Vol.4, No.1 Special Issue on New Dimensions in Economics, Accounting and Management
ISSN 1805-3602

# Inference on Selected Population under Generalized Stein Loss Function

**A. Pagheh[1*], N. Nematollahi[2]**
[1]Department of Statistics, Science and Research Branch, Islamic Azad University, Tehran, Iran
[2]Department of Statistics, Allameh Tabataba'i University, Tehran, Iran
*E-mail: pagheh@gmail.com

**Abstract**

Inference on selected population is concerned with the problem of selecting the best population among the given $k$ populations, and then doing inference on the parameter of selected population. Suppose independent random samples $(X_{i1},\dots,X_{in}), i=1,\dots,k$ are drawn from $U(0,\theta_i)-$ population, respectively. Let $X_i=\max(X_{i1},\dots,X_{in})$ and $X_{(1)}\le X_{(2)}\le\cdots\le X_{(k)}$ be the order statistics of $X_1,\dots,X_k$. The population corresponding to largest $X_{(k)}$ (or the smallest $X_{(1)}$) is selected and the problem of estimation the parameter $\theta_M$ (or $\theta_J$) of the selected population under generalized Stein loss function is considered. We obtain the Uniformly Minimum Risk Unbiased (UMRU) estimator of $\theta_M$ (and $\theta_J$) and show that the UMRU estimator of $\theta_M$ is inadmissible. For $k=2$, we derive the class of all linear admissible estimators of $\theta_M$ and $\theta_J$, respectively.

**Keywords:** Estimation after selection, Generalized Stein loss function; Natural estimator; UMRU estimator; Uniform population

## Introduction

Estimation after selection is an important estimation problem related to ranking and selection methodology, having wide practical applications. For example, we wish to select the most productive machine from $k$ different types of machines and then estimate the mean of the production of the selected machine. The problem of estimation after selection has received considerable attention by many researchers. Some references in this area include, Rubinstein (1961), Stein (1964), Sackrowitz and Samuel-Cahn (1987), Vellaisamy et al. (1988), Misra et al. (2006a, 2006b), Kumar and Gangopadhyay (2005), Vellaisamy (1992a, 1992b), Kumar and Kar (2001a,b), Vellaisamy and Punnen (2002), Vellaisamy (2003), Nematollahi and Motamed-Shariati (2009, 2012), Naghizadeh and Nematollahi (2012).

Let $\Pi_1,\dots,\Pi_k$ denote k ($\ge 2$) independent uniform population with associated probability density functions (p.d.f.)

$$f(x\mid\theta_i,\sigma)=\frac{1}{\theta_i},\ \ 0<x\le\theta_i,\ \ \theta_i>0,\ \ i=,\dots,k, \tag{1}$$

respectively, where $\theta_1,\dots,\theta_k$ are unknown scale parameters. Let $(X_{i1},\dots,X_{in})$ be a random sample of size n drawn from the ith population, and $X_i=\max(X_{i1},\dots,X_{in})$, $i=1,\dots,k$. Then $X_i$ is a complete sufficient statistic for $\theta_i$ and has p.d.f.

$$f(x_i|\theta_i,\sigma)=\frac{nx_i^{n-1}}{\theta_i^{n}},\ \ 0<x_i\le\theta_i,\ \ \theta_i>0,\ \ i=1,\dots,k. \tag{2}$$

Let $X_{(1)}\le\cdots\le X_{(k)}$ denote the order statistics of $X_1,\dots,X_k$. For selecting the population corresponding to the larger (or smaller) $\theta_i's$, we use natural selection rule and select the population corresponding to the $X_{(k)}$ (or $X_{(1)}$). Therefore, the scale parameter associated with the larger and smaller selected population are given by

$$\theta_M = \sum_{i=1}^{k} \theta_i \left\{ \prod_{j \neq i}^{k} I(X_i, X_j) \right\} \tag{3}$$

and

$$\theta_J = \sum_{i=1}^{k} \theta_i \left\{ \prod_{j \neq i}^{k} (1 - I(X_i, X_j)) \right\}, \tag{4}$$

respectively, where

$$I(a, b) = \begin{cases} 1 & a \geq b \\ 0 & a < b \end{cases}. \tag{5}$$

In this paper, we consider the estimation of $\theta_M$ and $\theta_J$ under the generalized Stein loss (GSL) function given by

$$L(h(\theta), \delta) = \left( \frac{\delta}{h(\theta)} \right)^q - q \ln\left( \frac{\delta}{h(\theta)} \right) - 1, \qquad q \neq 0 . \tag{6}$$

This loss is asymmetric and convex in $\Delta = \frac{\delta}{h(\theta)}$ when $q = 1$ and quasi-convex otherwise, has a unique minimum at $\Delta = 1$, is scale invariant and also is useful in situations where underestimation and overestimation have not a same penalty. The GSL function with negative $q$ values penalizes overestimation more than underestimation while with positive $q$ values acts vice-versa. As a special case, when $q = 1$, the loss function (6) reduce to Stein loss function and when $q = -1$, loss function (6) reduce to entropy loss function. For estimation the parameter of selected population under Stein and entropy loss functions, see Nematollahi and Motamed-Shariati (2009, 2012), respectively.

It is worth mentioning that near $\Delta = 1$,

$$\left( \frac{\delta}{h(\theta)} \right)^q - q \ln\left( \frac{\delta}{h(\theta)} \right) - 1 \approx \frac{1}{2} q^2 \left( \frac{\delta}{h(\theta)} - 1 \right)^2 ,$$

and for small $|q|$ values,

$$\left( \frac{\delta}{h(\theta)} \right)^q - q \ln\left( \frac{\delta}{h(\theta)} \right) - 1 \approx \frac{1}{2} q^2 \left( \ln \delta - \ln h(\theta) \right)^2 .$$

In this article, we consider the estimation of the selected parameter of uniform population, $\theta_M$ and $\theta_J$ given by (3) and (4), respectively, under the GSL function (6) with $h(\theta) = \theta_M$ or $h(\theta) = \theta_J$ .

