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OPTIMAL CONTROL OF PRODUCTION PROCESSES THAT INCLUDE LEAD-TIME DELAYS

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Abstract. *In this study, we investigate a method for optimal control of production processes that include lead-time delays. We propose a model that expresses lead-time lag in a strict mathematical model and a model with lead-time delay based on an average regression process, which is the Ornstein-Uhlenbeck process model that is used in mathematical finance. Optimal control is obtained using each state equation. Further, we present a simple example to verify the proposed optimal control. We additionally propose that the control system does not incorporate the lead-time delay into the control strategy and that it is simpler than the strict optimal control.*
Keywords: Lead time lag, Optimal control, Ornstein-Uhlenbeck process, Production process

1. **Introduction.** Based on mathematical and physical understandings of production engineering, we are conducting research aimed at establishing an academic area called mathematical production engineering. As our business size is a small-to-medium-sized enterprise, human intervention constitutes a significant part of the production process, and revenue can sometimes be greatly affected by human behavior. Therefore, when considering human intervention from outside companies, a deep analysis of the production process and human collaboration is necessary to understand the potential negative effects of such intervention.

With respect to mathematical modeling of deterministic systems, a physical model of the production process was constructed using a one-dimensional diffusion equation in 2012 [1]. However, the many concerns that occur in the supply chain are major problems facing production efficiency and business profitability. A stochastic bi-linear partial differential equation with time-delay was derived for outlet processes. The supply chain was modeled by considering with a time delay system [2]. With respect to the analysis of production processes in stochastic system based on financial engineering, we have proposed that a production throughput rate was able to be estimated by utilizing Kalman filter theory based on the stochastic differential equation [3]. We have also proposed a stochastic differential equation (SDE) for the mathematical model describing production processes from the input of materials to the end. We utilized a risk-neutral principal in stochastic calculus based on the SDE [4].

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On the other hand, fluctuations in the supply chain and market demand and the changes in the production volume of suppliers are propagated to other suppliers, and their effects are amplified. Therefore, because the amounts of stock are large, an increase or decrease of the suppliers' stock is modeled using differential equation. This differential equation is said as Billwhip model, representing a stock congestion [5, 6]. These studies are very interesting contents. The theory of constraints ("TOC") describes the importance of avoiding bottlenecks in production processes [15]. When using manufacturing equipment, delays in one production step are propagated to the next. Hence, the use of manufacturing equipment may lead to delays. "TOC" gives important suggestions for increasing efficiency of production projects. There is no research that mathematically models propagation of production density.

(1) Reducing the lead time, improving the throughput, and synchronizing the production process by the TOC.
(2) Sharing the demand information and performing mathematical evaluations.
(3) Analyzing the reduction and fluctuating demands of the subsystem (using nonlinear vibration theory).
(4) Basing the inventory management approach on stochastic demand.

When using manufacturing equipment, delays in one production step are propagated to the next. Hence, the use of manufacturing equipment itself may lead to delays. The improvement of production processes was presented that the "Synchronization with preprocess" method was the most desirable in practice using the actual data in production flow process based on the cash flow model by using the SDE of log-normal type [7]. In essence, we have proposed the best way, which is a synchronous method using the Vasicek model for mathematical finance [8]. Then, the supply chain theme, which was a time delay in the production processes, was proposed for the throughput improvement based on a stochastic differential equation of log-normal type [9].

Moreover, the analysis of the synchronized state indicated that this state was a much better method from the viewpoint of potential energy [9, 10]. We have also shown that the phase difference between stages in a process corresponded to the standard deviation of the working time [11]. When the phase difference was constant, the total throughput could be minimized. We showed that a synchronous process could be realized by the gradient system. The above problem is not limited to small- and medium-sized companies; in all cases, human interventions that directly affect the production process present a major challenge.

In general, we may reasonably consider that human interventions within and outside of the production system (internal and external forces, respectively) introduce uncertainties into the system's progress [4, 8]. The production system is formed by connecting both elements. When human intervention from outside companies involves an uncertainty, the noise element is frequently overlooked; instead, researchers have focused on efficient production or manufacturing the best system. Moreover, by including the noise element, we can recognize the unique advantage of the system.

In our previous study, we simulate a small-to-midsize firm without sufficient working capital to continue operations. Therefore, we need to raise working capital from financial institutions. Here, we call this cash flow. In essence, the rate of return (RoR) is at least proportional to the production lead time. In other words, if RoR forms a log-normal distribution, it is realistic to assume that the cash flow will also have the same log-normal distribution.

To evaluate the total production of a business, we utilize the actual throughput data of a firm with high productivity and implement a dynamic simulation for evaluation to confirm effectiveness of the synchronous and asynchronous processes.

With respect to an engineering system, transportation mechanisms such as substances, energy and signals, are described by the difference equation with time delay. This system is affected by the past state according to its transportation time. It often happens accompanied by phenomena such as diffusion of substances and energy, mixing of liquids, and chemical reactions inside the transport mechanism. The mathematical model of these systems is a distributed parameter system described by bilinear partial differential equation. From this point of view, it was pointed out that a system including a subsystem with a transport mechanism is appropriate to be handled as a mixed parameter system of lumped parameter system described by an ordinary differential equation and a distributed parameter system with transport type described by a bilinear partial differential equation [12].

In a previous research, Masuyama et al. have reported that a system, including a transport mechanism as a subsystem was formulated as a coupled system (Mixed Parameter System) of lumped parameter system and transport type distributed parameter system described by partial differential equations [13]. Mixed parameter system is formulated as a stochastic integral equation using the Ito's type stochastic integral along the characteristic curve by integrating the transport type distributed parameter system [13]. In other words, the coupling system in the case where additional disturbance is mixed in both the lumped parameter system and the transport type distributed parameter system was mathematically modeled as a stochastic process. However, solving simultaneous Riccati-type partial differential equations is a difficult task in a strict sense.

In this study, regarding the mathematical models of deterministic systems for the mixed-parameter system, we consider the problem of optimal control. However, for stochastic control problems, such as state estimation and optimal control, there has been little research about the case in which the coupled system to be controlled continued to operate even under additional disturbances. To develop this discussion, we must formulate a mathematical model of a coupled system that is working under additional disturbances as a stochastic process. Delayed production systems, for example, influence revenues in the form of logistics delays. It appears on the mathematical model at the boundary point of the distributed-parameter system [13]. The system is also influenced by an external disturbance. Further, the control method in the current research provides only the information at time t as feedback, and it generates an inventory quantity that is K times observed at the current time. In this control method, the optimum design of K is a major challenge [14]. Regarding the K, we propose the premium value K, which is derived by an empirical equation. As a result, the realistic optimal control of the mixed parameter system can be constructed. To the best of our knowledge, the optimal control is yet to be determined.

2. **Production Framework in Equipment Manufacturer.** We refer to the production system in manufacturing equipment industry studied in this paper. This is not a special system but "Make-to-order system with version control". Make-to-order system is a system which allows necessary manufacturing after taking orders from clients, resulting in "volatility" according to its delivery date and lead time. In addition, "volatility" occurs in lead time depending on the contents of make-to-order products (production equipment).

