THE BULLETIN OF OKAYAMA UNIVERSITY OF SCIENCE No. 39 A pp. 17-26 (2003)

Degeneracy of Herman Rings

Yasutoshi NOMURA

Department of Applied Science,

Faculty of Science,

Okayama University of Science,

Ridaicho 1—1, Okayama 700—0005, Japan

(Received November 7, 2003)

1. Introduction

One of the most interesting topics in complex dynamics is a study of rotation domains. N. Steinmetz has shown in his book [5] that the complex dynamics of the Blaschke product

$$f_S(z) = \exp(2\pi si)z\,\frac{z-a}{1-az}\cdot\frac{1+az}{z+a} \tag{1.1}$$

has a Herman ring if s is chosen appropriately and if a is sufficiently large, and further may have Siegel discs with centers 0 and ∞.

Furusawa [1] has shown that the above Steinmetz' function has a Herman ring only if $a>5^{1/2}+2$. In this article we give further examples of Blaschke products for which the similar phenomena occur, together with determination of parameter- range for Herman rings to be *degenerate into the unit circle*. Although it is not *a priori* evident that, for s giving a Herman ring, the Blaschke product has Siegel discs with 0 and ∞ as center, it seems to be most likely, by computer experiment, that there *coexist* a Herman ring and two Siegel discs for Blaschke products we have considered. At the end of this paper we show some pictures of a Herman ring and two Siegel disks of a Blaschke product which gives the rotation number $(13^{1/2}-3)/2 = [0; 3, 3\ 3, \cdots]$, a solution of $x^2+3x-1=0$, on the unit circle.

2. Rotation numbers and semi-reciprocal equations

We consider the Blaschke product

$$f(z) = \exp(2\pi si)\, z\prod_{j=1}^{p}\frac{z-a_j}{1-\overline{\overline{a_j}}z}\cdot\prod_{j=1}^{p}\frac{1-\overline{\overline{b_j}}z}{z-b_j} \tag{2.1}$$

which is a conformal map on a neighborhood U of the unit circle $|z|=1$. We assume that $f(z)=z\exp(ig(z))$ where g is holomorphic in U and real-valued on $|z|=1$.

Let $z=e^{it}$; then $f(z)=\exp(i(t+g(z)),\ f^2(z)=f(f(z))=\exp(i(t+g(z)+g(f(z))))$.

It follows by induction that

$$f^n(z)=\exp(iT_n(z)),\quad T_n(z)=t+g(z)+g(f(z))+\cdots+g(f^{n-1}(z)),$$

in other words, T_n is the lift of f^n with respect to the universal cover $t\to e^{it}$. Thus the **rotation number** μ is given by

$$\mu=\lim_{n\to\infty}\frac{T_n-t}{2n\pi}=\lim_{n\to\infty}\frac{g(z)+g(f(z))+\cdots+g(f^{n-1}(z))}{2\pi n}.$$

Since Herman rings (and Siegel discs) are bounded by two (or a) forward orbits of critical points [2, 3], the following proposition allows one to determine when a Herman ring is degenerate into the circle $|z|=1$.

Proposition 1.1. *A necessary and sufficient condition for which the equation*

$$a_0z^4+a_1z^3+a_2z^2+\overline{a_1}z+\overline{a_0}=0,$$

where a_2 is real, has a root $\exp(it)$ *of absolute value* 1 *is that the equation*

$$2\,\mathrm{Re}(a_0)\cos 2t-2\,\mathrm{Im}(a_0)\sin 2t+2\,\mathrm{Re}(a_1)\cos t-2\,\mathrm{Im}(a_1)\sin t+a_2=0$$

admits a real root t, or equivalently, that the equation

$$[a_2+2\,\mathrm{Re}(a_0)-2\,\mathrm{Re}(a_1)]\,u^4+[8\,\mathrm{Im}(a_0)-4\,\mathrm{Im}(a_1)]u^3+[2a_2-12\,\mathrm{Re}(a_0)]\,u^2$$
$$-[8\,\mathrm{Im}(a_0)+4\,\mathrm{Im}(a_1)]\,u+2\,\mathrm{Re}(a_0)+2\,\mathrm{Re}(a_1)+a_2=0$$

has a real root u.