To this end, in Section 2, we derive the Uniformly Minimum Risk Unbiased (UMRU) estimator of $\theta_M$ and $\theta_J$ under GSL function. In Section 3, we obtain some admissibility result in estimation of $\theta_M$ and $\theta_J$, and in Section 4, we find minimax estimator of $\theta_M$ when $k = 2$. In

Section 5, we show that the UMRU estimator of $\theta_M$ is inadmissible. Finally, a discussion is given in Section 6.

**UMRU estimation**

Let $\theta=(\theta_1,...,\theta_k)$, and $\boldsymbol{X}=(X_1,...,X_k)$ we want to estimate a function of $\theta$, say $h(\theta)$, by an estimator $\delta$ under the loss function $L(\mathrm{h}(\theta),\delta)$. Following Lehmann (1951), an estimator $\delta(\mathbf{X})$ of $h(\theta)$ is said to be risk-unbiased if it satisfies

$$E_\theta[\mathrm{L}(\mathrm{h}(\theta),\delta(\mathbf{X}))]\le E_\theta[\mathrm{L}(\mathrm{h}(\theta'),\delta(\mathbf{X}))],\quad \forall\theta'\ne\theta. \tag{7}$$

Under the GSL function (6), (7) reduces to $E_\theta[\delta^q(\mathbf{X})]=h^q(\theta)$. So, if $h(\theta)$ is a random parameter (e.g., $\theta_M$ or $\theta_J$), then the estimator $\delta(\mathbf{X})$ is a risk-unbiased estimator of $h(\theta)$ if

$$E_\theta[\delta^q(\mathbf{X})]=E_\theta[h^q(\theta)], \tag{8}$$

otherwise, it is biased and its bias is defined as

$$B(\delta)=E_\theta[\delta^q(\mathbf{X})]-E_\theta[h^q(\theta)]. \tag{9}$$

Nematollahi and Jafari Jozani (2014), considered the UMRU estimation of the random parameter $h(\theta)$ $(=\theta_M \text{ or } \theta_J)$ of the selected population under the general $\gamma$-loss function

$$L(\mathrm{h}(\theta),\delta)=(\gamma(\delta)-\gamma(h(\theta)))^2, \tag{10}$$

which has the risk-unbiased condition $E_\theta[\gamma(\delta(\mathrm{X}))]=E_\theta[\gamma(\mathrm{h}(\theta))]$. They showed that under the uniform model (2) and under the $\gamma$-loss function (6), the UMRU estimators of $\theta_M$ and $\theta_J$ are given by respectively.

$$\delta_M^U(\mathbf{X})=\gamma^{-1}\left(\gamma(\mathrm{X}_{(\mathrm{k})})+\frac{\gamma'(\mathrm{X}_{(\mathrm{k})})}{n}X_{(\mathrm{k})}\left(1-\left(\frac{\mathrm{X}_{(\mathrm{k}-1)}}{\mathrm{X}_{(\mathrm{k})}}\right)^n\right)\right) \tag{11}$$

and

$$\delta_J^U(\mathbf{X})=\gamma^{-1}\left(\gamma(\mathrm{X}_{(1)})+\sum_{i=1}^{k}\frac{\gamma'(\mathrm{X}_{(\mathrm{i})})}{n}X_{(\mathrm{i})}\left(\frac{X_{(1)}}{X_{(\mathrm{i})}}\right)^n\right), \tag{12}$$

Since the unbiased condition under GSL function (6) is equivalent to unbiased condition under $\gamma-$ loss function (10) with $\gamma(\mathrm{x})=x^q$, then from (11) and (12) the UMRU estimators of $\theta_M$ and $\theta_J$ in uniform population under GSL function are given by respectively.

$$\delta_M^U(\mathbf{X})=X_{(\mathrm{k})}\left(1+\frac{q}{n}-\frac{q}{n}\left(\frac{X_{(\mathrm{k}-1)}}{X_{(\mathrm{k})}}\right)^n\right)^{\frac{1}{q}} \tag{13}$$

and

$$\delta_J^U(\mathbf{X})=X_{(1)}\left(1+\frac{q}{n}\sum_{i=1}^{k}\left(\frac{X_{(1)}}{X_{(\mathrm{i})}}\right)^{n-q}\right)^{\frac{1}{q}}, \tag{14}$$

Remark 2.1. Nematollahi and Motamed-Shariati (2012) obtained UMRU estimators of $\theta_M$ and $\theta_J$ under entropy loss function. Their results can be obtain from (13) and (14) by taking $q=-1$, i.e.,

$$\delta_M^U(\mathbf{X})=\frac{nX_{(\mathrm{k})}}{\left(n-1+\left(\frac{X_{(\mathrm{k}-1)}}{X_{(\mathrm{k})}}\right)^n\right)}$$ and $$\delta_J^U(\mathbf{X})=\frac{nX_{(1)}}{\left(n-\sum_{i=1}^{k}\left(\frac{X_{(1)}}{X_{(\mathrm{i})}}\right)^{n+1}\right)}.$$

***Admissibility results***

Consider the case of two population, i.e., $k=2$. Let $X_1$ and $X_2$ be two independent random variables such that $X_i,\ i=1,\dots,\mathrm{k}$ has p.d.f (2). In estimation of unknown random parameter $\theta_M$ and $\theta_J$ under the scale invariant loss function (6), the problem is invariant under the scale group of transformation $(\mathrm{X}_1,\mathrm{X}_2)\to(\mathrm{cX}_1,\mathrm{cX}_2),\ \mathrm{c}>0$. Therefore, it is natural to consider only those estimators which are scale invariant, i.e., estimators satisfying $\delta(\mathrm{cX}_1,\mathrm{cX}_2)=c\delta(\mathrm{X}_1,\mathrm{X}_2)$ for all $\mathrm{c}>0$. We consider the subclasses

$$D_M=\left\{\delta_{1c}(\mathrm{X}_1,\mathrm{X}_2)=cX_{(2)}\ ,\ c>0\right\},$$ and $$D_J=\left\{\delta_{2c}(\mathrm{X}_1,\mathrm{X}_2)=cX_{(1)}\ ,\ c>0\right\},$$

of invariant estimators of $\theta_M$ and $\theta_J$, respectively. In this section, we will characterize admissible estimators of $\theta_M$ and $\theta_J$ within the subclasses $D_M$ and $D_J$, respectively, under the scale invariant loss function (6). The following lemma will be useful in driving the subsequent results.

