A syntactical comparison between pair sentential calculus **PSC** and Gupta's definitional calculus $\mathbf{C}_n$

Tadao Ishii*

Abstract

In this paper we will compare two logical systems **PSC** and $\mathbf{C}_n$ with a syntactical point of view. Because both notions of the pair-sentence with stage number in **PSC** and Gupta's sentence-definition with revision stage number in $\mathbf{C}_n$ are very similar, and both can deal with paradoxical sentences like a simple Liar sentence. His system was defined as a predicate calculus, but here we will introduce the propositional version of $\mathbf{C}_n$ for the comparison, and we had the following results: (1) $\mathbf{C}_0$ is a sublogic of **PSC**, or **PSC** is an extension of $\mathbf{C}_0$ under the two translations t_{C} and t_{P}. Similarly, $\mathbf{PSC}_n$ is an extension of $\mathbf{C}_n$. (2) If we extend the systems $\mathbf{C}_0$ and $\mathbf{C}_n$ by adding three properties: exchangeability, transitivity and relativity of revision indices, then two logics $\mathbf{C}_0$ and **PSC** (also $\mathbf{C}_n$ and $\mathbf{PSC}_n$) are syntactically equivalent. (3) We can calculate a cycle number of each pair sentence in **PSC**, but not in $\mathbf{C}_0$. (4) **PSC** can deal with multiple pair sentences, but difficult to deal with such multiple definitions in $\mathbf{C}_n$.

Keywords: **SCI**, pair-sentence, Liar paradox, Tarski's biconditional, revision theory.

1 Introduction

In the 1970's, R. Suszko attempted to formalize an ontology of facts in L. Wittgenstein's *Tractatus* on the basis of Fregean scheme, and called it *non-Fregean logic* [5, 6]. The sentential calculus with identity, **SCI** in short, is the most simplified version of his *non-Fregean logic* and obtained by adding the sentential identity connective $\equiv$ to the classical logic. Statements of the form $A \equiv B$ read as "A is identical with B" which means that the referent of two sentences are identical in the basis of Fregean scheme. **SCI** has the following identity axioms:

(E1) $A \equiv A$
(E2) $(A \equiv B) \to (B \equiv A)$
(E3) $(A \equiv B) \wedge (B \equiv C) \to (A \equiv C)$
(C1) $(A \equiv B) \to (\neg A \equiv \neg B)$
(C2) $(A \equiv B) \wedge (C \equiv D) \to (A \wedge C) \equiv (B \wedge D)$
(C3) $(A \equiv B) \wedge (C \equiv D) \to (A \vee C) \equiv (B \vee D)$
(C4) $(A \equiv B) \wedge (C \equiv D) \to (A \to C) \equiv (B \to D)$
(C5) $(A \equiv B) \wedge (C \equiv D) \to (A \equiv C) \equiv (B \equiv D)$
(SI) $(A \equiv B) \to (A \to B)$

Here (E1)–(E3) and (C1)–(C5) show that the identity connective $\equiv$ is an equivalence and congruence relation, respectively. From (SI) we get $A \leftrightarrow B \not\Rightarrow A \equiv B$ in general, which means **SCI** is non-Fregean logic because we can consider more than two situations (true and false) in **SCI**. Every equation in the logical theorems of **SCI** is only a trivial (i.e., $A \equiv A$). So, **SCI** is a very weak logical system. But many logical systems can be simulated on Suszko's theories of situation. For example, we define $\alpha \equiv \beta \iff \Box(\alpha \leftrightarrow \beta)$ in modal logic, then we have the

*Department of Information Systems, School of Information and Culture,
Niigata University of Information and International Studies, and visiting (2015.9 – 2016.8):
Department of Cognitive Science, Institute of Psychology, University of Łódź, Poland

correspondences $\mathbf{W}_{\mathrm{T}}$ – **S4** and $\mathbf{W}_{\mathrm{H}}$ – **S5**, where $\mathbf{W}_{\mathrm{T}}$ and $\mathbf{W}_{\mathrm{H}}$ are some elementary extensions of **SCI**. Also we define $\alpha \equiv \beta \iff (\alpha \Leftrightarrow \beta)$ in 3-valued Łukasiewicz logic where $\Leftrightarrow$ is a Ł$_3$ equivalence, then we have the correspondence **SCI** – **Ł$_3$**.