Proof. Observing that, for $z=e^{it}$,

$$a_0z^2+\overline{a_0}z^{-2}=2\,\mathrm{Re}(a_0e^{2it}),\qquad a_1z+\overline{a_1}z^{-1}=2\,\mathrm{Re}(a_1e^{it}),$$

the first equation follows by dividing by z^2. The second can be derived by substitutions $u=\tan(t/2)$, hence $\sin t=2u(1+u^2)^{-1}$, $\cos t=(1-u^2)(1+u^2)^{-1}$, $\sin 2t=4u(1-u^2)(1+u^2)^{-2}$, $\cos 2t=(u^4-6u^2+1)(1+u^2)^{-2}$.

3. The first example

We consider the function

$$f_1(z)=\exp(2\pi is)\,z\,\frac{z-a}{1-az}\cdot\frac{1-aiz}{z+ai}\qquad(a>0).\tag{3.1}$$

The derivative $f_1'(z)$ has, up to a constant factor, the numerator

$$q_1(z) = a^2 iz^4 - 2a(a^2 + i)z^3 + (a^2+1)^2 z^2 - 2a(a^2 - i)z - a^2 i,$$

whose zeros are critical points of $f_1(z)$ and can be calculated by the Ferrari method. The discriminant D_1 of $q_1(z)$ is given by, up to a constant factor,

$$2ia^6(a^2-1)^2(a^2+1)^2(a^8 - 14a^6 + 24a^4 - 14a^2 + 1).$$

The last factor becomes, by substitutions $a^2 = b$ and $b + b^{-1} = c$,

$$b^4 - 14b^3 + 24b^2 - 14b + 1 = b^2(c^2 - 14c + 22).$$

Thus $c = 7 + 27^{1/2}$ gives roots $b = (7 + 27^{1/2} \pm (72 + 14 \cdot 27^{1/2}))/2$, whence we see that

$$a_1 = 3.4804598236 \quad \text{and} \quad a_2 = 0.287318356$$

are the values of a for which $q_1(z) = 0$ has multiple roots.

Applying Proposition 1.1 to $q_1(z)$ one gets the equation

$$h_1(u) = ((a^2+1)^2 + 4a^3)u^4 + (8a^2 + 8a)u^3 + 2(a^2+1)^2 u^2 + (8a - 8a^2)u + (a^2+1)^2 - 4a^3 = 0,$$

whose discriminant is

$$2^{17}(a-1)^2(a+1)^3(a^2+1)^2 a^4(a^3 + 3a^2 - a + 1)(a^8 - 14a^6 + 24a^4 - 14a^2 + 1).$$

It follows that $f_1(z)$ has two critical points on the unit circle if and only if the equation $h_1(u) = 0$ has two real roots. Observe that

$$h_1(0) = (a^2+1)^2 - 4a^3 < 0 \quad \text{for } 1 < a \leqq 3.38,$$

$$h_1(-1) = 4(a^2+1)^2 - 16a < 0 \quad \text{for } 0.3 \leqq a < 1$$

$$h_1(-1.2) = (3721a^4 + 2684a^3 + 4802a^2 - 14640a + 3721)/625 < 0 \quad \text{for } 0.28733047 \leqq a \leqq 0.9833716$$

This shows that, for a with $a_2 < a < a_1$, $f_1(z)$ has two critical points on the unit circles and hence one concludes

Proposition 3.1. *Hermann rings of $f_1(z)$ degenerate for a in* $[a_2, a_1]$.

Naturally, in order that $f_1(z)$ for a not in $[a_2, a_1]$ can admit a Herman ring, one might take s for which the rotation number satisfies certain number-theoretic conditions such as Siegel, Arnold, Bryuno or Yoccoz found (see [4, pp. 119−123], [5, pp. 90−110]).

The following proposition will be useful in detecting Siegel discs at 0 and ∞.

Proposition 3.2. *$f_1(z)$ has multipliers*

i) $\exp 2\pi(s + 1/4)i$ at 0

ii) $\exp 2\pi(3/4 - s)i$ at ∞.

