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Stable Cycles of Odd Periods in Chaotic Dynamics

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1. Introduction

In a 1-dimensional dynamical system arising from iterates of a smooth function with one parameter, the behavior of periodic orbits of a fixed period changes as the parameter varies. For the parameter values at which stable cycles of periods $(2m+1)2^n$ are born (tangent bifurcation) and die (period-doubling bifurcation), Feigenbaum discovered and conjectured the universal asymptotic rate of their first differences with respect to n.

In this article I assert that similar phenomena occur with respect to m. In order to illustrate this assertion I perform the numerical experiments for the entire function $\lambda x e^x$ and few members of the polynomial family converging to this function.

2. Geometrically convergent sequences

Let u_m, $m = 1, 2, \cdots$ be a strictly monotone sequence with real terms; then the first difference

$$\Delta u_m = u_{m+1} - u_m$$

is a sequence whose terms have the same sign. So one may form the successive ratios

$$\frac{\Delta u_{m+1}}{\Delta u_m}$$

If this ratios converge to a limit r smaller than 1 as m tends to the infinity we call the sequence u_m *converges geometrically with ultimate ratio r.*

Proposition 2.1. *A geometrically convergent sequence converges.*

Proof. Observe that the above successive ratio consists of terms with the same sign. Hence we may apply the D'Alembert ratio test to Δu_m to conclude that the series $\Sigma \Delta u_m$ converges, that is, the sequence u_m has a limit.

3. Cascades in the bifurcation diagram

In this section we are concerned with a 1-parameter family of mappings of the interval

$$f_\lambda(x) = f(x, \lambda).$$

If a point a has period q, i.e. satisfies $f_\lambda{}^q(a) = a$, and has no smaller periods, then call it a *primitive q - periodic point* for f_λ. If its *multiplier*, i.e., the derivative of $f_\lambda{}^q$ at a has the absolute value smaller than 1, a is called an *attractive* or *stable* periodic point. In the (λ, x)-plane the locus of (λ, a) with stable a is known as the *bifurcation diagram*. Let

$$q = 2^m(2n+1) \qquad (m \geqq 0,\ n \geqq 0).$$

If a primitive q-periodic point a with multiplier 1 does not arise from stable primitive $2^{m-1}(2n+1)$-periodic points by bifurcation, then (λ, a) is the "root", or "origin" of the *cascade* with *proper* stable $2^t(2n+1)$- periodic points as ordinate in the bifurcation diagram.

Let $\omega(m, n)$ and $\alpha(m, n)$ denote the abscissa of the root for $2^m(2n+1)$ - periodic point with multiplier -1 (or 1) respectively. In other words $\alpha(m,n)$ is the value of λ for which the "first" stable periodic point of period $2^m(2n+1)$ arises and $\omega(m,n)$ the one for which it dies and stable periodic point of period $2^{m+1}(2n+1)$ is born.

According to Sharkovskii theorem we see that,

if $n < n'$ then $|\alpha(m, n)| > |\alpha(m, n')|$ and $|\omega(m, n)| > |\omega(m, n')|$

and that

if $m < m'$ and $n, n' > 0$ then

$|\alpha(m, n)| > |\alpha(m', n')|$ and $|\omega(m, n)| > |\omega(m', n')|$

Hence we have

Proposition 3.1. *The sequences* $\alpha(m, n)$ *and* $\omega(m, n)$ *are monotone both with respect to m and n, and* $|\omega(m, n)| > |\alpha(m, n)|$.

Feigenbaum observed that the sequences $\alpha(m,n)$ and $\omega(m, n)$ with respect to m geometrically converge with "universal" ultimate ratio, and afterwards this was proven by Lanford and Epstein for a wide class of f_λ. Thus successive ratios of their first differences

$$\frac{\omega(m+2,n) - \omega(m+1,n)}{\omega(m+1,n) - \omega(m,n)} \qquad \frac{\alpha(m+2,n) - \alpha(m+1,n)}{\alpha(m+1,n) - \alpha(m,n)}$$

converge, as m tends to an infinity, respectively to limits $\phi(\infty,n)$ and $\phi(\infty,n)$ whose reciprocal numbers are known by the name *Feigenbaum constants.*

By performing the computation with a program described in the next section I have obtained numerical evidences which supports the following conjecture:

Conjecture *For each m, sequences $\omega(m,n)$ and $\alpha(m,n)$ geometrically converge. More precisely, successive ratios of their first differences*

$$\frac{\omega(m,n+2)-\omega(m,n+1)}{\omega(m,n+1)-\omega(m,n)} \qquad \frac{\alpha(m,n+2)-\alpha(m,n+1)}{\alpha(m,n+1)-\alpha(m,n)}$$

are monotone for large n and converge to limits, as n tends to an infinity.