However, effective utilization of the production forecast information on the orders may suppress certain amount of "variation", but the complete suppression of variation will be difficult. In other words, “volatility” in monthly cash flow occurs and of course influences a rate of return in these companies. Production management systems, suitable for the separate make-to-order system which is managed by numbers assigned to each product upon order, is called as “product number management system” and is widely used. All productions are controlled with numbered products and instructions are given for each numbered product.

Thus, ordering design, logistics and suppliers are conducted for each manufacturer’s serial number in most cases except for semifinished products (unit incorporated into the final product) and strategic stocks. Therefore, careful management of the lead time or production date may not suppress “volatility” in manufacturing (production). The company in this study is the “supplier” in Figure 1 and “factory” here. Companies are under the assumption that there are N (numbers of) suppliers; however, this study deals with one company because no data is published for the rest of the companies $(N-1)$.

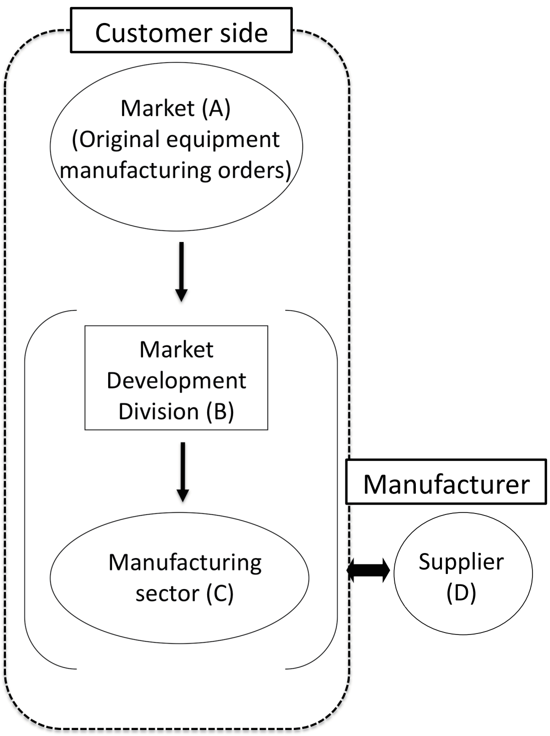

FIGURE 1. Business association chart of company of research target

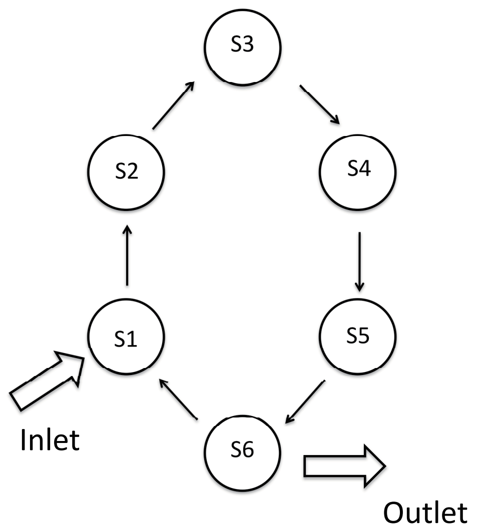

FIGURE 2. Production flow processes

2.1. **Production flow process.** A manufacturing process that is termed as a production flow process is shown in Figure 2. The production flow process, which manufactures low volumes of a wide variety of products, is produced through several stages in the production process. In Figure 2, the processes consist of six stages. In each step S1-S6 of the manufacturing process, materials are being produced. The direction of the arrows represents the direction of the production flow. Production materials are supplied through the inlet and the end-product is shipped from the outlet [7].

2.2. **Organization of production business.** We investigate a process structure in which the throughput is increased by altering the production system in a dynamic manner. Figure 3 illustrates the decision-making process of a company. A business monitors the perceived demand trends. When an order is received from a customer, the perceived

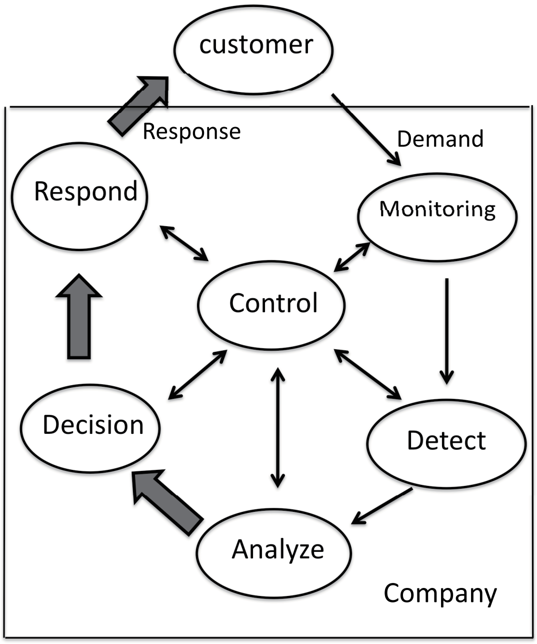

FIGURE 3. Decision-making process within the company

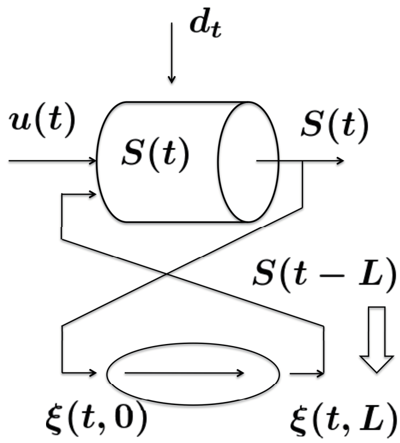

FIGURE 4. Concept of time-delay system

trend is analyzed. Based on the results of this analysis, the company decides the manner in which it should respond to the analyzed demand. The composition of the coupling system is shown in Figure 4. The lumped parameter system is influenced by the outlet end point of the distributed parameter system corresponding to the transport mechanism. The distributed parameter system is influenced by the lumped parameter system under the boundary condition. The coupling system is modeled with an external disturbance. With respect to a delay, Figure 4 illustrates the concept of a system that considers the lead time, L, as a delay time. We further intend to discuss whether to consider this delay time in the manner in which the delay time should be considered using the coefficient of optimal control. Therefore, we require a dynamic supply-chain management model.

A schematic focusing on the supply chain between an assembly manufacturer and a parts supplier is depicted in Figure 5. We propose a complex supply-chain model and a stochastic field (referred to as a production field) in the production process (refer to Figure 6). With respect to Figure 6, $S(t, x)$ represents the production density, $S(t-L)$ represents the production density at $x = 0$; i.e., it represents the order amount prior to the beginning of production time. In this situation, time delay is the interval between the time a manufacturer receives an order and the time production begins. A production process can include designing a product, ordering the necessary materials, actual production, and shipping the finished product to the customer. This series of operations involves a corresponding series of inherent time delays. We analyze the process stochastically on the assumption that it is not necessarily a flow of deterministic information. Rather, we assume that the operation flow is a supply chain [4].

We propose that the production processes, from product completion to customer delivery, are similar to that of a continuous-time model of thermal diffusion in physics. From this, we obtain the following.

