Mathematical Model of Plant Leaf Area Growth

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Abstract

In this paper, we introduce a mathematical model of plant growth, from the seedling to shipping of the adult plant. Our model has three parts. The first analyzes plant growth, the second models the fertilizer supply system, and the third identifies an optimal control strategy by matching the control system to the growth density. The mathematical model was defined using bilinear partial differential equations, but these were replaced by a strictly linear model.

Keyword: plant leaf, bilinear partial differential equations, strict linear model, optimal control

1 Introduction

Previous studies have attempted to develop a mathematical model of plant leaf growth[1, 2]. In particular, attention has focused on the relationship between growth and light intensity. In the field of physiological ecology, "growth analysis" involves several different measures, including the relative growth rate (RGR) and leaf area ratio (LAR). In a series of classical studies, Blackman et al. investigated the relationship between light intensity and plant growth and established relationships between the net assimilation rate (NAR), LAR, and RGR, and light intensity[3]. The essence of growth analysis is to understand the factors that produce differences in growth rates. Many factors cause the differences and identifying the exact factor that has caused the difference in each individual case is challenging. Factors such as photosynthesis and respiration are often analyzed, but these may not necessarily be the main causes of the differences. Bottom-up approaches examine each factor, one at a time, to clarify the role each plays in the growth rate. Growth analysis, in contrast, is a top-down method in which the growth rate is factorized. Growth rate analysis is rather a mathematical problem, but it is also an easy-to-understand theoretical framework that intuitively understands how plants grow.

In this paper, we report our mathematical modeling of the growth process, from seedling to shipping. Our proposed model has three parts. The first models the growth period (model 1), the second models the fertilizer supply system (model 2), and the third yields an optimal control strategy, in which the growth density is replaced by a control system. When plants are cultivated under LED lamps, the optimal input control uses a bang?bang-type feedback control in each time period. By controlling the optical power during the growth stage in the growth cycle, the harvest can be maximized. A rigorous linearization model is needed in such growth models.

2 Physical model of plant growth

2.1 Leaf growth

Figure 1 shows the processes by which plants absorb nutrients and trap energy from sunlight. Figure 2 shows the growth process from seedling to shipping. The labels 1, 2, and 3 denote the inputs at seeding, the seedling growth period, and shipping of growing plants, respectively.

Kleiber's law describes the growth of animals and plants based on the observation that there is a scaling relationship between the size of a body or organ and its structure or function[5]:

$$E = k \cdot M^b \tag{2.1}$$

where, k denotes a proportionality constant and b denotes a small plants as $b = 1$, a large plant as $b = 3/4$.

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This relationship has been established for single-celled organisms, thermo-controlled animals, and homeothermic animals, with the only difference being the value of the proportionality constant k. Clearly, it also applies to the human metabolism. In the case of photosynthesizing plants, the relationship between mass, unit density, and leaf area is given as follows:

$$\rho \times V_s = M_s \tag{2.2}$$

$$\rho \simeq M_s(\equiv F) \tag{2.3}$$

where, ρ, V_s, M_s and F denote a unit density, volume, a mass and a leaf area respectively, that is, the dynamic model of plant unit density is as follows.

$$\frac{\partial \rho(t,x)}{\partial t} + \frac{\partial}{\partial x}(v(t,x)\rho(t,x)) = \sigma(t,x) \tag{2.4}$$

where, $\sigma(t,x)$ denotes a mass increase rate.

Equations (2.2) - (2.4) denote a physical model of plant growth. Therefore, we try to replace this model with an engineering model.