We denote by $\rho(m,n)$ and $\tau(m,n)$ the above ratios and the conjectural ultimate ratios are denoted respectively by $\rho(m,\infty)$ and $\tau(m,\infty)$. It follows from Proposition 2.1 that $\omega(m,n)$ and $\alpha(m,n)$ converge to limits $\omega(m,\infty)$ and $\alpha(m,\infty)$ respectively as n tends to ∞. It is obvious that $\omega(m,\infty)=\alpha(m,\infty)$.

Here are some numerical results about them.

(1) For $f_\lambda(x)=1-\lambda|x|^q$ we have

i) if $q=2$, $\omega(0,\infty)=1.54368901269\cdots$, $\rho(0,\infty)=0.3549...$,
$\omega(1,\infty)=1.43035763\cdots$, $\rho(1,\infty)=0.3374...$,
$\omega(2,\infty)=1.4074051181\cdots$, $\rho(2,\infty)=0.3394145...$,
$\omega(3,\infty)=1.40249217\cdots$, $\rho(3,\infty)=0.33918...$,

ii) if $q=3$, $\omega(0,\infty)=1.6180339887079634960073 2\cdots$, $\rho(0,\infty)=0.290892...$,
$\omega(1,\infty)=1.5366213864468\cdots$, $\rho(1,\infty)=0.2633913...$,
$\omega(2,\infty)=1.5243173613341\cdots$, $\rho(2,\infty)=0.26789\cdots$,
$\omega(3,\infty)=1.5222788300\cdots$, $\rho(3,\infty)=0.2671\cdots$

iii) if $q=4$, $\omega(0,\infty)=1.667960707\cdots$, $\rho(0,\infty)=0.2529...$,
$\omega(1,\infty)=1.604070164985077307\cdots$, $\rho(1,\infty)=0.2203248...$,
$\omega(2,\infty)=1.59617687728\cdots$; $\rho(2,\infty)=0.2268\cdots$

iv) if $q=5$ $\omega(0,\infty)=1.70440752438785 48\cdots$, $\rho(0,\infty)=0.227149...$,
$\omega(1,\infty)=1.6518698802418709426 5\cdots$, $\rho(1,\infty)=0.191885\cdots$
$\omega(2,\infty)=1.6463079\cdots$, $\rho(2,\infty)=0.195\cdots$

(2) For $f_\lambda(x)=x^3+(\lambda-1)x-\lambda$ we have
$\omega(0,\infty)=-1.0589672552011187043 81\cdots$, $\rho(0,\infty)=0.36002818609\cdots$,
$\omega(1,\infty)=-0.906551500186\cdots$, $\rho(1,\infty)=0.338249504\cdots$,
$\omega(2,\infty)=-0.87603107\cdots$, $\rho(2,\infty)=0.33925\cdots$

(3) For $f_\lambda(x)=\lambda x\exp x$ with the critical point -1 we have
$\omega(0,\infty)=16.9990485568768866283192\cdots$, $\rho(0,\infty)=0.297777\cdots$,
$\omega(1,\infty)=15.24972\ 71308\cdots$, $\rho(1,\infty)=0.345508449\cdots$,
$\omega(1,\infty)=-2.248011886138047\cdots$, $\rho(1,\infty)=0.299188649152\cdots$
$\omega(2,\infty)=14.8673150743\cdots$, $\rho(2,\infty)=0.338423019\cdots$,
$\omega(2,\infty)=-2.1701396413\cdots$, $\rho(2,\infty)=0.3455368\cdots$

$\omega(3,\infty) = 14.78813\cdots$, $\rho(3,\infty) = 0.33930\cdots$

$\omega(3,\infty) = -2.15246080\cdots$, $\rho(3,\infty) = 0.33839\cdots$

The function $\lambda x \exp x$ *has only periodic points of even periods for* $\lambda < 0$, because non-zero x has the sign different from $f_\lambda(x)$ if $\lambda < 0$. Observe also that, *if* $\lambda > 1$ *then the function* $\lambda x \exp x$ *has no periodic point in* $x > 0$, for we have $f_\lambda(x) > x$ for $x > 0$.