Lemma 3.1. Let $\lambda=\frac{\min(\theta_1,\theta_2)}{\max(\theta_1,\theta_2)}$, then,

i) $$E\left(\frac{X_{(2)}}{\theta_M}\right)^q=\frac{n}{n+q}-\frac{n^2}{(n+q)(2\mathrm{n}+\mathrm{q})}\lambda^{n+q}+\frac{n}{2n+q}\lambda^n,\quad q>-n.$$

ii) $$E\left(\frac{X_{(1)}}{\theta_J}\right)^q=\frac{n}{n+q}-\frac{n}{2n+q}\lambda^n+\frac{n^2}{(n+q)(2\mathrm{n}+\mathrm{q})}\lambda^{n+q},\quad q>-n.$$

iii) $$E\left(\ln\left(\frac{X_{(2)}}{\theta_M}\right)\right)=-\frac{1}{n}-\frac{1}{2}\lambda^n\ln\lambda+\frac{1}{2n}\lambda^n.$$

The proof of Lemma 3.1 is given in the Appendix. In subsequent Theorems, we characterize the admissible estimators of $\theta_M$ and $\theta_J$ in the classes $D_M$ and $D_J$, respectively.

Theorem 3.1. Let $c_1^*=\left(\frac{2n+q}{2n}\right)^{\frac{1}{q}}$ and $c_2^*=\left(\frac{n+q}{n}\right)^{\frac{1}{q}}$, then under the GSL function, the estimators $\delta_{1c}(\mathrm{X}_1,\mathrm{X}_2)=cX_{(2)}$ are admissible within the subclass $D_M$ of invariant estimators of $\theta_M$, if and only if $c\in[\mathrm{c}_1^*,\mathrm{c}_2^*]$ and $q>-n$.

Proof: The risk function of $\delta_{1c}=cX_{(2)}$ is given by

$$R(\theta_M,\delta_{1c}) = \mathrm{E}\left(\left(\frac{cX_{(2)}}{\theta_M}\right)^q - q\ln\frac{cX_{(2)}}{\theta_M} - 1\right) = c^q E\left(\frac{X_{(2)}}{\theta_M}\right)^q - q\ln c - qE\left(\ln\frac{X_{(2)}}{\theta_M}\right) - 1 \quad (15)$$

For fixed $\lambda$, the risk function (15) takes its minimum at $c = c_1(\lambda)$, where

$$c = c_1(\lambda) = \left[E\left(\frac{X_{(2)}}{\theta_M}\right)^q\right]^{-\frac{1}{q}}. \quad (16)$$

From Lemma 3.1.i, it is easy to check that $c_1(\lambda)$ is a continuous function of $\lambda$ and is decreasing on $(0,1]$. Therefore,

$$\sup_{0<\lambda\le 1} c_1(\lambda) = \lim_{\lambda\to 0^+} c_1(\lambda) = \left(\frac{n+q}{n}\right)^{\frac{1}{q}}, \quad \text{and} \quad \inf_{0<\lambda\le 1} c_1(\lambda) = c_1(1) = \left(\frac{2n+q}{2n}\right)^{\frac{1}{q}}$$

Thus any value of $c \in [c_1^*, c_2^*)$ minimize the risk function $R(\theta_M,\delta_{1c})$ for some values of $0<\lambda\le 1$ and hence such a $c$ correspondence to an admissible estimator. The admissibility of the estimator $\delta_{1c_2^*}$ follows from the continuity of the risk function.

Note that for each fixed $0<\lambda\le 1$, the risk function $R(\theta_M,\delta_{1c})$ is an increasing function of $c$ if $c > c_1(\lambda)$ and it is a decreasing function of $c$ if $c < c_1(\lambda)$. Since $c_1^* \le c_1(\lambda) \le c_2^*$, $\forall\ 0<\lambda\le 1$, we conclude that the estimators $\delta_{1c}(X_1,X_2) = cX_{(2)}$ for $c \in \left(0,\left(\frac{2n+q}{2n}\right)^{\frac{1}{q}}\right) \cup \left(\left(\frac{n+q}{n}\right)^{\frac{1}{q}},\infty\right)$ are inadmissible in estimating $\theta_M$, which completes the proof.

Theorem 3.2. Let $c_2^* = \left(\frac{n+q}{n}\right)^{\frac{1}{q}}$ and $c_3^* = \left(\frac{(n+q)(2n+q)}{2n^2}\right)^{\frac{1}{q}}$, then under the GSL function, the estimators $\delta_{2c}(X_1,X_2) = cX_{(1)}$ are admissible within the subclass $D_J$ of invariant estimators of $\theta_J$, if and only if $c \in [c_2^*, c_3^*]$ and $q > -n$.

Proof: The proof is similar to the proof of Theorem 3.1.