2.2 Engineering model of plant growth

Figure 1 shows that the plant maintains physical equilibrium by absorbing nutrients and growing. We define the equilibrium of the potential is as follows:

Definition 2.1 *Equilibrium of the potential*

$$k^{'}[\phi_{N_u} - \phi_{W_a}] = P_\sigma \tag{2.5}$$

where, ϕ_{W_a}, ϕ_{N_u}, P_σ and $k^{'}$ denote a transpiration potential, a nutrient absorption potential, a growth potential and a growth coefficient respectively. ϕ_{W_a}, ϕ_{N_u} and P_σ are derived as follows[2]:

$$\phi_{W_a} = -\int_{\infty}^{x_b} f(t,x,a)dx \tag{2.6}$$

$$\phi_{N_u} = \int_{\infty}^{x_a} K_{com}(t,x)dx \tag{2.7}$$

$$P_\sigma = k^{'}[\phi_{N_u} - \phi_{W_a}] \tag{2.8}$$

where, $f(t,x,a)$ and $K_{com}(t,x)$ denote a growth function and an energy absorption function respectively.

$\xi(t,x)$ and $\theta_b(t,x)$ denotes the energy release and nutrient absorption respectively in Figure 5. The energy release between Δx is derived as follows:

$$\Delta Q = C_f \times P_f \times S_f \Delta\xi \Delta x \tag{2.9}$$

Nutrient absorption between Δx is derived as follows:

$$\Delta Q = K_p \cdot P_f \cdot P_s(\theta_b(t,x) - \xi(t,x))\Delta t \tag{2.10}$$

From Equations (2.9) and (2.10), let $\Delta t \to 0$, then we obtain as follows:

$$C_f \times P_f \times S_f \frac{d\xi}{dt} = \frac{\alpha A}{L} K_p \cdot P_f \cdot P_s(\theta_b - \xi) \tag{2.11}$$

where, α, A and L denote a fertilizer efficiency conversion coefficient, an effective reaction area between fertilizer stirring tank and growing machine and a length of cultivator respectively. Therefore, we obtain as follows[8]:

$$\frac{\partial \xi}{\partial t} + v\frac{\partial \xi}{\partial x} = \frac{d\xi}{dt} = \frac{K_p P_s \alpha A}{C_f \cdot S_f \cdot L}(\theta_b(t,x) - \xi(t,x)) \tag{2.12}$$

We obtain the following equation by modifing Equation (2.12):

$$\frac{\partial \xi}{\partial t} + v\frac{\partial \xi}{\partial x} = \frac{P(t)\alpha A}{L}(\theta_b(t,x) - \xi(t,x)) \equiv W(t)(\theta_b(t,x) - \xi(t,x)) \tag{2.13}$$

where, $P(t)$ is replaced as follows:

$$P(t) \cong \frac{K_p \cdot P_f \cdot P_s}{C_f \cdot S_f} \tag{2.14}$$

$$W(t) \equiv \frac{P(t)\alpha A}{L} \tag{2.15}$$

where, K_p, P_f, P_s, S_f and C_f denote an energy conversion coefficient, LED power density, a radiation area, a nutrient supply coefficient and a transmission efficiency respectively.