(4) The functions $f_\lambda(x) = \lambda x(1 + x/q)^q$, $q = 1, 2, \cdots$ converge uniformly on any finite interval to $\lambda x \exp x$, and has the critical point $-q/(q+1)$ with multiplicity one(see [2]).

i) if $q = 1$, $\omega(0,\infty) = 3.6785735104283222265\cdots$, $\rho(0,\infty) = 0.3549107\cdots$,
$\omega(1,\infty) = 3.67857351043\cdots$, $\rho(1,\infty) = 0.3549\cdots$,
$\omega(1,\infty) = -1.592572184107\cdots$, $\rho(1,\infty) = 0.337458936\cdots$,

ii) if $q = 2$, $\omega(0,\infty) = 5.5738193797302708856\cdots$, $\rho(0,\infty) = 0.334\cdots$,
$\omega(1,\infty) = 5.35836787\cdots$, $\rho(1,\infty) = 0.339307\cdots$,
$\omega(1,\infty) = -1.788010399634\cdots$, $\rho(1,\infty) = 0.318\cdots$,
$\omega(2,\infty) = 5.31286100\cdots$, $\rho(2,\infty) = 0.33926\cdots$
$\omega(2,\infty) = -1.7571576\cdots$, $\rho(2,\infty) = 0.34152\cdots$

iii) if $q = 3$, $\omega(0,\infty) = 7.0268502584429193 66\cdots$, $\rho(0,\infty) = 0.32795\cdots$,
$\omega(1,\infty) = 6.6817449480\ \cdots$, $\rho(1,\infty) = 0.34048\cdots$,
$\omega(2,\infty) = 6.608047241\cdots$, $\rho(2,\infty) = 0.33906\cdots$,
$\omega(1,\infty) = -2.00523\cdots$, $\rho(1,\infty) = 0.309\cdots$,
$\omega(2,\infty) = -1.8539571\cdots$, $\rho(2,\infty) = 0.3425989\cdots$

iv) if $q = 4$, $\omega(0,\infty) = 8.1656913903443185\cdots$, $\rho(0,\infty) = 0.32218\cdots$,
$\omega(1,\infty) = 7.70289053\cdots$, $\rho(1,\infty) = 0.341293\cdots$,
$\omega(2,\infty) = 7.603435\cdots$, $\rho(2,\infty) = 0.339\cdots$,
$\omega(2,\infty) = -1.91409\cdots$, $\rho(2,\infty) = 0.343\cdots$,

v) if $q = 5$, $\omega(0,\infty) = 9.0778522950724483\cdots$, $\rho(0,\ \infty) = 0.318255417104\cdots$,
$\omega(1,\infty) = 8.51149396\cdots$, $\rho(1,\infty) = 0.341878\cdots$,
$\omega(1,\infty) = -2.00523883603\cdots$, $\rho(1,\infty) = 0.309067977\cdots$,
$\omega(2,\infty) = -1.9550754\cdots$, $\rho(2,\infty) = 0.34364\cdots$,
$\omega(2,\infty) = 8.38930276\cdots$, $\rho(2,\infty) = 0.3388892\cdots$

(5) For $f_\lambda(x) = \lambda \sin x$ which has the negative Schwarzian derivative [6] we have

$\omega(0,\infty) = 2.8099906374581890561 31\cdots$, $\rho(0,\infty) = 0.34042429\cdots$,
$\omega(1,\infty) = 2.73856961157876\cdots$, $\rho(1,\infty) = 0.3391642188\cdots$,
$\omega(2,\infty) = 2.723419658\cdots$, $\rho(2,\infty) = 0.3392062\cdots$,

$$\omega(3,\infty) = -2.72018006\cdots, \qquad \rho(3,\infty) = 0.3392076\cdots$$

Proposition 3.2. *The function* $f_\lambda(x) = \lambda x(1 + x/q)^q$ *has no periodic point if* $\lambda > 1$ *and* $x > 0$, *or if* $|x| > q + q|\lambda|^{-1/q}$.