Here we consider to deal with a simple Liar sentence : "This sentence is not true" in **SCI**. Let's define A="This sentence is true", then $A \equiv \neg A$ because the referent of two sentences A and $\neg A$ are identical, but it's impossible logically by (SI). So, we can not deal with a Liar sentence in the normal **SCI**. In order to overcome the matter, we introduced a *referential relation* of pair-sentence similar to identity $\equiv$, i.e., $(A^0, \neg A^1)$: a *situation* of A at a stage 0 is *referential* to the *situation* of $\neg A$ at a stage 1 [2, 3]. And we proposed a pair sentential calculus, **PSC** in short, which was obtained from the classical sentential calculus by adding a new pair-sentence constructor $((\_)^i\ , (\_)^j)$, where i, j are some stage numbers.

As another approach to overcome the matter, A. Gupta and his colleagues studied the truth concept and paradox in the 1980's, and published a book, the title of which is "The Revision Theory of Truth" as the results so far obtained. In the book, Gupta proposed the theory of definitions which is the proper framework for the construction of a theory of truth [1]. At first as an analytical tool for the truth concept, Gupta based on *Tarski's biconditionals* for L in L' such that X is true in $L \iff p$ in L', where X is replaced by the standard name of a sentence of L and p is replaced by the translation in L' of the sentence. The language L for which the definition is constructed is called the *object language*, and the language L' in which the definition is given is called the *metalanguage*. For example, let's consider a simple Liar sentence:

(This sentence) "This sentence is not true".

Then at first we have the following identity.

(1) This sentence = "This sentence is not true",

and by the Tarski's biconditionals for this sentence, we get

(2) "This sentence is not true" is true $\iff$ This sentence is not true.

By substitutivity of (1) to (2), we get

(3) This sentence is true $\iff$ This sentence is not true,

which immediately yields a contradiction. So, the Liar paradox appears to show that the fundamental intuition is incoherent. In general, a central problem in the theory of truth is to resolve the paradox without damaging the fundamental intuition in any essential way.

Gupta viewed Tarski's biconditionals such that "A" is true $\iff A$ as procedures for determining whether a sentence A is true or not, and divided into two derivations:

$$\frac{A}{\text{"A" is true}}\ (\mathrm{T-Intro}) \qquad \frac{\text{"A" is true}}{A}\ (\mathrm{T-Elim})$$

where T-Intro and T-Elim mean T-Introduction and T-Elimination, respectively.

Similarly, the definition of a sentence A such that $A =_{df} D_A$, which may include a self-referential form, can be viewed as procedures for determining whether a sentence is A, and divided into two derivations:

$$\frac{D_A}{A}\ (\mathrm{Df-Intro}) \qquad \frac{A}{D_A}\ (\mathrm{Df-Elim})$$

Moreover, the sequence generated by each derivation was called a revision sequence for a sentence A and a revision stage number i was introduced to show the process of revision. The modified versions of (Df-Intro) and (Df-Elim) are as follows: for any integer number i,

$$\frac{{D_A}^i}{A^{i+1}}\ (\mathrm{Df-Intro}) \qquad \frac{A^i}{{D_A}^{i-1}}\ (\mathrm{Df-Elim})$$

where A is the definiendum and D_A is the definienda of A. Gupta proposed the definitional calculus based on the natural deduction system and called $\mathbf{C}_n$. The definitional calculus $\mathbf{C}_n$ consists of classical inference rules, two definition rules (Df-Intro) and (Df-Elim), and moreover the following two kinds of index shift rules: for any integer number i, j,

$$\frac{A^i}{A^j}\ (\mathrm{IS}) \qquad \frac{A^i}{A^{i\pm n}}\ (\mathrm{IS_n}) \quad (\exists n \in \mathbf{N})$$

where an occurrence of A^i in a derivation indicates the relative position of the step in a revision process. A must not contain any defined symbols in (IS), but may contain these in (IS$_\mathrm{n}$).

In this paper we will compare two logical systems **PSC** and $\mathbf{C}_n$ with a syntactical point of view. Because both notions of the pair-sentence with stage number in **PSC** and Gupta's sentence-definition with revision stage number in $\mathbf{C}_n$ are very similar, and both can deal with paradoxical sentences like a simple Liar sentence.