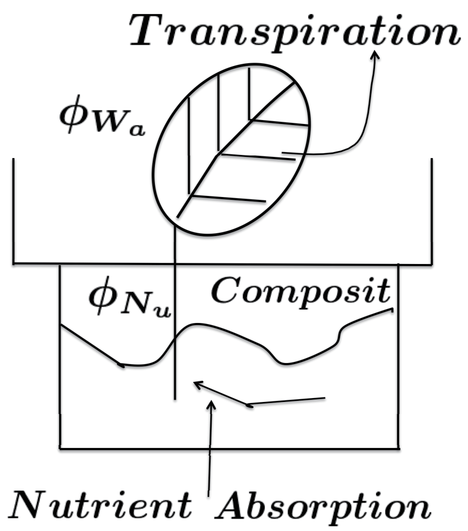


Figure 1: Leaf growth

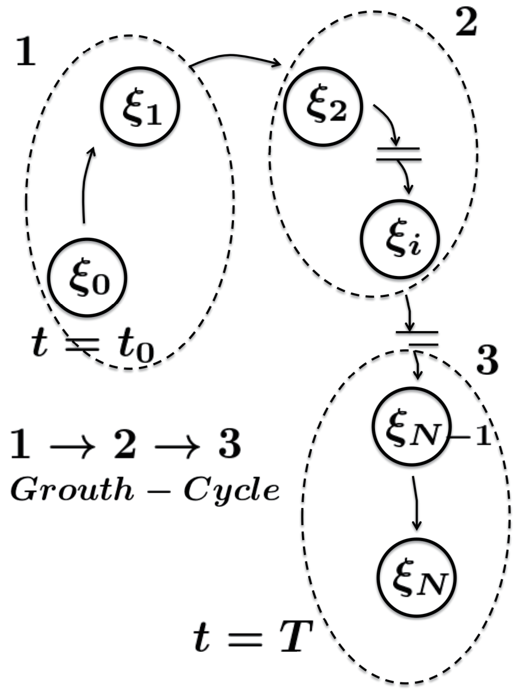


Figure 2: Growth process from seedling to shipping

3 Leaf growth period model of plant

Leaf growth period model of plant is derived as follows:

$$\frac{\partial \xi(t,\tau)}{\partial t} + \rho(t)\frac{\partial \xi(t,\tau)}{\partial \tau} - D\frac{\partial^2 \xi(t,\tau)}{\partial \tau^2} = W(t)\left\{K_{com}(t) - k_v(t)\xi(t,\tau)\right\} \tag{3.1}$$

where, $t \in (0,T)$, $\tau \in (0,L)$, $K_{com}(t) - k_v(t)\xi(t,\tau)$ denotes the exchange with fertilizer and $W(t)$ denotes the power by optical input.