Proof. If $\lambda > 1$ and $x > 0$ then $\lambda(1 + x/q)^q > 1$ implies $f_\lambda(x) > x$. In the latter case, since $|x| > q$, we have $|x/q + 1| \geqq |x|/q - 1 > 0$, hence

$$|f_\lambda(x)| \geqq |\lambda x|(|x|/q - 1)^q > |\lambda x|(|\lambda|^{-1/q})^q = |x|$$

by the assumed inequality.

4. Making Tables

A scheme to compute $\omega(m,n)$ and $\alpha(m, n)$ will be the following: First, one makes λ change in a fixed spacing h and then , for each λ, one search p-periodic point x by solving $f_\lambda{}^p(x) - x = 0$ using the Newton iteration method *starting a critical value of* f_λ *as the initial* x and, in the same time, calculates multiplier of $f_\lambda{}^p$ at x and, if this multiplier is equal to -1 (or 1) approximately, one has only to record such λ.

We perform the above scheme using Kida's UBASIC with the following program:

```
10  print "parameter c for diff=−1 of p-periodic point of x → cx(1+x/q)^q"
20  point 15
30  input " q, p, c0, c1, h="; q, p, c0, c1,h
40   c = c0
50   while c > c1
60      x = −q / (q+1): I=1
70       while I < 50
80             on error goto 220
90             if abs( fnh(x,c,q,p)−1) < 0.0000001 then goto 220
100            y = x−(fnf(x,c,q,p)−x) / (fnh(x,c,q,p)−1)
110            for j=1 to p¥2
120               if and{ p@j=0, abs(fnf(y,c,q,j)−y) < 0.000001} then cancel for : goto 220
130            next j
140            if abs(y−x) < 0.00000001 then goto 180
150            x=y: print ".";
160        I=I+1: wend
170      goto 220
180            print "c, diff = "; c, fnh(y,c,q,p)
190            if and{ fnh(y,c,q,p) < 0.0, fnh(y,c,q,p) > −1.0} then goto 230
```

```
200   rem     if fnh(y,c,q,p) > 1.0 then goto 230
210   rem     for j= 1 to p+1: print fnf(y,c,q,j);: next j
220   print "*";: c=c−h: wend
230   beep: print "p, c="; p; "    ";c :   end
300   fnk(x,c,q)
310     x = c*x*(1+x/q)^q
320     return(x)
350   fnf(x,c,q,k)
360     local z,i
370     z=x: for i=1 to k: z=fnk(z,c,q): next I
380     return(z)
400   fng(x,c,q)
410     x = c*(1+x/q)^(q−1)*(1+x/q+x)
420     return(x)
450   fnh(x,c,q,k)
460     local j, z
470     z = fng(x,c,q)
480       for j=1 to k−1: z= z*fng(fnf(x,c,q,j),c,q): next j
490     return(z)
```

5. Rational periodic points

R.Walde and P. Russo [7] have computed rational periodic points for the dynamics arising from the quadratic function $x^2 + c$. We aim here at a similar analysis for the function $\lambda x(1 + x/q)^q$, $q = 1, 2$.

The case where $f_\lambda(x) = \lambda x(1+x/2)^2$

Fixed points are obtained from

$$f_\lambda(x) - x = (x/4)(\lambda x^2 + 4\lambda x + 4\lambda - 4) = 0.$$

Hence one is $\xi = 2\lambda^{-1/2} - 2$ if $\lambda > 0$ and the other is $-4 - \xi$. ξ and λ are related by $(\xi + 2)^2 \lambda = 4$. It follows that *fixed points are rational if and only if λ is a rational square.*

Periodic points of period 2 are solutions of

$$0 = f_\lambda^2(x) - x = (\lambda^2 x/16)(x+2)^2\{(\lambda x/4)(x+2)^2 + 2\}^2 - x$$

$$= (x/256)\{\lambda^4 x^2(x+2)^6 + 16\lambda^3 x(x+2)^4 + 64\lambda^2(x+2)^2 - 256\}$$

$$= (x/256)(\lambda(x+2)^2 - 4)$$

$$\times\{x^2(x+2)^4\lambda^3 + 4x(x+2)^2(x+4)\lambda^2 + 16(x+2)^2\lambda + 64\}$$

Thus primitive 2-periodic points are obtained from

$$0 = (f_\lambda{}^2(x) - x) / (f_\lambda(x) - x)$$
$$= (x/64)\{x(x+2)\lambda + 4\}\{x(x+2)^3 \lambda^2 + 8(x+2)\lambda + 16\}$$