Remark 3.1. Nematollahi and Motamed-Shariati (2012) obtained the linear admissible estimators of $\theta_M$ and $\theta_J$ in the classes $D_M$ and $D_J$, respectively, under the entropy loss function. Their results can be obtain from Theorem 3.1 and 3.2 by taking $q = -1$, i.e., $\delta_{1c}(X_1,X_2) = cX_{(2)}$ is admissible in the class of linear invariant estimators of $\theta_M$, if and only if $\frac{2n}{2n-1} \le c \le \frac{n}{n-1}$, and $\delta_{2c}(X_1,X_2) = cX_{(1)}$ is admissible in the class of linear invariant estimators of $\theta_J$, if and only if $\frac{n}{n-1} \le c \le \frac{2n^2}{(n-1)(2n-1)}$.

**Mimimax estimator of $\theta_M$**

Let $X_1$ and $X_2$ be two independent random variables such that $X_i$ has a p.d.f as in (2). We want to find minimax estimator of $\theta_M$ under the loss function (6).

Following Sackrowitz and Samuel-Cahn (1987), we first find the Bayesestimator in component problem for $\theta_i$ , $i=1,2$. So, consider the following prior for $\theta_i$ , $i=1,2$.

$$\pi_i^{r,b}(\theta_i)=(\mathrm{r}-1)\,\mathrm{b}^{r-1}\theta_i^{-r}, \quad \theta_i>b, \quad r>1. \tag{17}$$

Since $X_i\,|\,\theta_i$ has p.d.f (2), the posterior density is

$$\pi(\theta_i\,|\,\mathrm{X}_i)=\begin{cases}(\mathrm{n}+\mathrm{r}-1)\theta^{-(\mathrm{n}+\mathrm{r})}b^{n+r-1} & x_i\le b\\ (\mathrm{n}+\mathrm{r}-1)\theta^{-(\mathrm{n}+\mathrm{r})}x_i^{\,n+r-1} & x_i>b\end{cases} \tag{18}$$

It is easy to see that the Bayes estimator of $\theta_i$ under loss function (6) is equal to

$$\delta_{\pi^{r,b}}(\mathrm{x}_i)=\left(E\left(\frac{1}{\theta_i^q}\,|\,x_i\right)\right)^{-\frac{1}{q}}=\begin{cases}\left(\dfrac{n+q+r-1}{n+r-1}\right)^{\frac{1}{q}}b & x_i\le b\\ \left(\dfrac{n+q+r-1}{n+r-1}\right)^{\frac{1}{q}}x_i & x_i>b\end{cases}, \tag{19}$$

and the posterior risk of $\delta_{\pi^{r,b}}(\mathrm{x}_i)$ is given by

$$r(\mathrm{x}_i,\delta_{\pi^{r,b}}(\mathrm{x}_i))=\mathrm{E}\left(\left(\frac{\delta_{\pi^{r,b}}(\mathrm{x}_i)}{\theta_i}\right)^q-q\ln\left(\frac{\delta_{\pi^{r,b}}(\mathrm{x}_i)}{\theta_i}\right)-1\,|\,x_i\right)=\ln\left(\mathrm{E}\left(\frac{1}{\theta_i^q}\,|\,x_i\right)\right)+q\,\mathrm{E}\left(\ln\theta_i\,|\,x_i\right)=\ln\left(\frac{n+r-1}{n+q+r-1}\right)+\frac{q}{n+r-1}.$$

Since the posterior risk does not depend on $x_i$ , the Bayes risk of $\delta_{\pi^{r,b}}(\mathrm{x}_i)$ is also

$$r^*(\pi_i^{r,\mathrm{b}},\delta_{\pi_i^{r,\mathrm{b}}})=\ln\left(\frac{n+r-1}{n+q+r-1}\right)+\frac{q}{n+r-1}, \quad i=1,2. \tag{20}$$

Now consider Bayes estimation of $\theta_M$ under GSL function. Suppose $\theta_i$ , $i=1,2$ are independent and has p.d.f (17). Then from (19) and using Lemma 3.2 of Sackrowitz and Samuel-Cahn (1987), the unique Bayes rule of $\theta_M$ is given by

$$\delta_{\pi^{r,b}}(X_1,\mathrm{X}_2)=\begin{cases}\left(\dfrac{n+q+r-1}{n+r-1}\right)^{\frac{1}{q}}b & X_{(2)}\le b\\ \left(\dfrac{n+q+r-1}{n+r-1}\right)^{\frac{1}{q}}X_{(2)} & X_{(2)}>b\end{cases},$$

where $\pi^{r,b}=(\pi_1^{r,b},\pi_2^{r,b})$. Since the posterior risk (20) for the component problem is independent of $x=(\mathrm{x}_1,\mathrm{x}_2)$, we conclude from Theorem 3.1 of Sackrowitz and Samuel-Cahn (1987) that the Bayes risk $r^*(\pi^{r,\mathrm{b}},\delta_{\pi^{r,\mathrm{b}}})$ of $\delta_{\pi^{r,b}}(X_1,\mathrm{X}_2)$ is the same as the one given in (20), i.e.,

$$r^*(\pi^{r,\mathrm{b}},\delta_{\pi^{r,\mathrm{b}}})=r^*(\pi_i^{r,\mathrm{b}},\delta_{\pi_i^{r,\mathrm{b}}})=\ln\left(\frac{n+r-1}{n+q+r-1}\right)+\frac{q}{n+r-1}, \quad i=1,2.$$

Now from Theorem 3.2 of Sackrowitz and Samuel-Cahn (1987), the estimator $\delta_M(X_1, X_2)$ is minimax for $\theta_M$ if

$$R(\theta_M, \delta_M) \le \lim_{\substack{r\to 1\\ b\to 0}} r^*(\pi^{r,b}, \delta_{\pi^{r,b}}) = \ln\left(\frac{n}{n+q}\right) + \frac{q}{n} \tag{21}$$

In the following Theorem we find the minimax estimator of $\theta_M$.

Theorem 4.1. Let $X_1$ and $X_2$ be two independent random variables such that $X_i$ has a p.d.f as in (2). Then $\delta_M(X_1, X_2) = (\frac{n+q}{n})^{\frac{1}{q}} X_{(2)}$ is a minimax estimator of $\theta_M$ under the loss function (6), when $q > -n$.