$$\xi(t,0) = \xi^0(t) \tag{3.2}$$

$$\xi(0,\tau) = \xi_v(\tau) \tag{3.3}$$

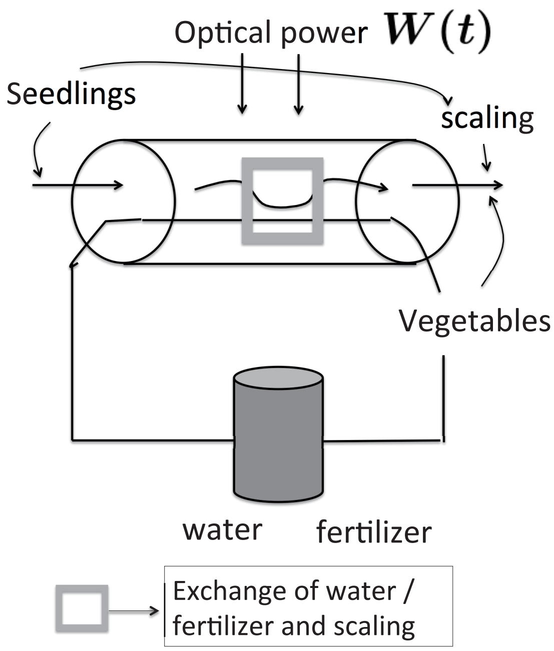


Figure 3: Growth period model

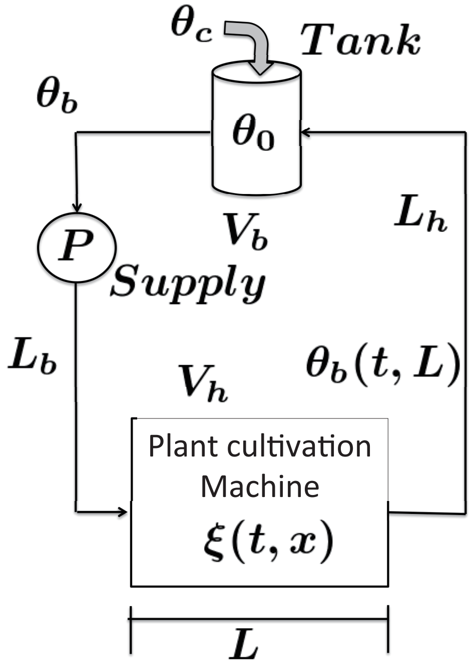


Figure 4: Plant cultivation model

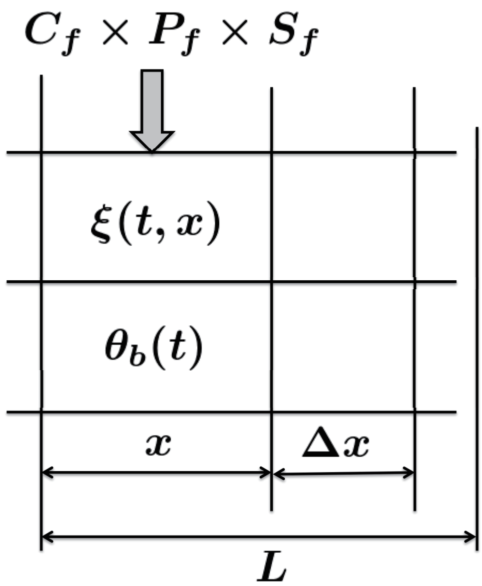


Figure 5: Energy release and nutrient absorption

Table 1: Physical meaning of each symbol

P	Supply of water and fertilizer
θ_0	Internal retention amount
θ_b	Supply amount of cultivator
θ_ξ	Return amount of cultivator
L_h	Pipe length
V_b	Tank capacity
V_h	Cultivation capacity
S	Piping cross section
q	Quantity of flow
c	Viscosity in piping
$\xi(t,x)$	Plant cultivation machine

where, Equations (3.2) and (3.3) denote the initial input (boundary condition) and the initial distribution (initial condition) respectively.

The relative growth rate (RGR) between times t_1 and t_2, the net assimilation rate (NAR: net assimilation rate) and leaf area ratio (LAR: leaf area ratio) can be drived as follows[3]:

$$RGR = \frac{1}{w} \cdot \frac{dw}{dt} \approx \frac{\ln w_2 - \ln w_1}{t_1 - t_2} \tag{3.4}$$

$$NAR = \frac{1}{u} \cdot \frac{dw}{dt} \approx \frac{\ln u_2 - \ln u_1}{u_1 - u_2} \tag{3.5}$$

$$LAR = \frac{u}{w} \approx \frac{\ln w_2 - \ln w_1}{w_2 - w_1} \tag{3.6}$$

Here w_1 and u_1 are the average individual weights at times t_1 and w_2, respectively, and u_2 is the average leaf area at time t_2. When calculating the NAR using Equation (3.5), the increase in leaf area (u) is not proportional to the weight (w) in the period from t_1 to t_2, and this produces a large error [4]. The RGR is therefore derived as follows[1]:

$$< RGR >= \frac{1}{W}\frac{dW}{dt} = \frac{d(\ln W)}{dt}[week^{-1}] \tag{3.7}$$

The Equation (3.7) is transformed as follows[2]:

$$< RGR >=< NAR > \cdot < LAR >= \left[\frac{1}{A}\frac{dW}{dt}\right] \times \left[\frac{A}{W}\right] \tag{3.8}$$

where, W and A denote a solid dry weight and a leaf dry weight respectively.

In mathematical terms, the growth of plants is an affine coupling between photosynthesis and plant form. The change in weight (growth) is a nonlinear combination of the optical power (which determines the photosynthetic capacity) and nutrient absorption. The latter reflects the nutrient potential, which depends on the morphology of the plant and the transpiration potential of the leaves. This can be expressed in strictly linearized terms as follows:

$$\frac{\partial S(t,\tau)}{\partial t} = h[K_{com}(t), k_v(t), S(t,\tau)] + f(t,\tau) \tag{3.9}$$

where, $h[K_{com}(t), k_v(t), S(t,\tau)]$ and $f(t,\tau)$ denote the nutrient absorption function and a discrete input such as

$$f(t,\tau) = \sum_j \varphi_j(\tau) f_j(t) \tag{3.10}$$

In the case of plant cultivation, let $\tau \equiv x$. Then, τ can be considered as same as a spatial movement direction x. Therefore, Equation (3.9) can be replaced as follows:

$$\begin{aligned} \frac{\partial S(t,x)}{\partial t} &= h[K_{com}(t), k_v(t), S(t,x)] + f(t,x) \\ f(t,x) &= \sum_j \varphi_j(x) f_j(t) \end{aligned} \tag{3.11}$$