The middle factor gives us $\lambda = -4x^{-1}(x+2)^{-1}$, i.e. $x^2 + 2x + 4\lambda^{-1} = 0$; thence $x = -1 \pm \lambda^{-1}\sqrt{(\lambda^2 - 4\lambda)}$. Thus we see that *2-periodic point x is rational if and only if λ is rational and $\lambda^2 - 4\lambda$ is a rational square.* It follows from the quadratic equation $\lambda^2 - 4\lambda - v^2 u^{-2} = 0$ where u and v are integers that the condition is converted into $4u^2 + v^2$ *is a square integer* w^2. Thus we may infer that *there exist integers m and n such that*

$$u = mn, \quad v = m^2 - n^2, \quad w = m^2 + n^2, \quad \lambda = (m^2 + n^2)(mn)^{-1}.$$

Now the last factor gives us

$$x(x+2)^3 \lambda = -4(x+2) \pm 4(x+2)(1 - 2x - x^2)^{1/2}$$

This implies that *λ is rational if and only if 2-period points x is rational and* $1 - 2x - x^2 = 2 - (x+1)^2$ *is a rational square* v^2/u^2, where $u > 0$ and v are integers. If u is chosen so that $x + 1 = w/u$ with an integer w, we may find integers m and n such that

$$u = m^2 + n^2 \quad \text{and} \quad v = m^2 - n^2 + 2mn,\ w = m^2 - n^2 - 2mn$$

(see [5, p. 13]). Since $-4 \pm 4(v/u) = \lambda(w^2 - u^2)(w+u)u^{-3}$ we have

$$\lambda = -(m^2+n^2)^2 m^{-2}(m-n)^{-2} \quad \text{or} \quad (m^2+n^2)^2(mn)^{-1}(m-n)^{-2}$$
$$x = -2n(m+n)(m^2+n^2)^{-1}$$

The case where $f_\lambda(x) = \lambda x(1 + x/q)^q$

Fixed points are obtained from

$$f_\lambda(x) - x = x(\lambda x + \lambda - 1) = 0.$$

Thus fixed point x are rational if and only λ is rational and $x = \lambda^{-1} - 1$.

2-periodic points are solutions of the last factor of the equation

$$0 = f_\lambda{}^2(x) - x = x\{\lambda^3 x^3 + 2\lambda^3 x^2 + (\lambda+1)\lambda^2 x + \lambda^2 - 1\}$$
$$= (\lambda x + \lambda - 1)\{\lambda^2 x^2 + \lambda(\lambda+1)x + \lambda + 1\},$$

the discriminant of which is $\lambda^2(\lambda^2 - 2\lambda - 3)$. Thus we see that *2-periodic point x is rational if and only if $\lambda^2 - 2\lambda - 3$ is a rational μ.* Putting $\lambda - 1 = u/w$ and $\mu = v/w$ we have $u^2 - v^2 = 4w^2$. Hence there exist integers m and n such that

$$u = m^2 + n^2, \quad w = mn, \quad v = m^2 - n^2$$

It follows that

$$\lambda = (m^2 + n^2 + mn)/mn, \quad x = m^2(m^2+n^2+mn)^{-1} \quad \text{or} \quad n^2(m^2+n^2+mn)^{-1}.$$

References

1. K.Briggs: A precise calculation of the Feigenbaum constants, Math. Comp. 57, 435—439 (1991)

2. N. Fagella: Surgery on the limbs of the Mandelbrot set, Progress in holomorphic dynamics edited by H. Kriete, Pitman Research Notes in Math. No. 387, Longman, 1998

3. P. E. Lanford III: Smooth transformation of intervals, Lecture Notes in Math. No. 901, (1980/81)

4. O. E. Lanford III: A computer-assisted proof of the Feigenbaum conjectures, Bull. Amer. Math. Soc. 6, 427—434 (1982)

5. L .J. Mordell : Diophantine equations, Academic Press, London and New York, 1959

6. A. Singer: Stable orbits and bifurcations of maps of the interval, SIAM J. Appl. Math. 35 260—267 (1978)

7. Walde and P. Russo: Rational periodic points of maps of the quadratic function $Q_c(x) = x^2+c$, Amer. Math. Monthly 101, 318—331 (1994)

