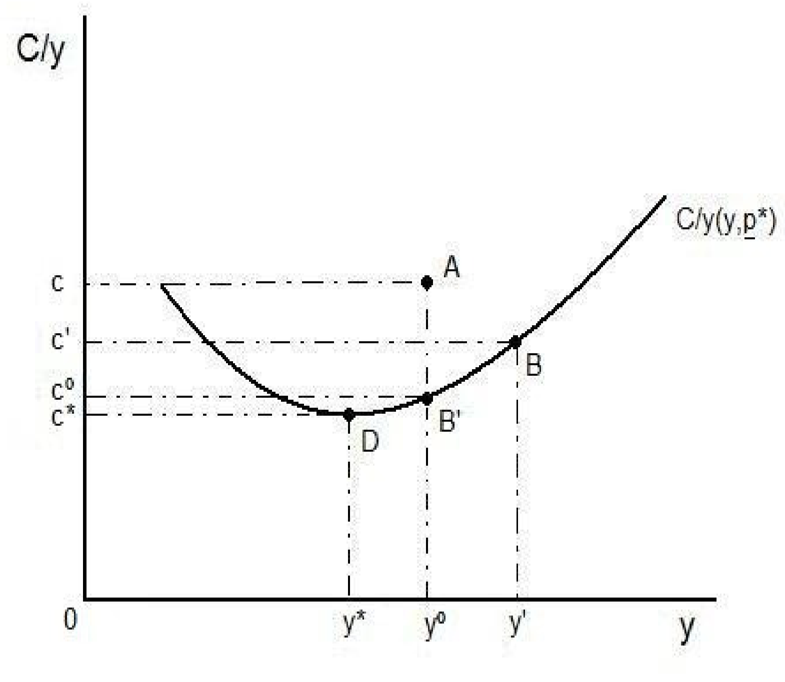


Figure 9. Economic efficiency measures in the indirect representation of the technology.

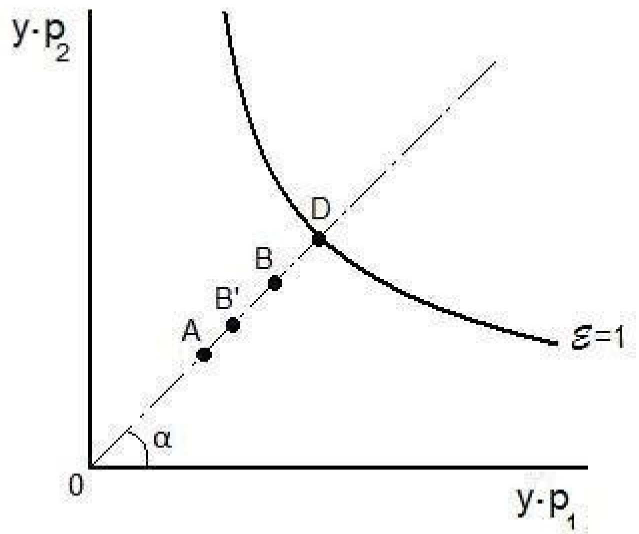
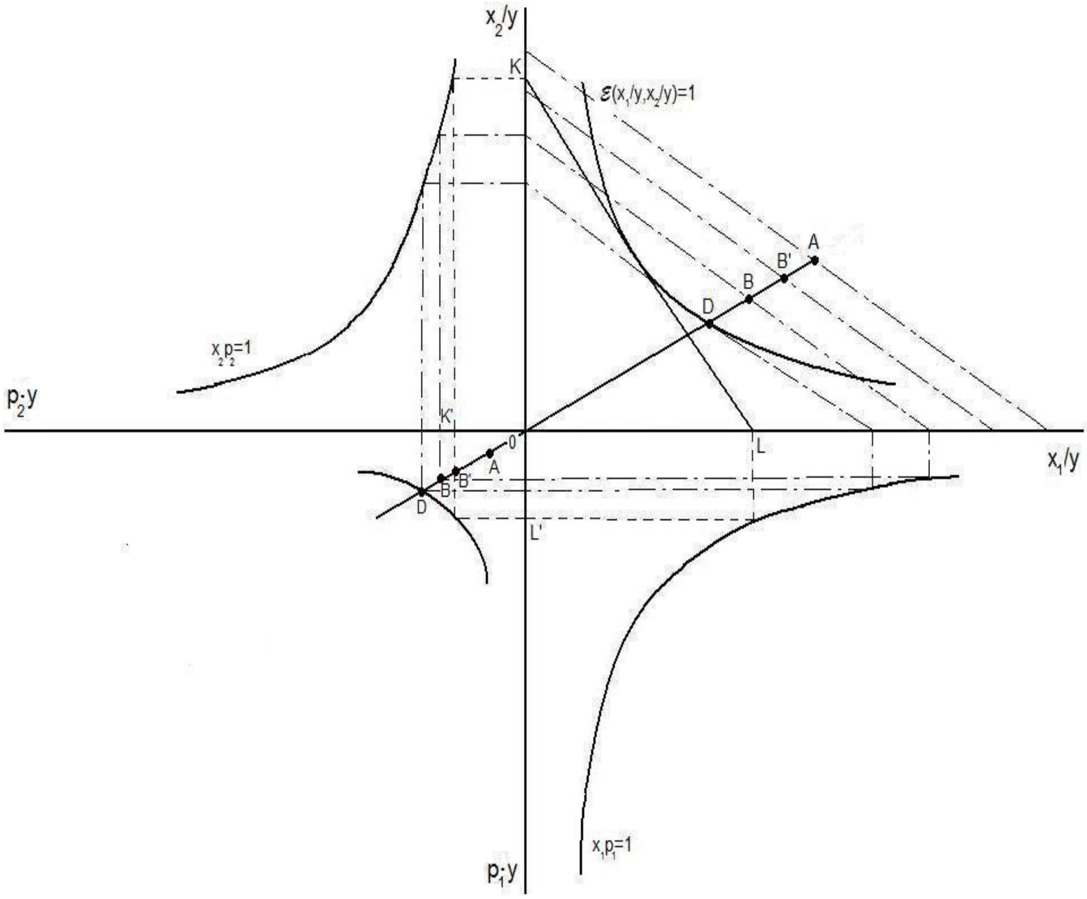


Figure 10. Revealing duality.



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