4 Plant cultivation model

In Figure 4, the fertilizer supply tank model can be expressed as follows:

$$(V_b + L_b S)\rho c \frac{d\theta_b}{dt} = \{-\theta_b + \theta_\xi(t,L)\} q\rho c + (\theta_0 - \theta_c)\kappa_0 v_b \rho c + P \tag{4.1}$$

We describe Equation (4.1) for simplicity as follows:

$$\begin{aligned} \frac{d\theta_b}{dt} &= h[\theta_b(t)] + k_1 m(t) \\ k_1 m(t) &\equiv (\theta_0 - \theta_c(t))\kappa_0 v_0 \rho c' + P \end{aligned} \tag{4.2}$$

where, $t \geq 0$.

On the other hand, the plant cultivation model is derived as follows:

$$\frac{\partial \xi(t,x)}{\partial t} + q(t)\frac{\partial S(t,x)}{\partial x} = W(t)\Big\{\frac{\kappa_0}{K_{com}}\theta_b(t) - \frac{\kappa_1}{(V_b + L_b S)\rho}\xi(t,x)\Big\} + \frac{1}{K_{com}}\{\kappa_0\theta_b(t) - A\xi(t,x)\} \tag{4.3}$$

where, A and ρ denote a cultivator area and a plant unit density respectively. κ_0 and κ_1 denote a growth coefficient. K_{com} denotes a fertilizer reaction coefficient.

Equation (4.3) can be replaced for simplicity as follows:

$$\frac{\partial \xi(t,x)}{\partial t} + q(t)\frac{\partial S(t,x)}{\partial x} = K_{com}W(t)[\theta_b(t) - \xi(t,x)] \tag{4.4}$$

where, $t \in (0,T)$, $x \in (0,L)$ and $K_{com} \equiv \{V_h, L_h, S, \rho, \kappa_0, A\}$.

Since Equation (4.4) is the bilinear partial differential equation, the solution is found as follows[6, 7]:

$$\begin{aligned} \xi(t,x) = \exp\Big(-\int_0^t K_{com}W(\sigma)d\sigma\Big) \times \Big\{\xi\Big(0, x - \int_0^t q(\sigma)d\sigma\Big)\Big\} \\ + \int_0^t K_{com}W(\sigma)\theta_b(\sigma) \times \exp\Big(\int_0^\sigma K_{com}W(\sigma)d\sigma\Big)d\sigma\Big\} \end{aligned} \tag{4.5}$$

Then, Equation (4.5) can be modified as follows:

$$\frac{\partial \xi(t,x)}{\partial t} = -q(t)\Big[\frac{K_{com}W(t)}{q_0}\Big(\theta_b(t) - \xi_{h0}(x)\Big|_{x = L - \int_0^t q(\sigma)d\sigma}\Big)\Big] \tag{4.6}$$

where, $\xi_{h0}(x)$ is the solution of the following equation.

$$q_0 \frac{\partial \xi_{h0}(x)}{\partial x} = K_{com}W(t)\Big[\theta_b(t) - \xi(t,x)\Big] \tag{4.7}$$

Equation (4.7) represents the steady growth characteristic. Moreover, Equation (4.7) can be modified as follows:

$$\frac{\partial \xi(t,x)}{\partial t} = K_{com}W(t)\Big[\frac{q(t)}{q_0}\Big\{\xi_{h0}\Big(L - \int_0^t q(\sigma)d\sigma\Big)\theta_b(t)\Big\} - \xi(t,L)\Big] \tag{4.8}$$