Proof: We have

$$R(\theta_M, \delta_M) = \mathrm{E}\left(\left(\frac{\delta_M}{\theta_M}\right)^q - q\ln\frac{\delta_M}{\theta_M} - 1\right) = \frac{n+q}{n} E\left(\frac{X_{(2)}}{\theta_M}\right)^q - \ln\frac{n+q}{n} - qE\left(\ln\frac{X_{(2)}}{\theta_M}\right) - 1$$

$$= \frac{n+q}{n}\left(\frac{n}{n+q} - \frac{n^2}{(n+q)(2n+q)}\lambda^{n+q} + \frac{n}{2n+q}\lambda^n\right) + \ln\left(\frac{n}{n+q}\right) + q\left(\frac{1}{n} + \frac{1}{2}\lambda^n \ln\lambda - \frac{1}{2n}\lambda^n\right) - 1$$

$$= \ln\left(\frac{n}{n+q}\right) + \frac{q}{n} - \frac{n}{2n+q}\lambda^{n+q} + \frac{n+q}{2n+q}\lambda^n + \frac{q}{2}\lambda^n\ln\lambda - \frac{q}{2n}\lambda^n = \ln\left(\frac{n}{n+q}\right) + \frac{q}{n}$$

$$+\lambda^{n+q}\left(-\frac{n}{2n+q} + \left(\frac{n+q}{2n+q} - \frac{q}{2n}\right)\lambda^{-q} + \frac{q}{2}\lambda^{-q}\ln\lambda\right) = \ln\left(\frac{n}{n+q}\right) + \frac{q}{n} + \lambda^{n+q} g(\lambda)$$

Note that $g'(\lambda) = \frac{1}{\lambda^{q+1}}\left\{\left(\frac{q^2(n+q)}{2n(2n+q)}\right) - \frac{q^2\ln\lambda}{2}\right\} > 0$ for $0 < \lambda \le 1$, $q > -n$, so $g(\lambda)$ is a strictly increasing function of $\lambda$ and $g(\lambda) \le g(1) = \frac{-q^2}{2n(2n+q)} < 0$. Thus which complete the proof.

$$R(\theta_M, \delta_M) < \ln\left(\frac{n}{n+q}\right) + \frac{q}{n},$$

Remark 4.1. Nematollahi and Motamed-Shariati (2012) obtained a minimax estimator of $\theta_M$ under entropy loss function. Their result can be obtain from Theorem 4.1 by taking $q = -1$, i.e., the estimator $\delta_M(X_1, X_2) = (\frac{n}{n-1})X_{(2)}$ is a minimax estimator of $\theta_M$ under the entropy loss function.

**Improving the UMRU estimator of $\theta_M$**

In this section, for the case of $k = 2$, we show that the UMRU estimator of $\theta_M$ under the loss function (6) is inadmissible. Let

$$D_U = \left\{\delta_\psi : \delta_\psi(X_1, X_2) = X_{(2)}\,\psi(Y)\right\}, \tag{22}$$

where $Y = \frac{X_{(1)}}{X_{(2)}}$ and $\psi(\cdot)$ is some real valued function defined on $(0,1]$. Let $Z = \frac{X_1}{X_2}$ then

$$\delta_{\psi}(X_1, X_2) = X_{(2)}\,\psi(Y) = \begin{cases} X_2\,\psi(\frac{X_1}{X_2}) & X_1 < X_2 \\ X_1\,\psi(\frac{X_2}{X_1}) & X_1 \geq X_2 \end{cases} = \begin{cases} X_2\,\psi(Z) & Z < 1 \\ X_2\,Z\psi(\frac{1}{Z}) & Z \geq 1 \end{cases}. \quad (23)$$

So, $\delta_{\psi}(X_1, X_2) = X_2\,\varphi(Z)$, where

$$\varphi(Z) = \begin{cases} \psi(Z) & Z < 1 \\ Z\,\psi(\frac{1}{Z}) & Z \geq 1 \end{cases} \quad (24)$$

In this section we use the technique of Brewester and Ziedeck (1974) to find estimators of the form $\delta_{\psi_S}(X_1, X_2) = X_{(2)}\,\psi_S(Y) = X_2\,\varphi_S(Z)$, which are dominate the estimators in class (22), $\delta_{\psi}(X_1, X_2) = X_{(2)}\,\psi(Y) = X_2\,\varphi(Z)$. The following theorem gives a sufficient condition for inadmissibility of the estimators $\delta_{\psi} \in D_U$ .

Theorem 5.1. Let $X_1$ and $X_2$ be two independent random variables such that $X_i$ has a p.d.f as in (2). Let $\delta_{\psi}(X_1, X_2) = X_2\,\varphi(Z) \in D_U$ be an invariant estimator of $\theta_M$ . If

$$\varphi_1(Z) = \begin{cases} \left(\frac{2n+q}{2n}\right)^{\frac{1}{q}} & Z < 1 \\ Z\left(\frac{2n+q}{2n}\right)^{\frac{1}{q}} & Z \geq 1 \end{cases}, \quad (25)$$

and $P_{\theta}(\varphi(Z) > \varphi_1(Z)) > 0,\ \forall\,\theta = (\theta_1, \theta_2) \in R^2$, then under loss function (6) the invariant estimator $\delta_{\psi}(X_1, X_2) = X_2\,\varphi(Z)$ is inadmissible for estimating $\theta_M$ and is dominated by $\delta_{\psi_S}(X_1, X_2) = X_2\,\varphi_S(Z)$, where

$$\varphi_S(Z) = \max(\varphi_1(Z), \varphi(Z)) = \begin{cases} \varphi_1(Z) & \varphi(Z) \leq \varphi_1(Z) \\ \varphi(Z) & \varphi(Z) > \varphi_1(Z) \end{cases} \quad (26)$$