Proof. This follows from the following computation:

$$\frac{f_1(z)}{\exp(2\pi si)} = \frac{z(z-a)(1-aiz)(1+az+a^2z^2+\cdots)}{ai(1-a^{-1}iz)}$$

$$= iz(1 - a^{-1}z)(1 - aiz)(1 + az + \cdots)(1 + a^{-1}iz + \cdots)$$

$$= \exp(\tfrac{1}{2}\pi i)\, z + \text{higher terms}$$

4. The second example

Consider the function

$$f_2(z) = \exp(2\pi s i)\, \frac{z(z-ai)(1-aiz)}{(1+aiz)(z+ai)} \qquad (a>0) \qquad (4.1)$$

The derivative $f_2{}'(z)$ has, up to a constant factor, the numerator

$$q_2(z) = a^2 z^4 + 2ai(a^2-1)z^3 + (a^4+1)z^2 + 2ai(1-a^2)z + a^2$$

Applying Proposition1.1 to $q_2(z)$ we see that $q_2(z)=0$ has two roots on the unit circle if and only if

$$h_2(u) = (a^2+1)^2 u^4 - 8a(a^2-1)u^3 + 2(a^4-6a^2+1)u^2 - 8a(a^2-1)u + (a^2+1)^2$$

has two real zeros. Setting $v = u + u^{-1}$ $h_2(u)$ becomes

$$\phi(v) = (a^2+1)^2 v^2 - 8a(a^2-1)^2 v - 16a^2$$

with discriminant $D/4 = 32a^2(a^4+1)$, so that $h_2(u)$ has two (real) zeros if and only if $\phi(v)$ has at least one non-zero real zero v such that $|v| > 2$. The negation of the latter is that $\phi(v)$ has roots α, β such that $-2 < \alpha, \beta < 2$. Examining inequalities

$$4-(\alpha+\beta) = 4(a^2+1)^{-2}(a^2(a-1)^2+a^2+2a) > 0,$$

$$(2-\alpha)(2-\beta) = 4(a^2+1)\ ^2(a^2-1)(a^2-4a-1) > 0,$$

$$4+(\alpha+\beta) = 4(a^2+1)(a^2(a+1)^2+(a-1)^2) > 0,$$

$$(2+\alpha)(2+\beta) = 4(a^2+1)^{-2}(a^2-1)(a^2+4a+1) > 0,$$

one has

Proposition 4.1. *Hermann rings of* $f_2(z)$ *degenerate for a in* $[5^{1/2}-2,\ 5^{1/2}+2]$.

The proof of the following proposition is similar to Proposition 3.2.

Proposition 4.2. $f_2(z)$ *has multipliers*

i) $\exp(2\pi(s+\tfrac{3}{4})i)$ *at* 0

ii) $\exp(2\pi(\tfrac{3}{4}-s)i)$ *at* ∞

5. Steinmetz' example

Consider the function (1.1) whose derivative has, up to a constant factor, the numerator

$$q(z) = a^2 z^4 - 2a(1-a^2)z^3 - (a^4+1)z^2 - 2a(1-a^2)z + a^2.$$

Setting $z + z^{-1} = \zeta$ gives rise to the equation

$$a^2\zeta^2 - 2a(1-a^2)\zeta - (a^2+1)^2 = 0\,. \qquad (5.1)$$

Its roots are real and given by

$$\zeta_{\pm} = a^{-1}(1-a^2 \pm (2(a^4+1))^{1/2}),$$

from which the equation $z^2 - \zeta_{\pm} z + 1 = 0$ yields the desired critical points

$$z = \tfrac{1}{2}(\zeta_+ \pm (\zeta_+^2 - 4)^{1/2}), \quad \tfrac{1}{2}(\zeta_- \pm (\zeta_-^2 - 4)^{1/2}).$$

For $\zeta^2 - 4 < 0$ one has non-real z with $|z|^2 = 1$, while for $\zeta^2 - 4 > 0$ one has (distinct) real roots with absolute value $\neq 1$. $\zeta^2 - 4 < 0$ can be solved as follows. It follows from (5.1) that $2a(1-a^2)\zeta + (a^2-1)^2 < 0$, i.e.,

$$(1-a^2)\{3(1-a^2) \pm 2[2(a^4+1)]^{1/2}\} < 0.$$

This yields that, whenever $a > 1$ or $0 < a < 1$, $a^4 - 18a^2 + 1 < 0$, from which one has $9 - 80^{1/2} < a^2 < 9 + 80^{1/2}$. Thus

$$5^{1/2} - 2 < a < 5^{1/2} + 2. \qquad (5.2)$$

Alternatively we proceed as follows. We see from Prop. 1.1 that, for $q(z) = 0$ to have (non-real) roots of absolute value =1 it is necessary and sufficient that the equation

$$(a^2-1)(a^2+4a-1)u^4 + 2(a^4+6a^2+1)u^2 + (a^2-1)(a^2-4a-1) = 0,$$

with discriminant $D/4 = 32a^2(a^4+1) > 0$, has positive root u^2, i.e., simultaneous inequalities

