Random Walks on Distance-Regular Graphs

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0. Introduction

In this paper we consider an isotropic random walk on a distance-regular graph. We describe the m-step transition probabilities of the walk in terms of the spectral data of the graph. The distance-regularity of the graph plays an essential role for deriving the spectral decomposition of the transition matrix of the walk explicitly. Several examples are listed.

1. The structure of distance-regular graphs

Let G be a connected graph with vertex set V of cardinality $|V|=n$. The graph distance of $x, y\in V$ is denoted by $d(x, y)$. For $x\in V$ and $j\in\mathbf{Z}_+$, we write $S_j(x)=\{y\in V\,;\, d(x, y)=j\}$. The diameter of G is denoted by d. We say that G is distance-regular if there exist b_j, $c_j\in\mathbf{Z}_+$ for $0\leqq j\leqq d$ such that $|S_1(y)\cap S_{j+1}(x)|=b_j$ and $|S_1(y)\cap S_{j-1}(x)|=c_j$ for any $x, y\in V$ at distance $d(x, y)=j$. Note that G is a regular graph with valency b_0, $c_1=1$, $c_0=b_d=0$, $b_j>0$ $(0\leqq j<d)$ and $c_j>0$ $(1\leqq j\leqq d)$. Put $a_j=b_0-b_j-c_j$ for $0\leqq j\leqq d$. Then $a_j=|S_1(y)\cap S_j(x)|$ where $d(x, y)=j$. Let $k_0=1$ and $k_j=b_0\cdots b_{j-1}/c_1\cdots c_j$ for $1\leqq j\leqq d$. Then $k_j=|S_j(x)|$ for any $x\in V$. We write $\iota(G)=\{b_0, \cdots, b_{d-1}\,;\, c_1, \cdots, c_d\}$ and call it the intersection array of G. Let $A_i (0\leqq i\leqq d)$ be the $n\times n$ matrix with entry $A_i(x, y)$, where $A_i(x, y)=1$ if $d(x, y)=i$ and 0 otherwise. Note that $A_0=I$ and A_1 is the adjacency matrix A of G. The matrix algebra $\mathcal{A}(G)$ of polynomials in A is called the adjacency algebra of G. It is known ([1]) that $(A_i)_{0\leqq i\leqq d}$ forms a basis of $\mathcal{A}(G)$ and

$$A_iA_j=\sum_{k=0}^{d}b_{ijk}A_k \quad (0\leqq i,\ j\leqq d) \tag{1}$$

where $b_{ijk}=|S_i(x)\cap S_j(y)|$ with $d(x, y)=k$. Let $(f_j(\lambda))_{0\leqq j\leqq d}$ be the sequence of polynomials in λ which is defined recursively by

(2) $$f_0(\lambda)=1,\quad f_1(\lambda)=\lambda \text{ and}$$
$$c_{j+1}f_{j+1}(\lambda)+a_jf_j(\lambda)+b_{j-1}f_{j-1}(\lambda)=\lambda f_j(\lambda)$$

for $1\leqq j\leqq d-1$. It follows from (1) with $i=1$

(3) $$A_j=f_j(A)\quad\text{for}\quad 0\leqq j\leqq d.$$

It is shown ([1]) that A has exactly $d+1$ distinct real eigenvalues $\lambda_0=b_0>\lambda_1>\cdots>\lambda_d$, whose multiplicities we denote respectively by $m_0=1$, m_1, $\cdots$, m_d. We write $\mathrm{Spec}\,G=\{\lambda_0, \cdots, \lambda_d\,;\, m_0, \cdots, m_d\}$ and call it the spectra of G. Note that $(\lambda_r)_{0\leqq r\leqq d}$ are the roots of $\lambda f_d(\lambda)=a_df_d(\lambda)+b_{d-1}f_{d-1}(\lambda)$ and the multiplicities $(m_r)_{0\leqq r\leqq d}$ are given by

(4) $$m_r=n/\sum_{j=0}^{d}k_j^{-1}f_j(\lambda_r)^2.$$

Let $(E_r)_{0\leqq r\leqq d}$ be the complete set of projections to the eigenspaces of A. Note that

(5) $$E_rE_s=\delta_{rs}E_r\ (0\leqq r,\ s\leqq d),\ AE_r=E_rA=\lambda_rE_r,$$
$$\mathrm{tr}\,E_r=m_r\ (0\leqq r\leqq d)\ \textit{and}\ I=\sum_{r=0}^{d}E_r.$$