Proof: We have

$$\Delta = R(\theta_M, \delta_{\psi}) - R(\theta_M, \delta_{\psi_S}) = E\left(\left(\frac{X_2\varphi(Z)}{\theta_M}\right)^q - \left(\frac{X_2\varphi_S(Z)}{\theta_M}\right)^q - q\ln\frac{\varphi(Z)}{\varphi_S(Z)}\right)$$

$$= E\left(\left(\frac{X_2}{\theta_M}\right)^q \left(\varphi^q(Z) - \varphi_S^q(Z)\right) + q\ln\frac{\varphi_S(Z)}{\varphi(Z)}\right) = E_{\theta}\left(D_{\theta}(Z)\right), \quad (27)$$

where $D_{\theta}(Z) = \left(\varphi^q(Z) - \varphi_S^q(Z)\right) E\left(\left(\frac{X_2}{\theta_M}\right)^q \mid Z\right) + q\ln\frac{\varphi_S(Z)}{\varphi(Z)}$ (28)

Similar to Vellaisamy et al. (1988) the conditional p.d.f of $X_2$ given $Z = \frac{X_1}{X_2} = z$ is given by

$$f_{X_2|Z}(x_2 \mid z) = \begin{cases} \frac{2n}{\theta_1^{2n}} x_2^{2n-1} & 0 < x_2 < \theta_2,\ 0 < z < \frac{\theta_1}{\theta_2} \\ \frac{2n}{\theta_1^{2n}} x_2^{2n-1} z^{2n} & 0 < x_2 < \frac{\theta_1}{z}, \quad z > \frac{\theta_1}{\theta_2} \end{cases}. \tag{29}$$

Note that,

$$E\left(\left(\frac{X_2}{\theta_M}\right)^q \mid Z = z\right) = \begin{cases} \frac{1}{\theta_2^q} E\left(X_2^q \mid Z = z\right) & Z < 1 \\ \frac{1}{\theta_1^q} E\left(X_2^q \mid Z = z\right) & Z \geq 1 \end{cases} \tag{30}$$

From (30) we have

$$E\left(X_2^q \mid Z = z\right) = \begin{cases} \int_0^{\theta_2} \frac{2n x_2^{2n+q-1}}{\theta_2^{2n}} dx_2 & Z < 1 \\ \int_0^{\frac{\theta_1}{z}} \frac{2n x_2^{2n+q-1}}{\theta_1^{2n}} z^{2n} dx_2 & Z \geq 1 \end{cases} = \begin{cases} \frac{2n}{2n+q} \theta_2^q & Z < \frac{\theta_1}{\theta_2} \\ \frac{2n}{2n+q} \frac{\theta_1^q}{z^q} & Z > \frac{\theta_1}{\theta_2} \end{cases} \tag{31}$$

For $\theta_1 < \theta_2$ we conclude that

$$E\left(\left(\frac{X_2}{\theta_M}\right)^q \mid Z = z\right) = \begin{cases} \frac{2n}{2n+q} & 0 < z \leq \lambda \\ \frac{2n}{2n+q} \frac{\lambda^q}{z^q} & \lambda < z \leq 1 \\ \frac{2n}{2n+q} \frac{1}{z^q} & z > 1 \end{cases},$$

where $\lambda = \frac{\min(\theta_1, \theta_2)}{\max(\theta_1, \theta_2)}$. Similarly for $\theta_1 > \theta_2$ we have

$$E\left(\left(\frac{X_2}{\theta_M}\right)^q \mid Z = z\right) = \begin{cases} \frac{2n}{2n+q} & z \leq 1 \\ \frac{2n}{2n+q} \lambda^q & 1 < z \leq \frac{1}{\lambda} \\ \frac{2n}{2n+q} \frac{1}{z^q} & z > \frac{1}{\lambda} \end{cases}.$$

In either cases, for $q < 0$ we have

$$\inf_{0<\lambda\leq 1} E\left(\left(\frac{X_2}{\theta_M}\right)^q \mid Z = z\right) = \begin{cases} \frac{2n}{2n+q} & z < 1 \\ \frac{2n}{2n+q} \frac{1}{z^q} & z > 1 \end{cases} = \frac{1}{\varphi_1^q(Z)}, \tag{32}$$

and for $q > 0$ we have

$$\sup_{0<\lambda\le 1} E\left(\left(\frac{X_2}{\theta_M}\right)^q \mid Z=z\right)=\begin{cases}\dfrac{2n}{2n+q} & z<1\\ \dfrac{2n}{2n+q} & z>1\end{cases}=\frac{1}{\varphi_1^q(Z)}. \tag{33}$$

Now from (26) and (27), if $\varphi_1(Z)\le\varphi(Z)$, then $D_\theta(Z)=0$. For $\varphi_1(Z)>\varphi(Z)$, we have $\varphi_S(Z)=\varphi_1(Z)$, and from (28), (32) and (33) we have

$$D_\theta(Z)\ge\left(\varphi^q(Z)-\varphi_1^q(Z)\right)\frac{1}{\varphi_1^q(Z)}-q\ln\frac{\varphi(Z)}{\varphi_1(Z)}=\left(\frac{\varphi(Z)}{\varphi_1(Z)}\right)^q-\ln\left(\frac{\varphi(Z)}{\varphi_1(Z)}\right)^q-1>0,$$

which completes the proof.