Appendix

Table $\omega(m,n)$, $\rho(m,n)$ for $\lambda x \exp x$

m	n	$\omega(m, n)$	$\rho(m, n-2)$
0	1	23.55985 47822 24106	
0	2	18.58139 14811 28502	
0	3	17.46907 54519 54268 662	0.22560381
0	4	17.14160 27296 5536	0.29440618
0	5	17.04198 92296 86161 292	0.304188694
0	6	17.01189 88296 46316 515	0.3020715068
0	7	17.00288 11930 08772 0016	0.29968483719
0	8	17.00019 00338 407435 59	0.2984328683
0	9	16.99938 83385 57850 044	0.297899612
...			
0	39	16.99904 85568 76886 62837 40975 74365 89012 87674 4002	0.29757777974297045206
0	40	16.99904 85568 76886 62833 55601 45143 01943 50988 5012	0.29757777974297045172
0	41	16.99904 85568 76886 62832 40922 62517 87570 68748 12918393	0.297577779742970451803
0	42	16.99904 85568 76886 62832 06796 75467 93244 87400 99956	0.29757777974297045169
0	43	16.99904 85568 76886 62831 96641 65390 43072 05664 18447 31342	0.···7974297045
0	44	16.99904 85568 76886 62831 93619 72156 26114 44468 57061 28539 2	0.···797429694611
0	45	16.99904 85568 76886 62831 92720 46164 58361 44321 38628 21053 81	0.···7975354717
0	46	16.99904 85568 76886 62831 92720 19404 60666 88967 95208 72431 97564 87	0.···7975354717
0	47	16.99904 85568 76886 62831 92640 56230 08105 52937 42653 44555 15374	0.···7975354717
0	48	16.99904 85568 76886 62831 92616 86566 28596 14352 64821 29699 76824 273	0.···7975000033
1	1	16.05625 33	
1	2	15.54072 775	
1	3	15.35534 4180	0.35960109
1	4	15.28749 8395	0.3659751
1	5	15.26302 57340	0.360710116
1	6	15.25436 33962 7	0.35395978
1	7	15.25133 53468 73	0.349564917
1	8	15.25028 37036 995	0.3473005341
1	9	15.24991 95605 032	0.3462611704
...			
1	23	15.24972 71309 03772 14126 55550 2	0.3455084503957
1	24	15.24972 71308 60242 44193 34671 39	0.3455084498176

1	25	15.24972 71308 45202 56300 53487 0309	0.34550844960768
1	26	15.24972 71308 40006 15775 57516 2752	0.34550844953159
1	1	−2.49780 21366 972	
1	2	−2.31422 20629 1050	
1	3	−2.26818 71866 59485	0.2507618354
1	4	−2.25419 99828 72251	0.3038392828
1	5	−2.24988 93270 68510	0.3081856723
1	6	−2.24857 70808 08075	0.3044191696
1	7	−2.24818 14004 59629 2	0.3015290348
1	8	−2.24806 26490 3920	0.30011957
1	9	−2.24802 70787 53115	0.2995356
…			
1	26	−2.24801 18861 38066 22145 75070 85825 75	0.299188649154088110
1	27	−2.24801 18861 38053 08813 45764 65076 38774 1	0.29918864915280117
1	28	−2.24801 18861 38049 15879 34299 70433 94532 3	0.2991886491522728716
1	29	−2.24801 18861 38047 98317 91602 92258 72715 51	0.29918864915227504835
1	30	−2.24801 18861 38047 63144 87150 23155 71922 39	0.29918864915223369113
2	1	15.03622 119	
2	2	14.92699 3800	
2	3	14.88858 8131	0.35161207
2	4	14.87477 5312	0.359655732
2	5	14.86988 82802	0.35380410037
2	6	14.86819 36627 01	0.34675580118
…			
2	11	14.86731 89992 78769 0	0.33853037697
2	12	14.86731 64026 71214 27	0.33846317506
2	13	14.86731 55238 78396 190	0.33843882817
2	14	14.86731 52264 69505 3177	0.33842890469
2	1	−2.20659 0318	
2	2	−2.18341 59807 377	
2	3	−2.17497 56782 32	0.364209001568
…			
2	21	−2.17013 96413 49925 49712 53800 48	0.3455368409159
2	22	−2.17013 96413 33170 26389 07419 52	0.34553683654489
2	23	−2.17013 96413 27380 71363 00237 021	0.345536834948349