Therefore, Equation (4.8) at $x = L$ is derived as follows:

$$\frac{\partial \xi(t,x)}{\partial t} = K_{com}W(t)\Big\{\hat{\kappa}_0(t)\theta_b(t) - \xi(t,L)\Big\} \tag{4.9}$$

where, $\hat{\kappa}_0(t)$ is derived as follows:

$$\hat{\kappa}_0(t) \equiv \left\{\frac{q(t)}{q_0}\xi_{h0}\Big(L - \int_0^t q(\sigma)d\sigma\Big)\right\} \tag{4.10}$$

where, $\hat{\kappa}_0(t)\theta_b(t)$ denotes the supply amount of fertilizer synchronized with steady growth.

Here, we assume $\hat{\kappa}_0(t)\theta_b(t)$ to a constant as follows:

Assumption 4.1

$$\hat{\kappa}_0(t)\theta_b(t) \equiv \hat{\kappa}_0\theta_b \tag{4.11}$$

Therefore, we obtain as follows:

$$\frac{\partial \xi(t,x)}{\partial t} = K_{com}W(t)\Big\{\hat{\kappa}_0\theta_b - \xi(t,L)\Big\} \tag{4.12}$$

Let $K_{com}W(t)$, which is a forcing term, be a constant (stationary irradiation) for simplicity. Then, Equation (4.12) is modified as follows:

$$\frac{\partial \xi_L(t)}{\partial t} = \theta_W\Big\{\hat{\kappa}_0\theta_b - \xi_L(t)\Big\} \tag{4.13}$$

where, $\xi(t,L) \equiv \xi_L(t)$ and $\theta_W \equiv K_{com}W(t)$.

Assuming that the growth rate is stochasitc due to surrounding influences, the following equation can be derived as follows:

$$d\xi_L(t) = \theta_W\Big(\hat{\kappa}_0\theta_b - \xi_L(t)\Big)dt + \sigma dZ(t) \tag{4.14}$$

where, $Z(t)$ denotes a Wiener process.

5 Optimal control of general bilinear distributed parameter system

The mathematical model of the first order bilinear distributed parameter system is given as follows:

$$\frac{\partial S}{\partial t} = f_0(t,x,S) + f_\alpha(t,x,S)W_\alpha(t) + f_k\frac{\partial S}{\partial x} \tag{5.1}$$

where, $(t,x) \in [t_0,t_f] \times \Omega$.

The initial and boundary conditions are as follows[6, 8]:

$$S(t_0,x) = S_0(x) \tag{5.2}$$

$$S(t,x)|_{x\in\partial\Omega} = S_b(t) \tag{5.3}$$

where, $S_0(x)$ and $S_b(t)$ denote each smooth function at $x \in \Omega$ and $(t,x) \in [t_0,t_f]$.

We define the evaluation function J to obtain the control function by the gradient method.

Definition 5.1

$$J = \int_{t_0}^{t_f} G(t,x,S(t,x)),u(t))\Big|_{x\in\Omega}dt \tag{5.4}$$

where, let $G(t,x,S(t,x)),u(t))$ be as follows:

$$G(t,x,S(t,x)),u(t)) = g(t,x,S) + g_\alpha(t,x,S)u \tag{5.5}$$

We formulate the gradient function using the maximum principle of Pontriyagin[7].

Definition 5.2 *Gradient function K*

$$K(t,x,S,S_x,P,W_\alpha) = P\{f_0(t,x,S) + f_\alpha(t,x,S)W_\alpha + f_k(t,x,S)S_k\} \tag{5.6}$$

where, $(t,x) \in [t_0,t_f] \times D$ *and* $S_x = \frac{\partial S}{\partial x}$.