To produce a product, the target company will design a product, order materials from a parts supplier, initiate production, and then ship the product to a customer. A series of such operations causes an inherent time delay. We stochastically analyze under the

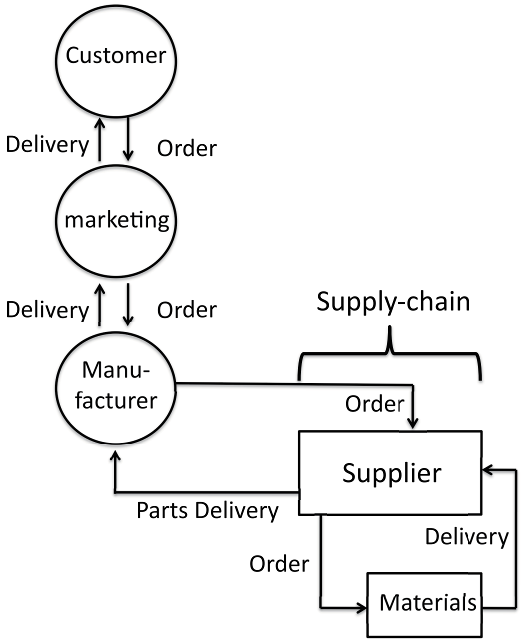

FIGURE 5. Supply chain management area

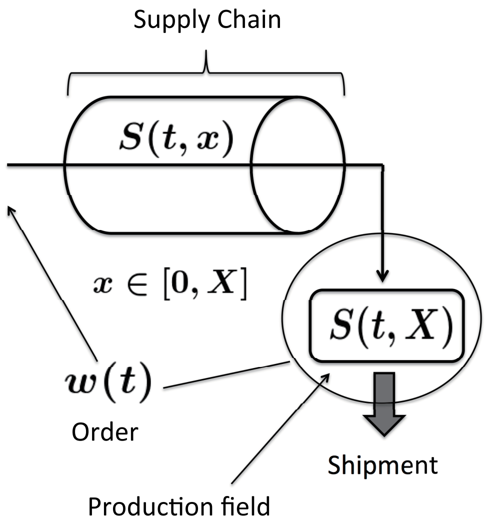

FIGURE 6. Supply chain and production field

assumption that it is not necessarily a flow of deterministic information. This flow of a series of operations is simply a supply chain [4].

3. **Optimal Control of Production Process with Lead Time Lag.** We derive the diffusion structure model of one-dimensional equation as follows:

$$\frac{\partial S(t,x)}{\partial t} = \mathcal{L}_{t,x} S(t,x) + \eta(t, \xi(t,L)) \tag{1}$$

where $S(t,x)$ and $\eta(t,\xi(t,L))$ denote a production density and a functional with delay respectively. $\mathcal{L}_{t,x}$ is derived as follows.

$$\mathcal{L}_{t,x} \equiv -\Gamma \frac{\partial}{\partial x} + D \frac{\partial^2}{\partial x^2} \tag{2}$$

where D and Γ represent parameters.

The state equation at the shipping point $x = X$ is as follows.

$$\frac{dS(t)}{dt} = f(t, \xi(t,L), u(t)) + G_1(t) d_t \tag{3}$$

where we assume that $S(t)$ and $\xi(t,s)$ satisfy the following condition.

Assumption 3.1.
1) $S(t)$and $\xi(t,s)$ are $\mathcal{B}_t$-measurable in each $t \in [0,T]$ and $s \in [0,L]$.
2) Existence of expected value $E[\bullet]$

$$E\left\{|S(t)|^2\right\} < \infty, \quad E\left\{|\xi(t,s)|^2\right\} < \infty \tag{4}$$

3) $S(t)$, $\xi(t,s)$ can be stochastically integrated and satisfy the boundary condition Equation (8) and initial condition Equation (9).
4) $S(t)$ is square-mean continuous in any $t \in [0,T]$. $\xi(t,s)$ is square-mean continuous in any $(t,s) \in [0,T] \times [0,X]$.
5) $\{S(t), \xi(t,s)\}$ exists with probability.
6) $\{S(t), \xi(t,s)\}$ follows the Markov process.

3.1. **Memory effect.**

$$\partial\xi(t,s) + g(t,s)\frac{\partial\xi(t,s)}{\partial s} = h(t,s,\xi(t,s),v(t,s)) + G_1(t,S)r(t) \tag{5}$$

where $0 \le t \le T$, $0 \le s \le L$ and the variables of Equations (1) and (5) are described as follows.

- $S(t,x)$: Production density related to two variables (t,x)
- $S(t)$: Shipping side state function
- $u(t)$: Controlling amount
- $v(t)$: Shipment side controlling amount
- $\xi(t,L)$: Delay amount with revenue lead time
- $g(t,s)$: Parameter
- d_t: Measurable disturbance (Depend on demand)
- $r(t)$: Shipment side disturbance
- $f(t,\bullet)$: Nonlinear operator
- $h(\bullet)$: Nonlinear operator

With respect to a linear system, $S(t)$, which is in accordance with the above description, the production process that considers delays in the arrival of the manufactured parts can be derived as follows:

$$dS(t) = [a(t)S(t) + b(t)\xi(t,L) + c(t)u(t)]dt + G_1(t)dW_t \tag{6}$$

where $b(t)$, $c(t)$ and $G_1(t)$ are parameters.

$$d\xi(t,s) = \left[-g(t,s)\frac{\partial\xi(t,s)}{\partial s} + k(t,s)\xi(t,s) + m(t,s)v(t,s)\right]dt + G_2(t,s)dW_t \tag{7}$$

where $k(t,s)$, $m(t,s)$ and $G_2(t,s)$ are parameters. The boundary condition and initial condition are as follows.

$$\xi(t,0) = L_1(t)S(t), \quad \xi(t,L) = L_2(t)S(t) \tag{8}$$

$$S(0) = S_0, \quad \xi(0,s) = \xi_0(s)(\equiv 0) \tag{9}$$

where $\xi(t,0)$ depicts an input amount according to the shipping function and $\xi(t,L)$ depicts an output quantity on variable revenue with lead-time delay.

The lead-time delay system is a distributed-parameter system corresponding to a system that is related to the revenue that fluctuates with delays such as logistics. The system is affected at the shipping point. The revenues are also affected by the production system based on the boundary conditions of the model equation [13]. Additionally, the logistics and production systems are also affected by the disturbances. Figures 1-5 show the change in the state of the shipping variable in the actual production system. Further, the probabilistic effect of demand and the revenue generated by the lead-time delay can be used to represent the stochastic characteristics.