Since $A_j=f_j(A)$, the spectral decomposition of A_j is given by

(6) $$A_j=\sum_{r=0}^{d}f_j(\lambda_r)E_r\quad\text{for}\quad 0\leqq j\leqq d.$$

Let $P=\sum_{j=0}^{d}p(j)A_j$ be any element of $\mathcal{A}(G)$ where $p(j)\in\mathbf{C}$ $(0\leqq j\leqq d)$. From (6), we can obtain

(7) $$P=\sum_{r=0}^{d}\hat{p}(r)E_r$$

where

(8) $$\hat{p}(r)=\sum_{j=0}^{d}f_j(\lambda_r)p(j)\quad\text{for}\quad 0\leqq r\leqq d.$$

To describe $P^m\,(m\geqq 0)$ as the linear combination of $(A_j)_{0\leqq j\leqq d}$, we need the following two lemmas.

Lemma 1. *Let F be the $(d+1)\times(d+1)$ matrix whose $(r,\ j)$-entry is $f_j(\lambda_r)$. Put $K=\mathrm{diag}\,(k_0, \cdots, k_d)$ and $M=\mathrm{diag}\,(m_0, \cdots, m_d)$. Then*

(9) $${}^tFMF=nK,\ \textit{that is,}\ \sum_r m_rf_i(\lambda_r)f_j(\lambda_r)=nk_i\delta_{ij}.$$

Proof. Since $\mathrm{tr}\,A_k=0$ for $k\neq 0$ and $b_{ij0}=k_i\delta_{ij}$, it follows from (1)

(10) $$\mathrm{tr}\,(A_iA_j)=nk_i\delta_{ij}.$$

On the other hand using (6) we have $A_iA_j=\sum_r f_i(\lambda_r)f_j(\lambda_r)E_r$. This yields

(11) $$\mathrm{tr}\,(A_iA_j)=\sum_r m_r f_i(\lambda_r)f_j(\lambda_r).$$

Combining (10) with (11), we get the lemma. □

Lemma 2. *The projections E_r can be written as*

(12) $$E_r=n^{-1}m_r\sum_{j=0}^{d}k_j^{-1}f_j(\lambda_r)A_j \quad \textit{for} \quad 0\leqq r\leqq d.$$

Proof. Since $E_r\in\mathcal{A}(G)$, it follows that $E_r=\sum_j C_{rj}A_j$ for some constants C_{rj}. From (6) we conclude that C_{rj} is the (r, j)*-entry* of ${}^tF^{-1}$. The lemma follows immediately since ${}^tF^{-1}=n^{-1}MFK^{-1}$ by Lemma 1. □

Theorem 1. *Let $P=\sum_{j=0}^{d}p(j)A_j$ be an arbitrary element of the adjacency algebra $\mathcal{A}(G)$ of a distance-regular graph G. Then for $m\in\mathbf{Z}_+$*

(13) $$P^m=\sum_{j=0}^{d}p_m(j)A_j$$

where

(14) $$p_m(j)=n^{-1}k_j^{-1}\sum_{r=0}^{d}m_rf_j(\lambda_r)\hat{p}(r)^m \quad (0\leqq j\leqq d)$$

and $$\hat{p}(r)=\sum_{r=0}^{d}f_j(\lambda_r)p(j) \quad (0\leqq r\leqq d).$$

Proof. It follows from (7) $P^m=\sum_r\hat{p}(r)^mE_r$. Using (12), we have

$$P^m=\sum_r\hat{p}(r)^mn^{-1}m_r\sum_j k_j^{-1}f_j(\lambda_r)A_j$$
$$=\sum_j\{n^{-1}k_j^{-1}\sum_r m_rf_j(\lambda_r)\hat{p}(r)^m\}A_j. \quad □$$

2. Isotropic random walks on distance-regular graphs

Let $(X_m)_{m\geqq 0}$ be an isotropic random walk on a distance-regular graph G. Namely $(X_m)_{m\geqq 0}$ is a Markov chain with state space V and the 1-step transition probabilities $\Pr[X_{m+1}=y\mid X_m=x]=P(x, y)$, which depend only on $d(x, y)$. The transition matrix $P=(P(x, y))_{x,y\in V}$ can be written as $P=\sum_{j=0}^{d}p(j)A_j$ where $p(j)=P(x, y)$ with $d(x, y)=j$. Note that $p(j)\geqq 0$ $(0\leqq j\leqq d)$ and $\sum_{j=0}^{d}k_jp(j)=1$.