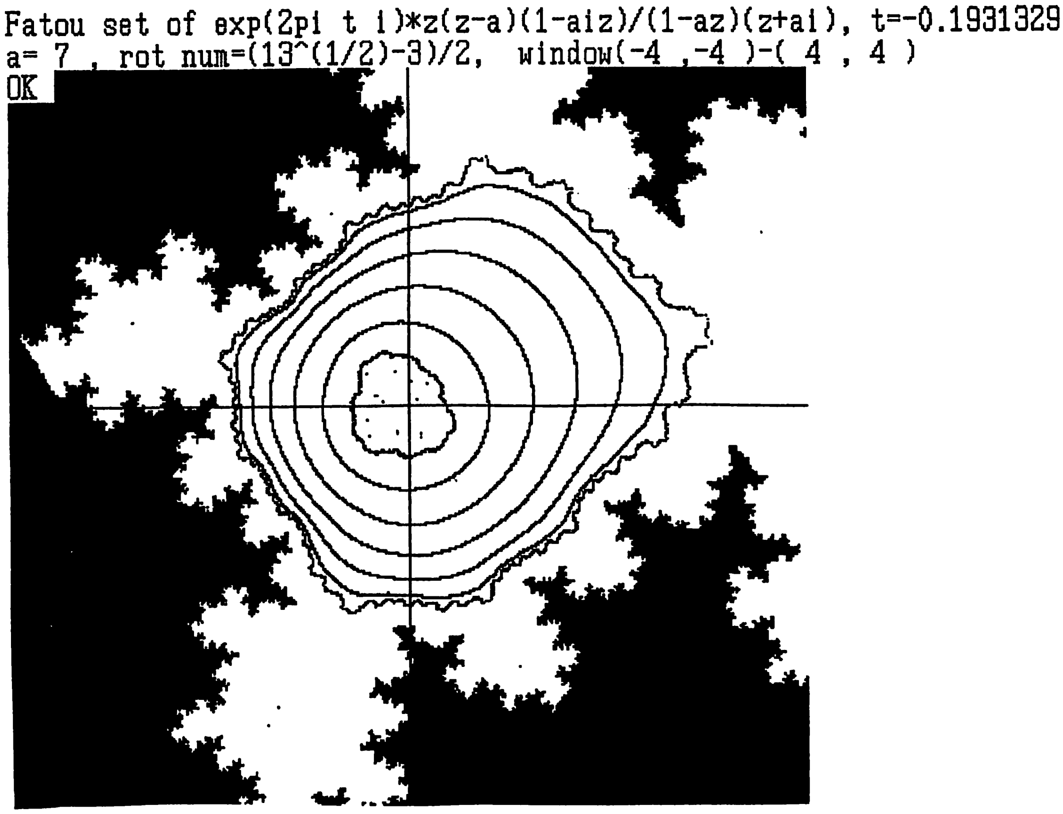
$$(a^2-1)(a^2+4a-1) < 0 \quad \text{and} \quad (a\,2-1)(a\,2-4a-1) < 0$$