Proposition 3.1. *The one-dimensional functional $V(t,a,\gamma)$ of the coupled system can be continuously differentiated two times with respect to $(a,\gamma) \in R \times L_2[0,L]$, and can be continuously differentiated with respect to time t. Let $\varphi(t) = [S(t),\xi(t,\bullet)]$ be the solution process of the stochastic integral equation of the coupled system, and then the following is derived as $0 \le t_1 < t_2 \le T$.*

$$\begin{aligned} &E\{V(t_2,S(x_2),\xi(t_2,\bullet))|S(x_1),\xi(t_1,\bullet)\} - V(t_1,S(x_1),\xi(t_1,\bullet)) \\ &= E\left\{\int_{t_1}^{t_2}\left[\frac{\partial V(\tau,S(\tau),\xi(\tau,\bullet))}{\partial\tau} + \mathcal{L}V(\tau,S(\tau),\xi(\tau,\bullet))\right]d\tau|S(t_1),\xi(t_1,\bullet)\right\} \end{aligned} \tag{10}$$

where $\mathcal{L}$ denotes the following operator.

$$\begin{aligned}
&\mathcal{L}V(t,S,\xi)\\
&= \frac{\partial V(t,S,\xi)}{\partial S} f(t,S,\xi_1,\xi_2,\xi) + \int_0^L \left[-g(t,s)\frac{\partial \xi}{\partial s} + k(t,s)\xi(t,s,\xi(\bullet,s),S,\xi) \right] ds \\
&\quad + \frac{1}{2}\left[G_1(t)\left[\frac{\partial^2 V(t,S,\xi)}{\partial S^2} G_1(t) + 2\int_0^L G_2(t,s)\frac{\partial^2 V(t,S,\xi)}{\partial \xi \partial S} G_1(t) ds \right] \right.\\
&\quad \left. + \int_0^L G_2(t,s)\frac{\partial}{\partial \xi}\int_0^L \frac{\partial V(t,S,\xi)}{\partial \xi} G_2(t,s')ds'ds \right] \qquad (11)
\end{aligned}$$

We prove Proposition 3.1 by using Taylor expansion as follows [13].

$$\begin{aligned}
&dV(t,S(t),\xi(t,\bullet))\\
&= V(t+dt, S(t+dt), \xi(t+dt,\bullet)) - V(t,S(t),\xi(t,\bullet))\\
&= \frac{\partial V}{\partial t}\cdot dt + \left(dS, \frac{\partial V}{\partial S}\right)_R + \left(d_t\xi, \frac{\partial V}{\partial \xi}\right)_{L2} + \frac{1}{2}\left(dS, \frac{\partial}{\partial S}\left(\frac{\partial V}{\partial t}\right)\right)_R\\
&\quad + \frac{1}{2}\left(d_t\xi, \frac{\partial}{\partial \xi}\left(\frac{\partial V}{\partial t}\right)\right)_{L2} dt + \frac{1}{2}\left(dS, \frac{\partial}{\partial S}\left(dS, \left(\frac{\partial V}{\partial S}\right)_R\right)_R\right)\\
&\quad + \frac{1}{2}\left(d_t,\xi, \frac{\partial}{\partial \xi}\left(dS, \frac{\partial V}{\partial S}\right)_R\right)_{L2[0,L]} + \frac{1}{2}\left(dS, \frac{\partial}{\partial S}\left(d_t\xi, \frac{\partial V}{\partial \xi}\right)_{L2[0,L]}\right)_R\\
&\quad + \frac{1}{2}\left(d_t\xi, \frac{\partial}{\partial \xi}\left(d_t\xi, \frac{\partial V}{\partial \xi}\right)_{L2[0,L]}\right)_R + \cdots \qquad (12)
\end{aligned}$$

where let $dw \cdot dt \cong (dt)^{3/2}$, $(dw)^2 \cong dt$. Leaving the term corresponding to $(dt)^{1/2}$ and dt becomes the following equation.

$$\begin{aligned}
&dV(t,S(t),\xi(t,\bullet))\\
&= \frac{\partial V}{\partial t}\cdot dt + \frac{\partial V}{\partial S}(f\cdot dt + G_1\cdot dw) + \int_0^L \frac{\partial V}{\partial \xi}\left(-g\frac{\partial \xi}{\partial s}dt + k\cdot dt + G_2\cdot dw\right)ds\\
&\quad + \frac{dt}{2}\cdot G_1\frac{\partial^2 V}{\partial S^2}G_1 + 2\int_0^L G_2\frac{\partial^2 V}{\partial \xi \partial S}G_1 ds + \int_0^L G_2\frac{\partial}{\partial \xi}\int_0^L \frac{\partial V}{\partial \xi}G_2 ds ds' \qquad (13)
\end{aligned}$$

Equation (14) is obtained by performing integral conversion on Equation (16) and performing expected value calculation $E\{\bullet|S(t_1),\xi(t_1,\bullet)\}$ (Proof finished) [13].

Therefore, the mathematical model of production system including lead time delay is derived as follows.

$$dS(t) = (\mu S(t) + b\xi(t,L) + cu(t))dt + \sigma dW_t \qquad (14)$$

$$d\xi(t,s) = \left[-g\frac{\partial \xi(t,s)}{\partial s} + k(t,s)\xi(t,s) + m(t,s)v(t,s)\right]dt + G_2(t,s)dW_t \qquad (15)$$

where μ and σ are an average and volatility respectively. Let $c(t) = c$ (constant).

The boundary condition and initial condition are Equations (8) and (9).

Definition 3.1. *System evaluation function*

$$\begin{aligned}
J = E\Bigg[\Bigg\{&\int_0^T Q_1(t)S^2(t) + R_1(t)u^2(t) + \int_0^L\int_0^L Q_2(t,s)\xi^2(t,s)ds\\
&+ \int_0^L R_2(t,s)v^2(t,s)ds\Bigg\}dt\Bigg] \qquad (16)
\end{aligned}$$

The optimal control of Equation (15) is derived as follows [12].

$$u_{opt}(t) = -\,R_1^{-1}(t)c\left[L_1(t)S(t) + \int_0^L L_2(t,\alpha)\xi(t,\alpha)d\alpha\right] \tag{17}$$

$$v_{opt}(t,s) = -\,R_2^{-1}(t,s)m(t,s)\left[L_2(t,s)S(t) + \int_0^L L_3(t,s,\sigma)\xi(t,\sigma)d\sigma\right] \tag{18}$$

Thus, the linear feedback at time t is given by the state functional $\varphi[S(t),\xi(t,\bullet)]$. $L_1(t)$, $L_2(t,s)$ and $L_3(t,s,s')$ are the solution of Riccati type equation [13].

$$\begin{aligned} 0 = {} & \frac{dL_1(t)}{dt} + 2\mu L_1(t) + Q_1 - L_1(t)c(t)R_1^{-1}c(t)L_1(t) \\ & - \int_0^L L_2(t,\sigma)m(t,\sigma)R_2^{-1}(t,\sigma)m(t,\sigma)L_2(t,\sigma)d\sigma \end{aligned} \tag{19}$$

$$\begin{aligned} 0 = {} & \frac{\partial L_2(t,s)}{\partial t} + \frac{\partial L_2(t,s)g(t,s)}{\partial s} + L_2(t,s)k(t,s) + \mu L_2(t,s) \\ & - L_1(t)c(t)R_1^{-1}c(t)L_2(t,s) - \int_0^L L_2(t,\sigma)m(t,\sigma)R_2^{-1}(t,\sigma)m(t,\sigma)L_3(t,\sigma,s)d\sigma \end{aligned} \tag{20}$$

$$\begin{aligned} 0 = {} & \frac{\partial L_3(t,s,s')}{\partial t} + \frac{\partial L_3(t,s,s')g(t,s)}{\partial s} + \frac{\partial L_3(t,s,s')g(t,s')}{\partial s'} \\ & + k(t,s)L_3(t,s,s') + L_3(t,s,s')k(t,s') - L_2(t,s)c(t)R_1^{-1}c(t)L_2(t,s') \\ & - \int_0^L L_3(t,\sigma,s)m(t,\sigma)R_2^{-1}(t,\sigma)m(t,\sigma)L_3(t,\sigma,s')d\sigma \end{aligned} \tag{21}$$

where the boundary condition and initial condition are derived as follows.