Corollary 5.1. For $k=2$, the UMRU estimator given in (13) is inadmissible and it dominated by

$$\delta_M^D(X)=\max\left(X_{(2)}\left(\frac{2n+q}{2n}\right)^{\frac{1}{q}},\ \delta_M^U(X)\right)=X_{(2)}\max\left(\left(\frac{2n+q}{2n}\right)^{\frac{1}{q}},\ \left(1+\frac{q}{n}-\frac{q}{n}\left(\frac{X_{(1)}}{X_{(2)}}\right)^n\right)^{\frac{1}{q}}\right) \tag{34}$$

Proof: Let

$$\varphi(Z)=\begin{cases}\left(1+\dfrac{q}{n}-\dfrac{q}{n}Z^n\right)^{\frac{1}{q}} & Z\le 1\\ Z\left(1+\dfrac{q}{n}-\dfrac{q}{n}\dfrac{1}{Z^n}\right)^{\frac{1}{q}} & Z>1\end{cases} \tag{35}$$

Then $\delta_M^U(\mathbf{X})=X_2\,\varphi(Z)=\delta_\psi(X)$. Now, if $\left(\frac{1}{2}\right)^{\frac{1}{n}}<Z<1$ then $\left(1+\frac{q}{n}-\frac{q}{n}Z^n\right)^{\frac{1}{q}}<$ $\left(\frac{2n+q}{2n}\right)^{\frac{1}{q}}$ and if $1<Z<2^{\frac{1}{n}}$ then $Z\left(1+\frac{q}{n}-\frac{q}{n}Z^{-n}\right)^{\frac{1}{q}}<Z\left(\frac{2n+q}{2n}\right)^{\frac{1}{q}}$. So from Theorem 5.1, $\delta_M^U(\mathbf{X})=\delta_\psi(X)$ is

inadmissible and dominated by

$$\delta_M^D(X)=X_2\max(\varphi_1(Z),\varphi(Z))=\begin{cases}X_2\max\left(\left(\dfrac{2n+q}{2n}\right)^{\frac{1}{q}},\left(1+\dfrac{q}{n}-\dfrac{q}{n}Z^n\right)^{\frac{1}{q}}\right) & Z\le 1\\ X_2\max\left(Z\left(\dfrac{2n+q}{2n}\right)^{\frac{1}{q}},Z\left(1+\dfrac{q}{n}-\dfrac{q}{n}\dfrac{1}{Z^n}\right)^{\frac{1}{q}}\right) & Z>1\end{cases}=X_{(2)}\max\left(\left(\frac{2n+q}{2n}\right)^{\frac{1}{q}},\left(1+\frac{q}{n}-\frac{q}{n}\left(\frac{X_{(1)}}{X_{(2)}}\right)^n\right)^{\frac{1}{q}}\right)$$

Remark 5.1. Nematollahi and Motamed-Shariati (2012) found improved estimator of $\theta_M$ under entropy loss function. Their result can be obtain from Theorem 5.1 by taking $q=-1$, i.e., the estimator dominate the UMRU estimator of $\theta_M$ under entropy loss function.

$$\delta_{\varphi}^{*}=X_{(2)}\max\left(\frac{2n}{2n-1},\ \frac{n}{n-1+\left(\frac{X_{(1)}}{X_{(2)}}\right)^{n}}\right)$$

**Discussion and Conclusion**

In this paper, we find UMRU estimator and admissible linear estimators of $\theta_M$ and $\theta_J$ under the GSL function (6) in selected uniform population. Also, minimax estimator of $\theta_M$ and dominating estimator of UMRU estimator of $\theta_M$ were obtained under GSL function. For selected Pareto distribution, similar results could be obtained. Also, when $q=1$, the GSL function reduce to Stein loss function and in this case we have the following results

**(i)** $\delta_M^U(\mathbf{X})=\frac{X_{(\mathrm{k})}}{n}\left(n+1-\left(\frac{X_{(\mathrm{k}-1)}}{X_{(\mathrm{k})}}\right)^{n}\right)$ is UMRU estimator of $\theta_M$ (Misra and Mulen , 2001).

**(ii)** $\delta_J^U(\mathbf{X})=\frac{X_{(1)}}{n}\left(n-\sum_{i=1}^{k}\left(\frac{X_{(1)}}{X_{(\mathrm{i})}}\right)^{n-1}\right)$ is UMRU estimator of $\theta_J$ (Misra and Mulen, 2001).

**(iii)** $\delta_{1c}(\mathrm{X}_1,\mathrm{X}_2)=cX_{(2)}$ is admissible in the class of linear invariant estimators of $\theta_M$ , if and only if $\frac{2n+1}{2n}\le c\le\frac{n+1}{n}$.

**(iv)** $\delta_{2c}(\mathrm{X}_1,\mathrm{X}_2)=cX_{(1)}$ is admissible in the class of linear invariant estimators of $\theta_J$ , if and only if $\frac{n+1}{n}\le c\le\frac{(n+1)(2n+1)}{2n^2}$.

**(v)** $\delta_M(\mathrm{X}_1,\mathrm{X}_2)=(\frac{n+1}{n})X_{(2)}$ is minimax estimator of $\theta_M$ .

**(vi)** The UMRU estimator of $\theta_M$ , i.e., $\delta_M^U(\mathbf{X})=\frac{X_{(2)}}{n}\left(n+1-\left(\frac{X_{(1)}}{X_{(2)}}\right)^{n}\right)$ is inadmissible and dominated by

$$\delta_M^D=X_{(2)}\max\left(\frac{2n+1}{2n},\frac{1}{n}\left(n+1-\left(\frac{X_{(1)}}{X_{(2)}}\right)^{n}\right)\right).$$

**References**

Brewester, J. F., & Ziedeck, J. V. (1974). Improving on equivariant estimators. Ann. Statist, 2, 21-38.

Kumar, S., & Kar, A. (2001a). Estimating quantiles of a selected exponential population. Statistics & Probability Letters, 52, 9-19.

Kumar, S., & Kar, A. (2001b). Minimum variance unbiased estimation of quantile of a selected exponential population. Amer. J. Math. Management Sci, 21, (1-2), 183-191.

Kumar, S., & Gangopadhyay, A. K. (2005). Estimating parameters of a selected pareto population. Statistical methodology, 2, 121-130.

Lehmann, E.L. (1951). A general concept of unbiasedness, Ann. Math. Statist, 22, 578–592.