Theorem 2. *The m-step transition probabilities $P_m(x, y)$ of the isotropic random walk $(X_m)_{m\geqq 0}$ on a distance-regular graph G are*

$$P_m(x, y)=n^{-1}k_j^{-1}\sum_{r=0}^{d}m_rf_j(\lambda_r)\hat{p}(r)^m \quad \textit{where } d(x, y)=j.$$

Proof. Since $P_m(x, y)$ is the (x, y)-entry of P^m, the assertion of the theorem is a direct consequence of Theorem 1. □

Fix an arbitrary $x_0 \in V$ and put $Y_m = d(x_0, X_m)$. Then $(Y_m)_{m \geq 0}$ is a Markov chain with state space $\{0, \cdots, d\}$, whose 1-step transition probabilities $\Pr[Y_{m+1} = j \mid Y_m = i] = Q(i, j)$ are given by $Q(i, j) = \sum_{y \in S_j(x_0)} P(x, y)$ where $x \in S_i(x_0)$. Note that

(15) $$Q(i, j) = \sum_{h=0}^{d} \sum_{y \in S_j(x_0) \cap S_h(x)} P(x, y) = \sum_{h=0}^{d} b_{jhi} p(h).$$

Theorem 3. *Let* $(Y_m)_{m \geq 0} = (d(x_0, X_m))_{m \geq 0}$ *be the Markov chain on* $\{0, \cdots, d\}$ *where* $(X_m)_{m \geq 0}$ *is the isotropic random walk on a distance-regular graph G. Then the transition matrix* $Q = (Q(i, j))_{0 \leq i, j \leq d}$ *of* $(Y_m)_{m \geq 0}$ *can be written as*

$$Q = F^{-1} \hat{P} F = n^{-1} K^{-1}\, {}^t F M \hat{P} F$$

where the matrices F, K, M are as in Lemma 1, *and* $\hat{P} = \mathrm{diag}(\hat{p}(0), \cdots \hat{p}(d))$. *Furthermore the m-step transition probabilities* $Q_m(i, j)$ *are given by*

(16) $$Q_m(i, j) = n^{-1} k_i^{-1} \sum_{r=0}^{d} m_r \hat{p}(r)^m f_i(\lambda_r) f_j(\lambda_r).$$

Proof. Using (14), with $m=1$ and (15), we can write

$$Q(i, j) = n^{-1} \sum_r m_r \hat{p}(r) \sum_h b_{jhi} k_h^{-1} f_h(\lambda_r).$$

From (12) and (1),

$$A_j E_r = n^{-1} m_r \sum_h k_h^{-1} f_h(\lambda_r) A_j A_h$$

$$= n^{-1} m_r \sum_i \left(\sum_h b_{jhi} k_h^{-1} f_h(\lambda_r) \right) A_i.$$

On the other hand

$$A_j E_r = f_j(\lambda_r) E_r = n^{-1} m_r \sum_i k_i^{-1} f_i(\lambda_r) f_j(\lambda_r) A_i.$$

Hence

(17) $$\sum_h b_{jhi} k_h^{-1} f_h(\lambda_r) = k_i^{-1} f_i(\lambda_r) f_j(\lambda_r).$$

This yields

$$Q(i, j) = n^{-1} k_i^{-1} \sum_r m_r \hat{p}(r) f_i(\lambda_r) f_j(\lambda_r) \quad (0 \leq i, j \leq d),$$

which implies $Q = n^{-1} K^{-1}\, {}^t F M \hat{P} F = F^{-1} \hat{P} F$. The second assertion is clear from $Q^m = F^{-1} \hat{P}^m F$. □

3. Examples

Keeping the preceding notations, we consider the several examples. In most of examples, we only afford the matrix F and its inverse F^{-1}. If you want to compute the m-step transition probabilities $p_m={}^t(p_m(0), \cdots, p_m(d))$ by giving $p={}^t(p(0), \cdots, p(d))$, you compute $\hat{p}={}^t(\hat{p}(0), \cdots, \hat{p}(d))=Fp$ and then you get $p_m=F^{-1}\,{}^t(\hat{p}(0)^m, \cdots, \hat{p}(d)^m)$.