$$L_1(t)b_1(t) = 0, \quad L_2(t,s)b_1(t) = 0 \tag{22}$$

$$L_1(T) = 0, \quad L_2(T,s) = 0, \quad L_3(T,s,s') = 0 \tag{23}$$

$$0 \le s \le L, \quad s \ge 0, \quad s' \le L \tag{24}$$

The optimal control is as follows in case of Q_1, R_1, Q_2, g, m and k are all constant data.

$$u_{opt}(t) = -R_1^{-1}c\left[L_1(t)S(t) + \int_0^L L_2(t,0)\xi(t,s)ds\right] \tag{25}$$

$$v_{opt}(t) = -R_2^{-1}m\left[L_2(t,s)S(t) + \int_0^L L_3(t,s,s')\xi(t,s')ds'\right] \tag{26}$$

Here, Ricotti type differential equations are as follows [13].

$$0 = \frac{dL_1(t)}{dt} + 2\mu L_1(t) + Q_1 - cR_1L_1^2(t) - \int_0^L m^2R_2^{-1}L_2^2(t,s)ds \tag{27}$$

$$\begin{aligned} 0 = {} & \frac{\partial L_2(t,s)}{\partial t} + g\frac{\partial L_2(t,s)}{\partial s} + k\cdot L_2(t,s) + \mu L_2(t,s) - c^2R_1^{-1}L_1 + L_2(t,s) \\ & - \int_0^L m^2R_2^{-1}L_2^2(t,s)ds \end{aligned} \tag{28}$$

$$\begin{aligned} 0 = {} & \frac{\partial L_3(t,s,s')}{\partial t} + g\frac{\partial L_3(t,s,s')}{\partial s} + g\frac{\partial L_3(t,s,s')}{\partial s'} + k\cdot L_3(t,s,s') + \mu L_3(t,s,s') \\ & - c^2R_1^{-1}L_1 + L_2 - \int_0^L m^2R_2^{-1}L_2^2(t,s)ds \end{aligned} \tag{29}$$

In other words, the optimal controls, $u_{opt}(t)$ and $v_{opt}(t, s)$, exist in a form that considers the memory effect accompanying the lead-time delay. Nishihira describes the control, which is a time-delay system for inventory management. In the present study, the orders of stock quantity K times depend on the feedback that is obtained based on only the information at time t at the current time. However, the optimal design of K is a major issue. Therefore, in general, Masuyama's control method can be assumed to be strictly optimal. However, solving simultaneous Riccati-type partial differential equations is a major issue.

4. **Numerical Example of Optimal Control.** We provide feedback based on only the information, $S(t)$, at time t multiplied by a factor, K, according to the magnitude of the delay in advance. Therefore, no information is required about the lead-time delay. Here, we outline the stationary problem.

We assume the following items as follows.

Assumption 4.1.

- *We need to decide K_p in advance.*
- *The system is asymptotically stable.*
- *It is possible to make a production plan that includes prediction of income delays.*

The production model on the shipping side is the same as the model with memory effect and is derived as follows (Refer Figure 7).

$$\frac{dS}{dt} = aS(t) + b\xi(t, L) + cu(t) + \sigma_s \eta_s \tag{30}$$

$$\frac{\partial \xi}{\partial t} + p\frac{\partial \xi}{\partial x} = w\left(\bar{\xi} - \xi(t, x)\right) + \sigma_\xi \eta_\xi \tag{31}$$

$$d\xi(t, L) = w\left(\bar{\xi} - \xi(t, L)\right) dt + \sigma dW_t \tag{32}$$

Equation (30) depicts the process model, Equation (31) depicts the lead-time lag model, whereas Equation (32) depicts the memory-less (ML) model. Regarding Equation (32), we use the estimated data, $\bar{\xi}(t, L)$, instead of the observed data, $\xi(t, L)$. We use the premium equation, which is an empirical equation, to determine the premium value, K_p, which can be derived as follows:

$$K_p = \frac{\kappa_p}{\xi_p}\left\{(R + k) + \xi_f \cdot \bar{\xi}\right\} \tag{33}$$

where κ_p, ξ_p, R, k and ξ_f are parameters. $\bar{\xi}$ denotes the estimated data of $S(t)$.

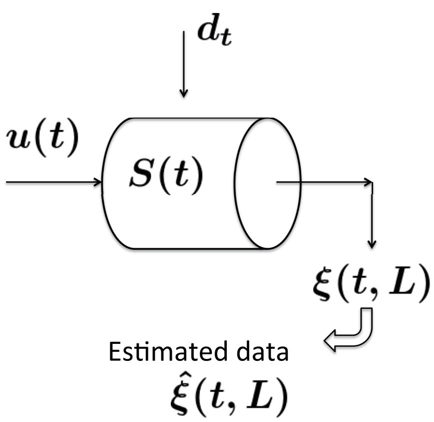

FIGURE 7. Estimation of production processes

The optimal control is derived as follows.

$$u_{opt}(t) \equiv -rK_p q^{-1} cL^* S(t), \quad r >> 1 \tag{34}$$

where r ($>> 1$) is called a premium correction coefficient. The value of r is needed to be chosen appropriately.

Riccati type equation is derived as follows [13].

$$\frac{dL}{dt} + 2aL - ac^2L^2 - P^2 = 0 \tag{35}$$

L^* is derived from Equation (35).

$$J = E\left\{\int \left(pS^2 + qu^2\right) dt + \int \xi^2(t, L)dt\right\} \tag{36}$$

4.1. **Comparison of strict control considering lead time lag and control with memoryless.**

- Figures 8 and 9 show the optimal feedback function and response considering lead time lag.
- Figures 10 and 11 show the optimal feedback function and response with memoryless.

With respect to the parameter settings, $a = 0.73$, $c = 0.1$, $p = 0.3$, $q = 1$, $B = 0.3$, $(R + k) = 0.6$, $\kappa = 0.8$, $\xi_f = 0.6$, $\bar{\xi} = 0.4$, $\sigma = \sigma_s = \sigma_\xi = 0.2$, $w = 0.5$, $\eta_s = \eta_\xi = 1.0$, the evaluation function value $J = 7.1025$ and the premium value $K_p \approx 0.629$. With respect to L^*, $L^* = 0.0272$ in case of memoryless, then $K_{opt} = q^{-1}cL^* = 0.00272$. On the other hand, $L^*(T) = 0.0342$ in case of lead time lag included, then $K_{opt} = q^{-1}cL^* = 0.00342$. Then, the evaluation value $J \approx 3.85073$ in case of memoryless and $J \approx 3.84951$.