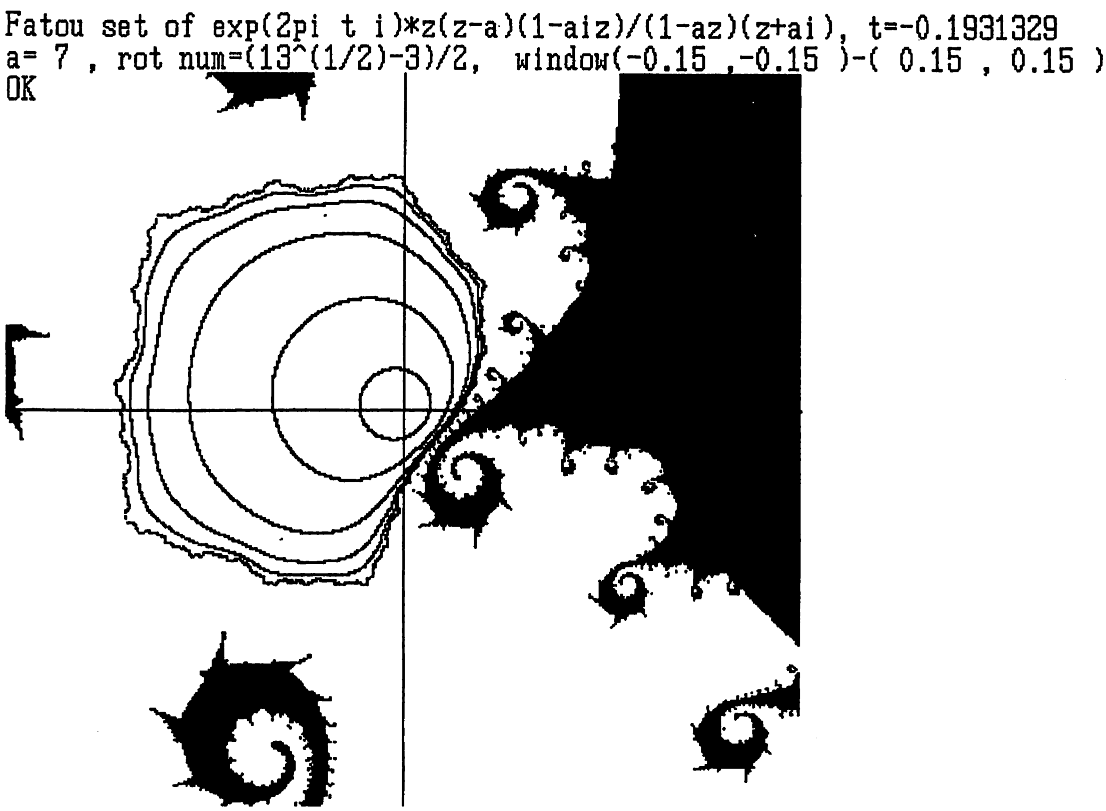
are impossible, which gives rise to (5.2). Thus we have

Proposition 5.1. ***Herman rings of $f_S(z)$ degenerate for a in*** **$[5^{1/2}-2,\ 5^{1/2}+2]$.**

References

1. Y. Furusawa: On stable domains in complex dynamics, Master thesis in Okayama Univ. of Sciences (2002).
2. M.R. Herman: Are ther critical points on the boundary of singular domain ?, Comm. Math.Phys. 99 (1985), 593-612.
3. H.Kriete: Herman's proof of the existence of critical points on the boundary of singular domains, Progress in holomorphic dynamics. Pitman Research Notes on Math. Ser. 387 (1998).
4. J. Milnor: Dynamics in one complex variable: Introductory Lectures. Vieweg (1999).
5. N. Steinmetz: Rational Iteration: Complex Analytic Dynamical Systems. Walter de Gruyter , Berlin, N.Y. (1993).


Fatou set of exp(2pi t i)*z(z-a)(1-aiz)/(1-az)(z+ai), t=-0.1931329
a= 7 , rot num=(13^(1/2)-3)/2, window(-4 ,-4)-(4 , 4)
OK


Fatou set of exp(2pi t i)*z(z-a)(1-aiz)/(1-az)(z+ai), t=-0.1931329
a= 7 , rot num=(13^(1/2)-3)/2, window(-0.15 ,-0.15)-(0.15 , 0.15)
OK