Example 1. The complete graph K_n.

$\iota(K_n)=\{n-1\ ;\ 1\}$, $|V|=n$, $\mathrm{Spec}(K_n)=\{n-1,\ -1\ ;\ 1,\ n-1\}$,

$$F=\begin{bmatrix} 1 & n-1 \\ 1 & -1 \end{bmatrix} \quad \text{and} \quad F^{-1}=n^{-1}F.$$

Hence

$$\begin{bmatrix} p_m(0) \\ p_m(1) \end{bmatrix} = n^{-1} \begin{bmatrix} (p(0)+(n-1)p(1))^m+(n-1)(p(0)-p(1))^m \\ (p(0)+(n-1)p(1))^m-(p(0)-p(1))^m \end{bmatrix}.$$

Example 2. The complete bipartite graph $K_{n,n}$.

$\iota(K_{n,n})=\{n,\ n-1\ ;\ 1,\ n\}$, $|V|=2n$, $\mathrm{Spec}(K_{n,n})=\{n,\ 0,\ -n\ ;\ 1,\ 2n-2,\ 1\}$,

$$F=\begin{bmatrix} 1 & n & n-1 \\ 1 & 0 & -1 \\ 1 & -n & n-1 \end{bmatrix},\ F^{-1}=1/2n \begin{bmatrix} 1 & 2n-2 & 1 \\ 1 & 0 & -1 \\ 1 & -2 & 1 \end{bmatrix}.$$

Example 3. The triangular graph $T(n)=J(n,\ 2)$.

$\iota(T(n))=\{2n-4,\ n-3\ ;\ 1,\ 4\}$, $|V|=\binom{n}{2}$, $\mathrm{Spec}(T(n))$

$=\{2n-4,\ n-4,\ -2\ ;\ 1,\ n-1,\ n(n-3)/2\}$,

$$F=\begin{bmatrix} 1 & 2n-4 & (n-2)(n-3)/2 \\ 1 & n-4 & -(n-3) \\ 1 & -2 & 1 \end{bmatrix},$$

$$F^{-1}=\binom{n}{2}^{-1} \begin{bmatrix} 1 & n-1 & n(n-3)/2 \\ 1 & (n-1)(n-4)/2(n-2) & -n(n-3)/2(n-2) \\ 1 & -2(n-1)/(n-2) & n/(n-2) \end{bmatrix}.$$

Example 4. The Hamming graph $H(d,\ q)$ $(d,\ q\geqq 2)$.

$\iota(H(d,\ q))=\{b_j=\ (q-1)(d-j)\ (0\leqq j\leqq d-1)\ ;\ c_j=j\ \ (1\leqq j\leqq d)\}$, $|V|=q^d$,

$\mathrm{Spec}(H(d,\ q))=\{\lambda_r=(q-1)d-qr\ (0\leqq r\leqq d)\ ;\ m_r=(q-1)^r\binom{d}{r}\ (0\leqq r\leqq d)\}$,

$F=(f_j(\lambda_r))_{0\leq r,\ j\leq d}$ where $f_j(\lambda_r)=\sum_{a=0}^{j}(-1)^a(q-1)^{j-a}\binom{r}{a}\binom{d-r}{j-a}$ and $F^{-1}=q^{-d}F$.

Example 5. The Petersen graph.

$\iota(G)=\{3,\ 2\ ;\ 1,\ 1\}$, $|V|=10$, $\mathrm{Spec}\,G=\{3,\ 1,\ -2\ ;\ 1,\ 5,\ 4\}$,

$$F=\begin{bmatrix}1&3&6\\1&1&-2\\1&-2&1\end{bmatrix},\quad F^{-1}=1/10\begin{bmatrix}1&5&4\\1&5/3&-8/3\\1&-5/3&2/3\end{bmatrix}.$$

Example 6. The Johnson graph $J(14,\ 7)$.