4.2. **Comparison of test runs 1 and 2/3.** The production throughput is evaluated using the number of equipment pieces in comparison with the target number of equipment pieces (production ranking) and that observed in the simulation of asynchronous and synchronous production (see Appendix A). The asynchronous method is prone to worker fluctuations that are caused due to various delays, whereas worker fluctuations in the

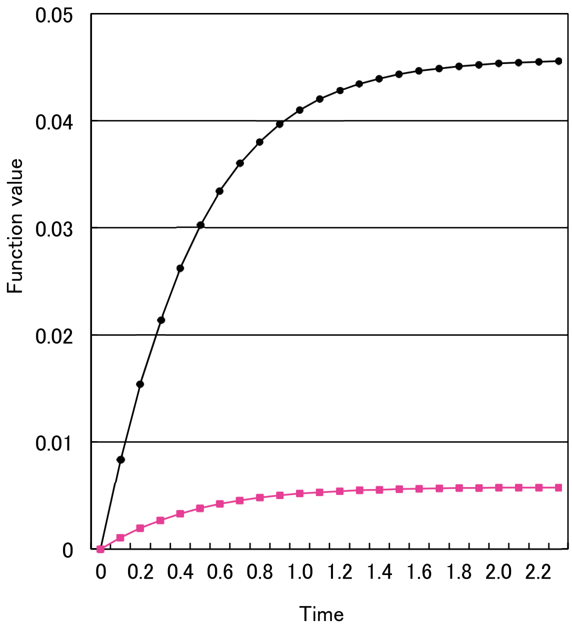

FIGURE 8. Calculation of optimal feedback function

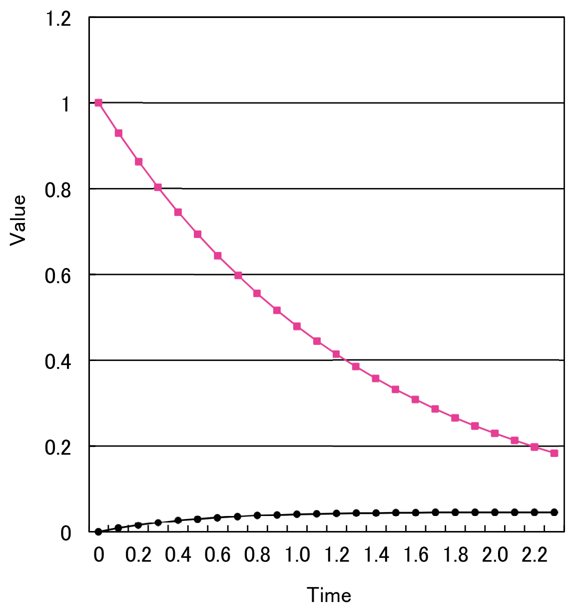

FIGURE 9. Response of optimal feedback system

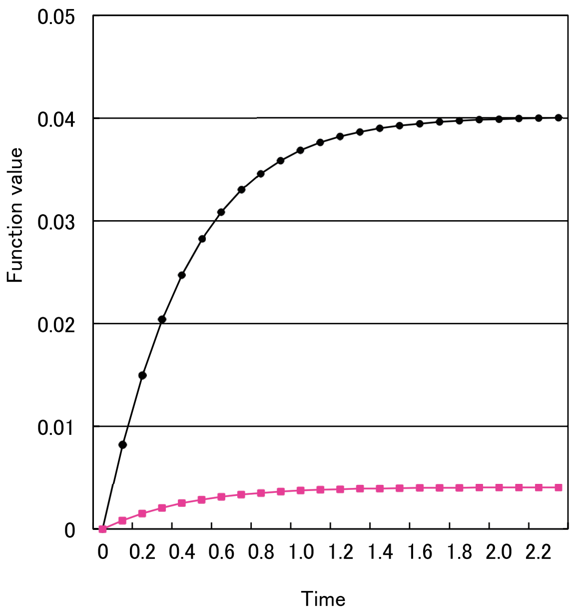

FIGURE 10. Calculation of optimal feedback function

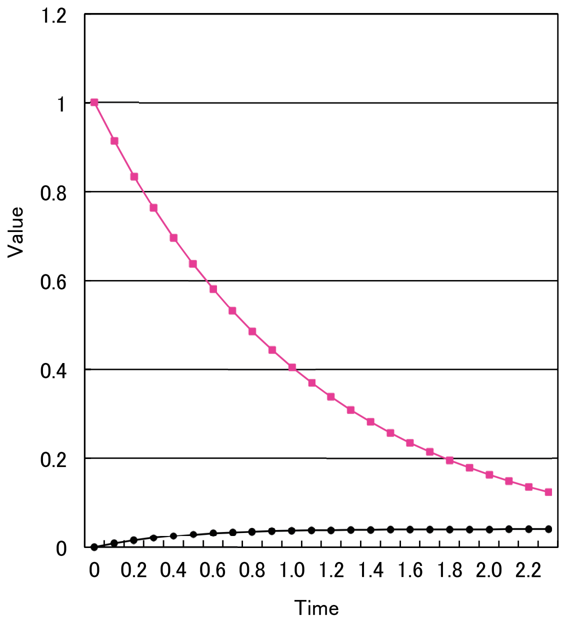

FIGURE 11. Response of optimal feedback system

synchronous method are observed to be minor. Based on the production lead times presented in Appendix A, the productivity ranking tests indicate that test run 3 > test run 2 > test run 1, where test run 1 is asynchronous and test runs 2 and 3 are synchronous. Here, the throughput values that are calculated using the throughput probability obtained from test runs 1-3 are as follows:

- Test run 1: 4.4 (pieces of equipment)/6 (pieces of equipment) = 0.73
- Test run 2: 5.5 (pieces of equipment)/6 (pieces of equipment) = 0.92
- Test run 3: 5.7 (pieces of equipment)/6 (pieces of equipment) = 0.95

Regarding test runs 1-3, the comparisons of test runs 1 and 2/3 in the case of ML are listed below. The actual data of test runs 1-3 are given in Appendix A.

- Test run 1, the performance value = 7.1102 in case of memoryless
- Test run 2/3, the performance value = 5.9714 in case of memoryless

In the case of ML, the premium value can be calculated as follows. To perform parameter selection, it is necessary to consider various factors such as the compensation that is inherent to the system and correction of the estimated value; however, the premium value is determined mainly by κ. Here, κ is based on the throughput values of test runs 1 (= 0.73) and 2 (= 0.92).

- Test run 1, the premium value = 0.5966 ($\kappa = 0.73$)
- Test run 2, the premium value = 0.7854 ($\kappa = 0.92$)

As described above, we compared the ML and lead-time delay using a simple example. The premium value was calculated using the value that was estimated on the shipping side.

4.3. **Dynamic simulation of production processes.** We attempted to perform a dynamic simulation of the production process by utilizing the simulation system that NTT DATA Mathematical Systems Inc. (www.msi.co.jp) has developed. With respect to the meaning of the individual parts in Figure 12, we conducted a simulation of the following procedure. For more information, please refer our previous study [16].

- When the simulation began, it generated one of the products on a “generate” parts go to “finish”.
- In each process, including the six workers in parallel, the slowest worker waited till the work was completed.
- When the work of each process was completed, it moved to the next process.
- Simultaneously as each process was completed, it recorded the working time of each process.