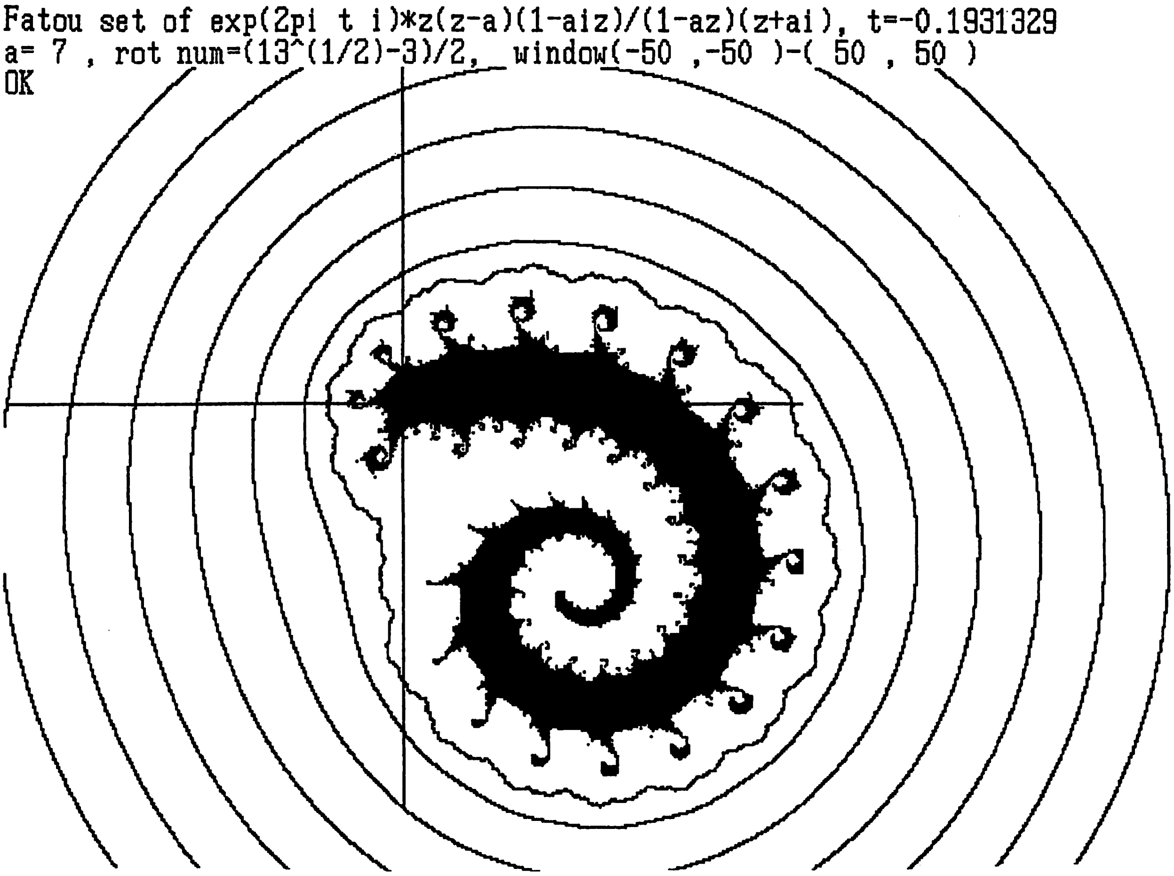

Fatou set of exp(2pi t i)*z(z-a)(1-aiz)/(1-az)(z+ai), t=-0.1931329
a= 7 , rot num=(13^(1/2)-3)/2, window(-50 ,-50)-(50 , 50)
OK

Table 1 s giving the rotation number $(\sqrt{5}-1)/2$ in $\exp(2\pi is)z(z-a)(1-aiz)/(1-az)(z+ai)$

a	s	a	s
0.12	0.61613 60	0.16	0.61510 805
0.2	0.61398 64	0.24	0.61286 45
0.28	0.61188 03	0.286	0.61178 21
4	0.11119 077	14	0.11710 621
5	0.11337 88	15	0.11718 439
6	0.11461 39	16	0.11724 869
7	0.11538 799	17	0.11730 221
8	0.11590 648	18	0.11734 721
9	0.11596 749	19	0.11738 541
10	0.11623 209	20	0.11741 810
11	0.11673 591	21	0.11744 631
12	0.11688 936	22	0.11747 080
13	0.11700 988	23	0.11749 220

Table 2 s giving the rotation number Euler number in $\exp(2\pi is)z(z-a)(1-aiz)/(1-az)(z+ai)$

a	s	a	s
0.1	0.57630 29	0.12	0.57610 71
0.16	0.57581 80	0.2	0.57584 25
0.24	0.31454 60	0.28	0.31784 37
0.286	0.57817 709	0.287	0.57823 412
4	0.07546 68	15	0.07658 638
5	0.07529 25	16	0.07662 460
6	0.07548 97	17	0.07665 678
7	0.07572 56	18	0.07668 411
8	0.07593 09	19	0.07670 750
9	0.07609 69	20	0.07672 766
10	0.07622 87	21	0.07674 517
11	0.07633 34	22	0.07676 045
12	0.07641 72	23	0.07677 388
13	0.07648 512	500	0.07692 645
14	0.07654057		