P in Equation (5.6) is defined as follows:

Definition 5.3 *Function P*

$$\frac{\partial P(t,x)}{\partial t} = -\frac{\partial K(t,x,S,S_x,P,W_\alpha)}{\partial S} + \frac{\partial}{\partial x}\Big[\frac{\partial K(t,x,S,S_x,P,W_\alpha)}{\partial S}\Big] \tag{5.7}$$

where, $P(t_f,x) = 0$, $\frac{\partial P}{\partial x}\Big|_{x\in\partial D_2} = 0$ *and* $\partial D = \partial D_1 + \partial D_2$.

Then, Hamiltonian H is defined as follows:

Definition 5.4 *Hamiltonian H*

$$H = -G\Big|_{x\in\partial D_2} + \int_D K dx \tag{5.8}$$

The necessary condition for being the optimum operation amount $\bar{W}_\alpha$ is as follows.

$$H(t,x,\bar{S},\bar{S}_x,\bar{P},\bar{W}_\alpha) = \max_{W_\alpha\in\Omega} H(t,x,\bar{S},\bar{S}_x,\bar{P},\bar{W}_\alpha) \tag{5.9}$$

where, $t \in [t_0,t_f]$.

Then, the gradient function h_W as defined as follows:

Definition 5.5

$$h_W = \frac{\partial H}{\partial W_\alpha} = -g\alpha(t,x)\Big|_{x\in\partial D_2} + \int_D P(t,x) f_\alpha(t,x,S) dx \tag{5.10}$$

where, $t \in [t_1,t_2] \subset [t_0,t_f]$

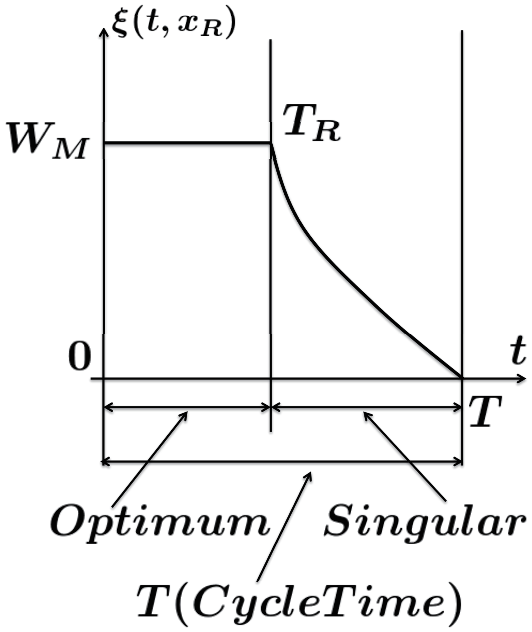
The optimal control function can be obtained from the gradient method using Equation (5.10).

As described above, in the case of cultivating plants by light power input by LED, it is understood that the optimal input for obtaining the desired harvest amount is the Bang-Bang control per period time $t \in [0,T]$, $T_R \leq T$. That is, by applying the optical power adapted to the period cycle according to the above-described control strategy, a desired harvest can be obtained in the final period.

6 Optimal control law by strict linear model approximation

We find the optimal solution using the strict linear model to make the bilinear partial differential equations easier to handle. We rewrite the leaf growth period model of plant as follows:

$$\frac{\partial \xi(t,x)}{\partial t} = W(t)\Big\{k_0\theta_b(t) - k_1\xi(t,x)\Big\} \tag{6.1}$$



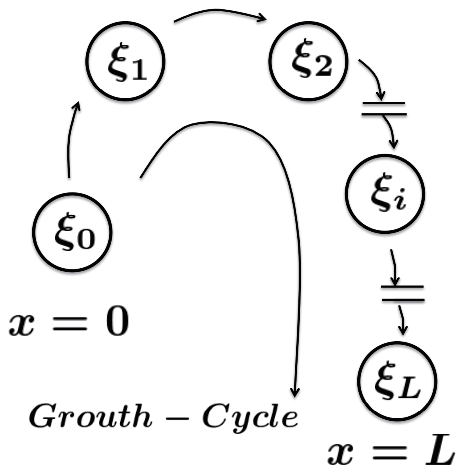


Figure 6: Optimal and Singular control

Figure 7: Continuum approximation

Equation (6.1) is derived at $x = L$ as follows:

$$\frac{d\xi_L(t)}{dt} = \frac{\hat{W}(t)A}{L(c_f\rho_f S_f)}\Big\{k_0\theta_b(t) - k_1\xi_L(t)\Big\} = \frac{1}{L(c_f\rho_f S_f)}\Big\{\xi_0(t) - k_1\xi_1(t)\Big\} \tag{6.2}$$