With respect to Table 1 and Table 2,

- Process No. indicates each process (1-6).
- Average indicates the average time.
- STD indicates the standard deviation of process time (sec).
- Worker efficiency (WE) indicates the efficiency of six workers.

“record” calculates the worker’s operating time, which is obtained by multiplying the specified WE data for the log-normally distributed random numbers in Table 1.

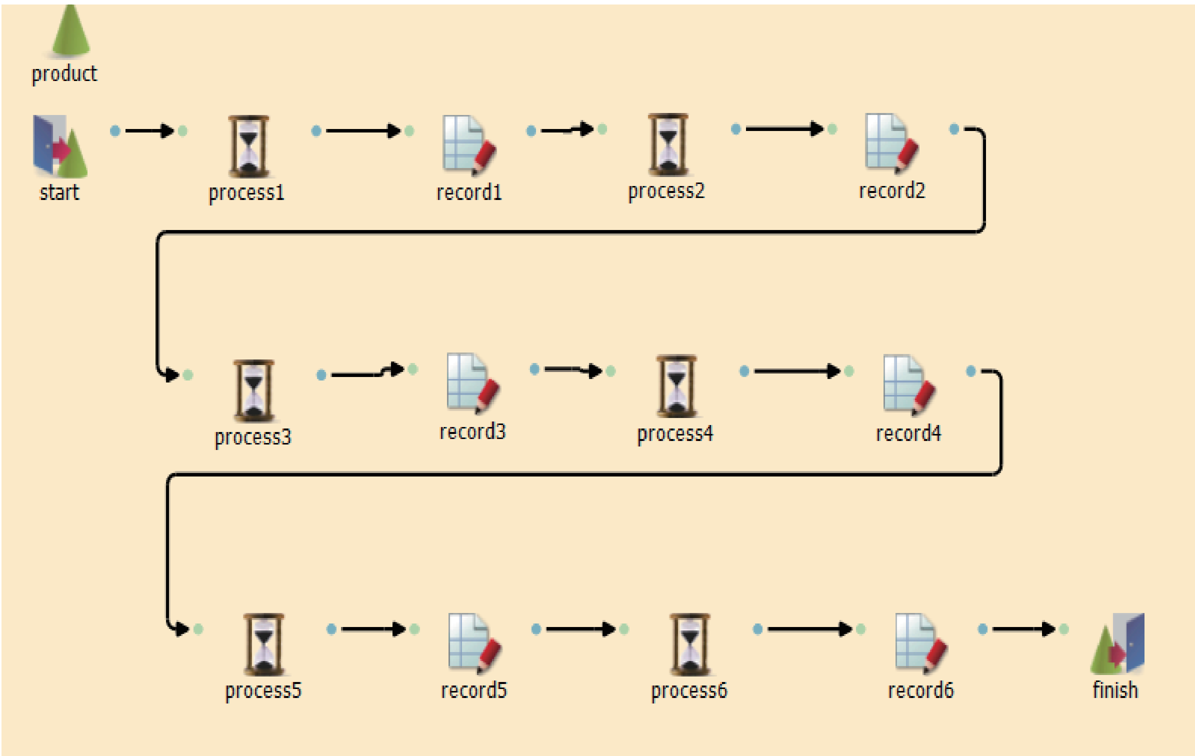

FIGURE 12. Simulation model of production flow system

Figure 13 shows the operating time of processes 1-6 (record1-record6). As the working time of the synchronous process is less volatile, the work efficiency became higher than the asynchronous process. In Figure 13, the total working time of asynchronous and synchronous processes are 1241.7(sec) and 586.4(sec) respectively. The synchronous process shows more better production efficiency than the asynchronous process.

5. **Conclusion.** The control system was constructed by using the product of the correction coefficient as a feedback coefficient. Consequently, an improvement in the evaluation function was observed in the ML case. The evaluation can be understood using the ML feedback coefficient that was proposed by Nishihira. The constriction of the control system easily is meaningful to make mathematically. However, applying it to a distributed-parameter system is a future task.

TABLE 1. Working data for six production asynchronous processes

Process No.	No.1	No.2	No.3	No.4	No.5	No.6
Average	20	22	25	22	25	21
STD	2.1	2.5	1.6	1.9	2.0	1.9
WE 1	0.83	1.0	0.66	0.76	0.88	0.91
WE 2	1.27	1.26	1.21	1.31	1.17	1.20
WE 3	0.96	1.11	1.01	1.12	0.88	0.89
WE 4	0.92	0.96	1.06	0.98	0.91	0.9
WE 5	1.2	1.03	1.07	0.89	1.03	1.1
WE 6	1.09	1.1	1.2	0.98	1.13	0.89

TABLE 2. Working data for six production synchronous processes

Process No.	No.1	No.2	No.3	No.4	No.5	No.6
Average	20	20	20	20	20	20
STD	1.1	1.5	1.2	1.4	1.0	1.4
WE 1	1.0	1.0	1.0	1.0	1.0	1.0
WE 2	1.0	1.0	1.2	1.3	1.1	1.2
WE 3	1.7	1.1	1.0	1.1	1.0	1.0
WE 4	1.0	1.0	1.0	1.0	1.0	1.0
WE 5	1.0	1.0	1.0	1.0	1.0	1.0
WE 6	1.0	1.3	1.2	1.0	1.1	1.0

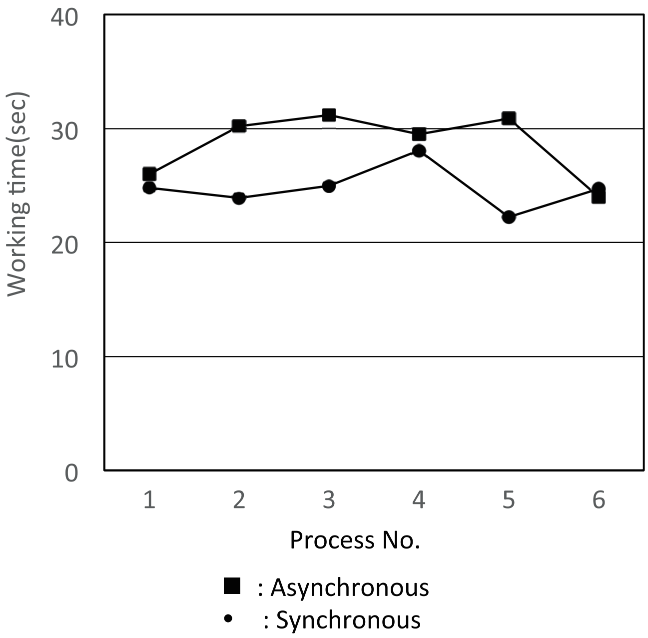

FIGURE 13. Working time for process number one through six

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Appendix A. **Analysis of Actual Data in the Production Flow System.** Figure 2 represents a manufacturing process called a production flow system, which is a manufacturing method employed in the production of control equipment. The flow production system, which in this case has six stages, is commercialized by the production of material in steps S1-S6 of the manufacturing process.

The direction of the arrow represents the direction of the production flow. In this system, production materials are supplied from the inlet and the end product will be shipped from the outlet.