2 PSC Logic

Let $\mathcal{L}_P =< \boldsymbol{FOR_P}, \neg, \wedge, \vee, \rightarrow, ((\_)^i\ , (\_)^j), \top, \bot >$ be a language of the sentential calculus with pair-sentence constructor to construct a pair sentence formula (A^i, B^j) which means "A at a stage i is referential to B at stage j", where i and j are referential stage numbers that A and B hold, respectively. Then the formulas $\boldsymbol{FOR_P}$ of a language $\mathcal{L}_P$ are generated in the usual way from an infinite set $\boldsymbol{VAR_P}$ of sentential variables and constants $\top$ (true), $\bot$ (false) by the standard truth functional connectives $\neg$(negation), $\wedge$ (conjunction), $\vee$ (disjunction) and $\rightarrow$ (material implication) as well as the pair-sentence constructor $((\_)^i\ , (\_)^j)$, where i, j are some stage numbers. So, we have:
(1) $\boldsymbol{VAR_P} = \bigcup_{i\in\mathbf{N}} \boldsymbol{VAR}^i$, where $\boldsymbol{VAR}^i = \{p^i, q^i, r^i, \ldots\}$
(2) $\boldsymbol{VAR_P} \subseteq \boldsymbol{FOR_P}$
(3) $\forall A^i, B^i, C^j \in \boldsymbol{FOR_P} \Longrightarrow \neg A^i, A^i \wedge B^i, A^i \vee B^i, A^i \rightarrow B^i, (A^i, C^j) \in \boldsymbol{FOR_P}$

Also we may use the same parentheses as auxiliary symbols even assume that the priority of each connective and constructor are weak as $\neg$, $\wedge$, $\vee$, $\rightarrow$, $((\_)^i\ , (\_)^j)$ in order. Throughout this paper the letters p, q, r, p^0, $p^1 \ldots$ are used to denote any variables; the letters A, B, C, $\ldots$ denote formulas of a **PSC** language $\mathcal{L}_P$; the letters X, Y denote sets of formulas.

We will introduce several terminologies with the pair-sentence as follows:
(1) If A is a subformula of B, then we say that the pair-sentence (A^i, B^j) is a *circular referential relation*. Otherwise, is a *non-circular referential relation*.
(2) For a circular referential relation (A^i, B^j) such that the referential recursive pattern $A^i\ B_0^j\ B_1^{j+1}\ B_2^{j+2}\ \cdots B_0^{j+(n-1)}\ \cdots$ holds, the total referential stage number n of B_0^j being recursively returned to itself $B_0^{j+(n-1)}$ is called a *referential cycle number* of B relative to a circular referential relation (A^i, B^j), and $\tau(B) = n$ in symbol. Otherwise, for a non-circular referential relation (A^i, B^j), the referential stage numbers i, j are ineffective, so we will eliminate each referential stage number like (A, B), and $\tau(B) = 0$.
(3) For any circular referential relation (A^i, B^j), if $\tau(B) = 1$ then we say that B is *categorical*. Otherwise, if $\tau(B) \geq 2$ then we say that B is *paradoxical*.

We define the referential stage numbering of composed formulas as the following: $\forall i, j \in \mathbf{N}$,
(1) $(\neg A)^i \iff \neg A^i$
(2) $(A \wedge B)^i \iff A^i \wedge B^i$
(3) $(A \vee B)^i \iff A^i \vee B^i$
(4) $(A \rightarrow B)^i \iff A^i \rightarrow B^i$
(5) $(A, B)^i \iff (A^i, B^i)$
(6) $(A^i)^j \iff A^j$

The axiomatic system **PSC** for the language $\mathcal{L}_P$ is defined by the following way: $\forall i, j, k \in \mathbf{N}$,
(A1) $A^i \rightarrow (B^i \rightarrow A^i)$
(A2) $(A^i \rightarrow (B^i \rightarrow C^i)) \rightarrow ((A^i \rightarrow B^i) \rightarrow (A^i \rightarrow C^i))$
(A3) $A^i \wedge B^i \rightarrow A^i$
(A4) $A^i \wedge B^i \rightarrow B^i$
(A5) $A^i \rightarrow (B^i \rightarrow (A^i \wedge B^i))$
(A6) $A^i \rightarrow A^i \vee B^i$
(A7) $B^i \rightarrow A^i \vee B^i$
(A8) $(A^i \rightarrow C^i) \rightarrow ((B^i \rightarrow C^i) \rightarrow (A^i \vee B^i \rightarrow C^i))$
(A9) $\neg A^i \rightarrow (A^i \rightarrow B^i)$
(A10) $A^i \vee \neg A^i$