Equation (6.2) is transformed as follows:

$$\frac{d\xi_L(t)}{dt} = W(t,L)\Big\{k_0\theta_b(t) - k_1\xi_L(t)\Big\} + f[\xi_L(t)] = W(t,L)g\{\xi_L(t)\} + f[\xi_L(t)] \tag{6.3}$$

where, the functions $W(t,L)$, $g\{\xi_L(t)\}$ and $f\{\xi_L(t)\}$ in Equation (6.3) as follows[6, 7]:

$$W(t,L) \equiv \frac{\hat{W}(t)A}{L(c_f\rho_f S_f)} = w(t) \tag{6.4}$$

$$g\{\xi_L(t)\} = k_0\theta_b(t) - k_1\xi_L(t) \tag{6.5}$$

$$f\{\xi_L(t) = \frac{1}{L(c_f\rho_f S_f)}\Big\{\xi_0(t) - k_1\xi_1(t)\Big\} \tag{6.6}$$

From Equations (6.3) $-$ (6.6), we obtain as follows:

$$\frac{dC_L(t)}{dt} = mC_L(t) + bu(t) \tag{6.7}$$

where, $C_L(t)$, m, b, $w(t)$ and $\hat{W}(t)$ are as follows:

$$C_L(t) \equiv k_1\xi_L(t) - k_0\theta_b(t) \tag{6.8}$$

$$m = \frac{1}{L(c_f\rho_f S_f)} + \frac{A}{L(c_f\rho_f S_f)^2} \tag{6.9}$$

$$b \equiv \frac{1}{L(c_f\rho_f S_f)} \tag{6.10}$$

$$w(t) \equiv \frac{1}{\xi_0(t) - k_1\xi_L(t)}\left\{\frac{1}{L(c_f\rho_f S_f)}\{\xi_0(t) - k_1\xi_L(t)\} + u(t)\right\} \times \frac{A}{(c_f\rho_f S_f)} \tag{6.11}$$

$$\hat{W}(t) \equiv k_p P_p(t) P_s \cdot \alpha \tag{6.12}$$

where, $\xi_L(t)$ satisfy the following equation.

$$\frac{d\xi_L(t)}{dt} = \frac{\hat{W}(t)A}{L(c_f\rho_f S_f)}\left\{k_0\theta_b(t) - k_1\xi_L(t)\right\} + \frac{1}{L(c_f\rho_f S_f)}\left\{\xi_0(t) - k_1\xi_L(t)\right\} \tag{6.13}$$

From above description, we obtain as follows:

$$\frac{dC_L(t)}{dt} = mC_L(t) + bu(t) \tag{6.14}$$

Equation (6.14) denotes the exact linear model exchanged. Then, the optimal input is derived as follow:

$$u_{opt}(t) = -kC(t) \tag{6.15}$$

When the optimal input $u_{opt}(t)$ is applied, we obtain as follows:

$$\frac{dC_{L_{opt}}(t)}{dt} = (m-k)C_{L_{opt}}(t) \tag{6.16}$$

Therefore, due to the optimal input $u_{opt}(t)$ for the mathematical model of plant Leaf area growth , the mathematical model is represented by the equation (6.16).

7 Results

In this paper, mathematical modeling was applied to plant growth, from the seedling to the shipment of the adult plant. We observed that it was theoretically possible to derive an optimal control method by starting with bilinear partial differential equations in the growth period and developing a rigorous linear model. In future work, we will apply the model to real data, to test the accuracy of the model.

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