Assumption A.1. *The production structure is nonlinear.*

Assumption A.2. *The production structure is a closed structure; that is, the production is driven by a cyclic system (production flow system).*

Assumption A.1 indicates that the determination of the production structure is considered a major factor, which includes the generation value of production or the throughput generation structure in a stochastic manufacturing process (hereafter called the manufacturing field). Because such a structure is at least dependent on the demand, it is considered to have a nonlinear structure.

Because the value of such a product depends on the throughput, its production structure is nonlinear. Therefore, Assumption A.1 reflects the realistic production structure and is somewhat valid. Assumption A.2 is completed in each step and flows from the next step until stage S6 is completed. Assumption A.2 is reasonable because new production starts from S1.

Based on the control equipment, the product can be manufactured in one cycle. The production throughput required to maintain 6 pieces of equipment/day is as follows:

$$\frac{(60 \times 8 - 28)}{3} \times \frac{1}{6} \simeq 25 \text{ (min)} \tag{37}$$

where the throughput of the previous process is set as 20 (min). In Equation (37), “8” represents the throughput of the previous process plus the idle time for synchronization. “8” is the number of processes and the total number of all processes is “8” plus the previous process. “60” is given by 20 (min) $\times$ 3 (cycles).

One process throughput (20 min) in full synchronization is

$$T_s = 3 \times 120 + 40 = 400 \text{ (min)} \tag{38}$$

Therefore, a throughput reduction of about 10% can be achieved. However, the time between processes involves some asynchronous idle time.

As a result, the above test-run is as follows.

- (test run 1) Each throughput in every process (S1-S6) is asynchronous, and its process throughput is asynchronous. Table 3 represents the manufacturing time (min) in each process. Table 4 represents the variance in each process performed by workers. Table 3 represents the target time, and the theoretical throughput is given by $3 \times 199 + 2 \times 15 = 627$ (min).

  In addition, the total working time in stage S3 is 199 (min), which causes a bottleneck. Figure 14 shows a graph illustrating the measurement data in Table 3, and it represents the total working time for each worker (K1-K9). The graph in Figure 15 represents the variance data for each working time in Table 3.
- (test run 2) Set to synchronously process the throughput.

  The target time in Table 5 is 500 (min), and the theoretical throughput (not including the synchronized idle time) is 400 (min). Table 6 represents the variance data of each working process (S1-S6) for each worker (K1-K9).
- (test run 3) The process throughput is performed synchronously with the reclassification of the process. The theoretical throughput (not including the synchronized idle time) is 400 (min) in Table 7.

  Table 8 represents the variance data of Table 7. “WS” in the measurement tables represents the standard working time. This is an empirical value obtained from long-term experiments.

Table 3. Total manufacturing time at each stage for each worker

	WS	S1	S2	S3	S4	S5	S6
K1	15	20	20	25	20	20	20
K2	20	22	21	22	21	19	20
K3	10	20	26	25	22	22	26
K4	20	17	15	19	18	16	18
K5	15	15	20	18	16	15	15
K6	15	15	15	15	15	15	15
K7	15	20	20	30	20	21	20
K8	20	29	33	30	29	32	33
K9	15	14	14	15	14	14	14
Total	145	172	184	199	175	174	181

Table 4. Volatility of Table 3

K1	1.67	1.67	3.33	1.67	1.67	1.67
K2	2.33	2	2.33	2	1.33	1.67
K3	1.67	3.67	3.33	2.33	2.33	3.67
K4	0.67	0	1.33	1	0.33	1
K5	0	1.67	1	0.33	0	0
K6	0	0	0	0	0	0
K7	1.67	1.67	5	1.67	2	1.67
K8	4.67	6	5	4.67	5.67	6
K9	0.33	0.33	0	0.33	0.33	0.33

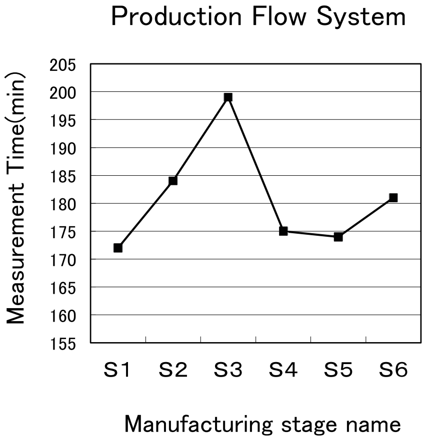

FIGURE 14. Total work time for each stage (S1-S6) in Table 3

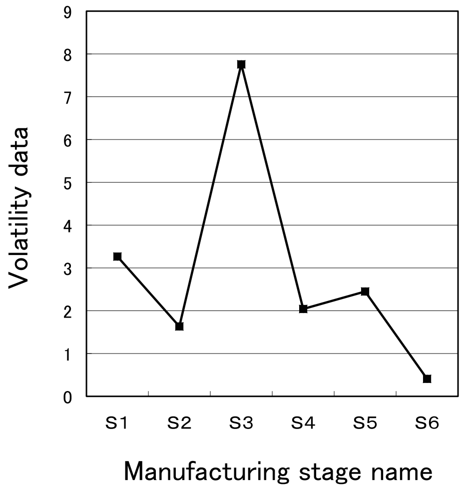

FIGURE 15. Volatility data for each stage (S1-S6) in Table 3

TABLE 5. Total manufacturing time at each stage for each worker

	WS	S1	S2	S3	S4	S5	S6
K1	20	20	24	20	20	20	20
K2	20	20	20	20	20	22	20
K3	20	20	20	20	20	20	20
K4	20	25	25	20	20	20	20
K5	20	20	20	20	20	20	20
K6	20	20	20	20	20	20	20
K7	20	20	20	20	20	20	20
K8	20	27	27	22	23	20	20
K9	20	20	20	20	20	20	20
Total	180	192	196	182	183	182	180

TABLE 6. Volatility of Table 5

K1	0	1.33	0	0	0	0
K2	0	0	0	0	0.67	0
K3	0	0	0	0	0	0
K4	1.67	1.67	0	0	0	0
K5	0	0	0	0	0	0
K6	0	0	0	0	0	0
K7	0	0	0	0	0	0
K8	2.33	2.33	0.67	1	0	0
K9	0	0	0	0	0	0

TABLE 7. Total manufacturing time at each stage for each worker

	WS	S1	S2	S3	S4	S5	S6
K1	20	18	19	18	20	20	20
K2	20	18	18	18	20	20	20
K3	20	21	21	21	20	20	20
K4	20	13	11	11	20	20	20
K5	20	16	16	17	20	20	20
K6	20	18	18	18	20	20	20
K7	20	14	14	13	20	20	20
K8	20	22	22	20	20	20	20
K9	20	25	25	25	20	20	20
Total	180	165	164	161	180	180	180

TABLE 8. Variance of Table 7

K1	0.67	0.33	0.67	0	0	0
K2	0.67	0.67	0.67	0	0	0
K3	0.33	0.33	0.33	0	0	0
K4	2.33	3	3	0	0	0
K5	1.33	1.33	1	0	0	0
K6	0.67	0.67	0.67	0	0	0
K7	2	2	2.33	0	0	0
K8	0.67	0.67	0	0	0	0
K9	1.67	1.67	1.67	0	0	0