(E1) (A^i, A^j) where $i = j$ if A is related to others as a circular referential relation.
(E2) $(A^i, B^j) \to (B^j, A^i)$
(E3) $(A^i, B^j) \wedge (B^j, C^k) \to (A^i, C^{k)})$
(C1) $(A^i, B^j) \to ((\neg A)^i, (\neg B)^j)$
(C2) $(A^i, B^j) \wedge (C^i, D^j) \to ((A \wedge C)^i, (B \wedge D)^j)$
(C3) $(A^i, B^j) \wedge (C^i, D^j) \to ((A \vee C)^i, (B \vee D)^j)$
(C4) $(A^i, B^j) \wedge (C^i, D^j) \to ((A \to C)^i, (B \to D)^j)$
(C5) $(A^i, B^j) \wedge (C^i, D^j) \to ((A, C)^i, (B, D)^j)$
(P1) $(A^i, B^j) \to (A^i \to B^j)$
(P2) $(A^i, B^j) \wedge (B \leftrightarrow C)^j \to (A^i, C^j)$
(P3) $(A^i, B^j) \wedge (A \leftrightarrow C)^i \to (C^i, B^j)$
(P4) $(A^i, B^j) \to (A^{i \pm n}, B^{j \pm n})$ where $\exists n \geq 0$
(Mp) $\dfrac{A^i \quad A^i \to B^j}{B^j}$

Here the axioms of (A1)–(A10) with modus ponens(Mp) as a single rule (and of course not exist any pair sentence, so we can eliminate any upper index) will give an axiomatic system $\boldsymbol{CL}$ for the classical sentential logic, and if we will restrict the pair-sentence formula (A^i, B^j) to a non-circular referential relation, i.e, (A, B), then **PSC** is collapsed into an extension $\mathbf{W}_{\mathrm{B}}$ of **SCI** system because if we regard (A, B) as $A \equiv B$, then any axioms of **SCI** can be drived from **PSC** and also we must demand to have two additional axioms $(A \equiv B) \wedge (B \leftrightarrow C) \to (A \equiv C)$ and $(A \equiv B) \wedge (A \leftrightarrow C) \to (C \equiv B)$ in **SCI**. Also we call a system which is obtained from **PSC** by adding an axiom (P5) $(A^i, A^{i \pm n})$ where $\exists n \geq 0$, $\mathbf{PSC}_n$.

Definition 2.1 (Derivability) *Let X be a set of formulas in a language $\mathcal{L}_P$, A a formula and* **PSC** *a system in $\mathcal{L}_P$. Then we say that:*
(1) (A_1^i, A_2^j) is derivable *from X in* **PSC***, we write $X \vdash_{\mathrm{PSC}} (A_1^i, A_2^j)$ iff there is a sequence of formulas $B_1, B_2, \ldots, B_n (n \geq 0)$ such that every formula in the sequence $B_1, B_2, \ldots, B_n, (A_1^i, A_2^j)$ is either a theorem of* **PSC***, or belongs to X, or is obtained by (Mp) rule from formulas occurring before it in the sequence, where if $X = \emptyset$, we write $\vdash_{\mathrm{PSC}} (A_1^i, A_2^j)$, and we say that (A_1^i, A_2^j) is a* theorem *of* **PSC**.
(2) A is derivable *from X in* **PSC***, we write $X \vdash_{\mathrm{PSC}} A$ iff there is a sequence of formulas $B_1, B_2 \ldots, B_n$ such that $B_1^0, B_2^0 \ldots, B_n^0 \vdash_{\mathrm{PSC}} A^0$. If $X = \emptyset$, we write $\vdash_{\mathrm{PSC}} A$, and we say that A is a* theorem *of* **PSC**.

For example, let's consider a simple Liar sentence: " This sentence is not true". If we define A = "This sentence is true". Then a simple Liar sentence is expressed by a pair sentence formula $(A^0, \neg A^1)$, and we can prove that $(A^0, \neg A^1) \vdash_{\mathrm{PSC}} (A^0, A^2)$ as follows:

1 $[(A^0, \neg A^1)]$	[Hypothesis]
2 $(A^0, \neg A^1) \to (\neg A^0, \neg\neg A^1)$	[(C1)]
3 $(\neg A^0, \neg\neg A^1)$	[1, 2 and (Mp)]
4 $(\neg\neg A \leftrightarrow A)^1$	[Tautology of $\boldsymbol{CL}$]
5 $(\neg A^0, \neg\neg A^1) \wedge (\neg\neg A \leftrightarrow A)^1 \to (\neg A^0, A^1)$	[(P2)]
6 $(\neg A^0, A^1)$	[3, 4, 5 and (Mp)]
7 $(\neg A^0, A^1) \to (\neg A^1, A^2)$	[(P4)]
8 $(\neg A^1, A^2)$	[6, 7 and (Mp)]
9 $(A^0, \neg A^1) \wedge (\neg A^1, A^2) \to (A^0, A^2)$	[(E3)]
10 (A^1, A^2)	[1, 8, 9 and (Mp)]