$\iota(J(14,\ 7))=\{49,\ 36,\ 25,\ 16,\ 9,\ 4,\ 1\ ;\ 1,\ 4,\ 9,\ 16,\ 25,\ 36,\ 49\}$, $|V|=3432$,

$\mathrm{Spec}(J(14,\ 7))=\{49,\ 35,\ 23,\ 13,\ 5,\ -1,\ -5,\ -7\ ;\ 1,\ 13,\ 77,\ 273,\ 637,\ 1001,\ 1001,\ 429\}$,

$$F=\begin{bmatrix}1&49&441&1225&1225&441&49&1\\1&35&189&175&-175&-189&-35&-1\\1&23&51&-75&-75&51&23&1\\1&13&-9&-45&45&9&-13&-1\\1&5&-21&15&15&-21&5&1\\1&-1&-9&25&-25&9&1&-1\\1&-5&9&-5&-5&9&-5&1\\1&-7&21&-35&35&-21&7&-1\end{bmatrix},$$

$$F^{-1}=1/3432\begin{bmatrix}1&13&77&273&637&1001&1001&429\\1&65/7&253/7&507/7&65&-143/7&-715/7&-429/7\\1&39/7&187/21&-39/7&-91/3&-143/7&143/7&143/7\\1&13/7&-33/7&-351/35&39/5&143/7&-143/35&-429/35\\1&-13/7&-33/7&351/35&39/5&-143/7&-143/35&429/35\\1&-39/7&187/21&39/7&-91/3&143/7&143/7&-143/7\\1&-65/7&253/7&-507/7&65&143/7&-715/7&429/7\\1&-13&77&-273&637&-1001&1001&-429\end{bmatrix}.$$

Example 7. The Foster graph, bipartite, antipodal 3-cover of the incidence graph of $GQ(2,\ 2)$.

$\iota(G)=\{3,\ 2,\ 2,\ 2,\ 2,\ 1,\ 1,\ 1\ ;\ 1,\ 1,\ 1,\ 1,\ 2,\ 2,\ 2,\ 3\}$, $|V|=90$,

$\mathrm{Spec}\,G=\{3,\ \sqrt{6},\ 2,\ 1,\ 0,\ -1,\ -2,\ -\sqrt{6},\ -3\ ;\ 1,\ 12,\ 9,\ 18,\ 10,\ 18,\ 9,\ 12,\ 1\}$,

$$F=\begin{bmatrix} 1 & 3 & 6 & 12 & 24 & 24 & 12 & 6 & 2 \\ 1 & \sqrt{6} & 3 & \sqrt{6} & 0 & -\sqrt{6} & -3 & -\sqrt{6} & -1 \\ 1 & 2 & 1 & -2 & -6 & -4 & 2 & 4 & 2 \\ 1 & 1 & -2 & -4 & 0 & 4 & 2 & -1 & -1 \\ 1 & 0 & -3 & 0 & 6 & 0 & -6 & 0 & 2 \\ 1 & -1 & -2 & 4 & 0 & -4 & 2 & 1 & -1 \\ 1 & -2 & 1 & 2 & -6 & 4 & 2 & -4 & 2 \\ 1 & -\sqrt{6} & 3 & -\sqrt{6} & 0 & \sqrt{6} & -3 & \sqrt{6} & -1 \\ 1 & -3 & 6 & -12 & 24 & -24 & 12 & -6 & 2 \end{bmatrix},$$

$$F^{-1}=1/90\begin{bmatrix} 1 & 12 & 9 & 18 & 10 & 18 & 9 & 12 & 1 \\ 1 & 4\sqrt{6} & 6 & 6 & 0 & -6 & -6 & -4\sqrt{6} & -1 \\ 1 & 6 & 3/2 & -6 & -5 & -6 & 3/2 & 6 & 1 \\ 1 & \sqrt{6} & -3/2 & -6 & 0 & 6 & 3/2 & -\sqrt{6} & -1 \\ 1 & 0 & -3/2 & 0 & 5/2 & 0 & -3/2 & 0 & 1 \\ 1 & -\sqrt{6}/2 & -3/2 & 3 & 0 & -3 & 3/2 & \sqrt{6}/2 & -1 \\ 1 & -3 & 3/2 & 3 & -5 & 3 & 3/2 & -3 & 1 \\ 1 & -2\sqrt{6} & 6 & -3 & 0 & 3 & -6 & 2\sqrt{6} & -1 \\ 1 & -6 & 9 & -9 & 10 & -9 & 9 & -6 & 1 \end{bmatrix}.$$

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