Table 3 s giving the rotation number $(\sqrt{13}-3)/2$ in $\exp(2\pi i)z(z-a)(1-aiz)(1-az)^{-1}(z+ai)^{-1}$

a	s	a	s
0.12	0.30587 97	0.16	0.30833 186
0. 2	0.31131 38	0.24	0.31454 60
0.28	0.31784 37	0.287	0.31845 51
4	0.18738 43	16	0.19650 093
5	-0.18993 99	17	−0.19659 948
6	-0.19182 35	18	−0.19668 234
7	-0.19313 29	19	−0.19675 266
8	-0.19404 98	20	−0.19681 283
9	0.19470 74	21	0.19686 471
10	-0.19519 180	22	−0.19690 976
11	-0.19555 735	23	−0.19694 911
12	−0.19583 934	500	−0.19737476
13	−0.19606 112		
14	−0.19623 8503		
15	−0.19638 250		
16	−0.19650 093		

Table 4 Roots of $a^2 iz^4 - 2a(a^2+i)z^3 + (a^2+1)z^2 - 2a(a^2-i)z + a^2 i = 0$

a	roots			
0.1	0.045687+0.009744i	-0.045687-0.214004i	-0.954088-4.469122i	20.954132+4.473382i
0.12	0.054843+0.011627i	-0.054956-0.259212i	-0.782724-3.691903i	17.449504+3.699487i
0.14	0.064083+0.013475i	-0.064339-0.305975i	-0.658126-3.129852i	14.944096+3.142353i
0.16	0.073369+0.015281i	-0.073899-0.354816i	-0.562589-2.701185i	13.06312+2.720721i
0.18	0.082709+0.017039i	-0.08374-0.406474i	-0.486203-2.360019i	11.598345+2.389453i
0.20	0.09211+0.0187451i	-0.09403-0.462061i	-0.422908-2.078153i	10.4248282.12147i
0.24	0.11113+0.021972i	-0.117442-0.594003i	-0.320327-1.62016i	8.659973+1.712191
0.28	0.130495+0.024916i	-0.156568-0.820001i	-0.224658-1.176615i	7.393588+1.4117000i
0.286	0.133434+0.025331i	-0.172914-0.91086i	-0.201164-1.089675i	7.233651+1.373205I
3.481	-0.025367-0.133696i	0.926798+0.175848i	1.04149+0.197609i	-1.369855-7.219761i
4	-0.022736-0.115936i	0.634889+0.124508i	1.516744+0.297449i	-1.628898-8.306022i
5	-0.022736-0.115936i	0.634889+0.124508i	1.516744+0.297449i	-1.628898-8.306022i
6	-0.015873-0.076476i	0.371678+0.077142i	2.579391+0.535355i	-2.601862-12.536021i
7	-0.013736-0.065406i	0.312811+0.065692i	3.061789+0.642969i	-3.07515-14.643274i
8	-0.012093-0.057149i	0.270739+0.057289i	3.535299+0.748074i	-3.543945-16.748214i
9	-0.010794-0.05075i	0.238953+0.050825i	4.003788+0.8516i	-4.009725-18.851675i
10	-0.009744-0.045643i	0.214004+0.045687i	4.469122+0.954088i	-4.473382-20.954132i
11	-0.008878-0.041472i	0.19386+0.041499i	4.932327+1.055846i	-4.935492-23.055873i
12	-0.008152-0.038002i	0.177235+0.038019i	5.394021+1.157069i	-5.396437-25.157086i
13	-0.007534-0.035068i	0.163269+0.035079i	5.854598+1.257887i	-5.856486-27.257899i
14	-0.007004-0.032555i	0.151365+0.032563i	6.314325+1.358391i	-6.315829-29.358399i
15	-0.006542-0.030379i	0.141093+0.030384i	6.773388+1.458646i	-6.774606-31.458652i
16	-0.006137-0.028476i	0.132138+0.02848i	7.231924+1.5587i	-7.232924-33.558704i
17	-0.00578-0.026797i	0.124258+0.0268i	7.690032+1.65859i	-7.690863-35.658593i
18	-0.005461-0.025306i	0.117271+0.025308i	8.147788+1.758343i	-8.148487-37.758346i
19	-0.005176-0.023972i	0.111032+0.023973i	8.605251+1.857983i	-8.605844-39.857984i
20	-0.004919-0.022771i	0.105426+0.022772i	9.062467+1.957525i	-9.062974-41.957526i
21	-0.004686-0.021685i	0.100362+0.021686i	9.619473+2.056985i	-9.519911-44.056986i
22	-0.004474-0.020698i	0.095763+0.020699i	9.9763+2.155373i	-9.97668-46.156374i
23	-0.00428-0.019797i	0.091569+0.019798i	10.432971+2.255699i	-10.433303-48.2557i