Misra, N., & Meulen, E. C. V. (2001). On estimation following selection from non-regular distributions. Commun. Statist. Theory and Methods, 30 (12), 2543-2561.

Misra, N., Van der Meulen, E.C., & Branden, K.V. (2006a). On estimating the scale parameter of the selected gamma population under the scale invariant squared error loss function. Journal of Computational and Applied Mathematics, 186, 268-282.

Misra, N., van der Meulen, E.C., & Branden, K.V. (2006b). On some inadmissibility results for the scale parameters of selected gamma populations. Journal of Statistical Planning and Inference, 136, 2340-2351.

Naghizadeh Qomi, M., & Nematollahi, N. (2012). Estimation following selection under LINEX loss function. Proceeding of the 11th Iranian statistical conference, 398-404.

Nematollahi, N., & Jafari Jozani, M. (2014). On risk unbiased estimation after selection. Accepted to be appear in Brazilian Journal of Probability and Statistics.

Nematollahi, N., & Motamed-Shariati, F. (2009). Estimation of the scale parameter of the selected gamma population under the entropy loss function. Commun. Statist. Theory and Methods, 38, 208-221.

Nematollahi, N., & Motamed-Shariati, F. (2012). Estimation of the parameter of the selected uniform population under the entropy loss function. Journal of Statistical Planning and Inference, 142, 2190-2202.

Rubinstein, D. (1965). Estimation of failure rates in a dynamic reliability program. Rep.65-RGO-7, Tech. Info. Service, GE Co.

Sackrowitz, H.B., & Samuel-Cahn, E. (1987). Evaluating the chosen population. A Bayesian minimax approach. Adaptive statistical procedures and related topics, Institute of Mathematical Statistics, Hayward, CA, 386- 395.

Stein, C. (1964). Contribution to the discussion of Bayesian and non Bayesian decision theory. Handout, IMS Meeting, Amherst, MA.

Vellaisamy, P. (1992a). Average worth and simultaneous estimation of the selected subset, Ann. Inst. Statist. Math, 44, 551-562.

Vellaisamy, P. (1992b). Inadmissibility results for the selected scale parameters. Ann. Statist., 20, 2183-2191.

Vellaisamy, P. (2003). Quantile estimation of the selected exponential population. Journal of Statistical Planning and Inference, 115, 461-470.

Vellaisamy, P., & Punnen, A. P. (2002). Improved estimators for the selected location parameters. Statist. Papers, 43, 291-299.

Vellaisamy, P., Kumar, S., & Sharma, D. (1988). Estimating the mean of the selected uniform population. Commun. Statist. Theory and Methods, 17, 3447–3475.

**Appendix**

Proof of lemma 3.1.

(i) If $\theta_1 > \theta_2$ then,

$$E\left(\frac{X_{(2)}}{\theta_M}\right)^q = \int_0^{\theta_2}\int_{x_2}^{\theta_1}\left(\frac{x_1}{\theta_1}\right)^q \frac{nx_1^{n-1}}{\theta_1^n}\frac{nx_2^{n-1}}{\theta_2^n}dx_1dx_2 + \int_0^{\theta_2}\int_0^{x_2}\left(\frac{x_2}{\theta_2}\right)^q \frac{nx_1^{n-1}}{\theta_1^n}\frac{nx_2^{n-1}}{\theta_2^n}dx_1dx_2$$

$$= \frac{n}{n+q} - \frac{n^2}{(\mathrm{n+q})(2\,\mathrm{n+q})}\left(\frac{\theta_2}{\theta_1}\right)^{n+q} + \frac{n}{2n+q}\left(\frac{\theta_2}{\theta_1}\right)^n .$$

Similarly, for $\theta_1 < \theta_2$ we have

$$E\left(\frac{X_{(2)}}{\theta_M}\right)^q = \frac{n}{n+q} - \frac{n^2}{(\mathrm{n+q})(2\,\mathrm{n+q})}\left(\frac{\theta_1}{\theta_2}\right)^{n+q} + \frac{n}{2n+q}\left(\frac{\theta_1}{\theta_2}\right)^n .$$

So,

$$E\left(\frac{X_{(2)}}{\theta_M}\right)^q = \frac{n}{n+q} - \frac{n^2}{(\mathrm{n+q})(2\,\mathrm{n+q})}\lambda^{n+q} + \frac{n}{2n+q}\lambda^n ,$$

where $\lambda = \dfrac{\min(\theta_1,\theta_2)}{\max(\theta_1,\theta_2)}$.

(ii) If $\theta_1 > \theta_2$ then

$$E\left(\frac{X_{(1)}}{\theta_J}\right)^q = \int_0^{\theta_2}\int_{x_2}^{\theta_1}\left(\frac{x_2}{\theta_2}\right)^q \frac{nx_1^{n-1}}{\theta_1^n}\frac{nx_2^{n-1}}{\theta_2^n}dx_1dx_2 + \int_0^{\theta_2}\int_0^{x_2}\left(\frac{x_1}{\theta_1}\right)^q \frac{nx_1^{n-1}}{\theta_1^n}\frac{nx_2^{n-1}}{\theta_2^n}dx_1dx_2$$

$$= \frac{n}{n+q} + \frac{n^2}{(\mathrm{n+q})(2\,\mathrm{n+q})}\left(\frac{\theta_2}{\theta_1}\right)^{n+q} - \frac{n}{2n+q}\left(\frac{\theta_2}{\theta_1}\right)^n .$$

Similar result can be obtain for $\theta_1 < \theta_2$. Therefore,

$$E\left(\frac{X_{(1)}}{\theta_J}\right)^q = \frac{n}{n+q} + \frac{n^2}{(\mathrm{n+q})(2\,\mathrm{n+q})}\lambda^{n+q} - \frac{n}{2n+q}\lambda^n$$

(iii) See Nematollahi and Motamed-Shariati (2012).