So, we know that $\tau(A) = 2$ relative to a circular referential relation $(A^0, \neg A^1)$ and A is paradoxical. Also we can prove that $(A^0, \neg A^1) \vdash_{\mathrm{PSC}_3} \perp$ as follows:

1 $[(A^0, \neg A^1)]$	[Hypothesis]
2 $[A^0]$	[Hyothesis]
3 $\neg A^1$	[(P1):$A^0 \wedge (A^0, \neg A^1) \to \neg A^1$]
4 $[\neg A^2]$	[Hypothesis]
5 A^1	[4 and (P1):$\neg A^2 \wedge (A^0, \neg A^1) \to A^1$]

6 $(\bot)^1$ [3 and 5]
7 $(\bot)^2$ [(E1):$(\bot^1,\bot^2)$ and (P1)]
8 A^2 [4 and 7]
9 $\neg A^3$ [8, (P4) and (P1):$A^2\wedge(A^0,\neg A^1)\to\neg A^3$]
10 $\neg A^0$ [(P5):$\neg A^3\to\neg A^0$]
11 $\neg A^0$ [2 and 10]
12 A^1 [11, (C1) and (P1):$\neg A^0\wedge(\neg A^1,A^2)\to A^1$]
13 $\neg A^2$ [12 and (P1):$A^1\wedge(A^0,\neg A^1)\to A^2$]
14 $\neg A^3$ [11 and (P5):$\neg A^0\to\neg A^3$]
15 A^2 [14, (P4) and (P1):$\neg A^3\wedge(A^0,\neg A^1)\to A^2$]
16 $(\bot)^2$ [13 and 15]
17 $(\bot)^0$ [15, (E1):$(\bot^2,\bot^0)$ and (P1)]

Similarly, we have the following results:
(1) $\{(S^0,\neg S^1),(P^0,S^1)\}\vdash_{\mathrm{PSC}}(S^0,S^2)$ and (P^0,P^2). So, both of S and P are paradoxical and $\tau(S)=\tau(P)=2$.
(2) $\{(S^0,\neg P^1),(P^0,S^1)\}\vdash_{\mathrm{PSC}}(S^0,S^4)$, (P^0,P^4) and $((S\wedge P)^0,(S\wedge P)^4)$. So, all of S, P and $S\wedge P$ are paradoxical and $\tau(S)=\tau(P)=4$.
(3) $\{(S^0,\neg P^1),(P^0,\neg S^1)\}\vdash_{\mathrm{PSC}}(S^0,S^2)$, (P^0,P^2) and $((S\wedge\neg P)^0,(S\wedge\neg P)^1)$. So, both of S and P are paradoxical and $S\wedge\neg P$ is categorical.
(4) $\{(S^0,\neg P^1),(P^0,\neg A^1),(A^0,S^1)\}\vdash_{\mathrm{PSC}}((S\wedge P\wedge A)^0,(S\wedge P\wedge A)^3)$,
$((S\wedge P\wedge\neg A)^0,(S\wedge P\wedge\neg A)^3)$, $((S\wedge\neg P\wedge\neg A)^0,(S\wedge\neg P\wedge\neg A)^3)$,
$((\neg S\wedge P\wedge A)^0,(\neg S\wedge P\wedge A)^3)$, $((\neg S\wedge\neg P\wedge A)^0,(\neg S\wedge\neg P\wedge A)^3)$,
$((\neg S\wedge\neg P\wedge\neg A)^0,(\neg S\wedge\neg P\wedge\neg A)^3)$, $((S\wedge\neg P\wedge A)^0,(S\wedge\neg P\wedge A)^1)$ and
$((\neg S\wedge P\wedge\neg A)^0,(\neg S\wedge P\wedge\neg A)^1)$.
(5) $\{(S^0,\neg P^1),(P^0,\neg A^1),(A^0,\neg S^1)\}\vdash_{\mathrm{PSC}}((S\wedge P\wedge\neg A)^0,(S\wedge P\wedge\neg A)^6)$,
$((S\wedge\neg P\wedge A)^0,(S\wedge\neg P\wedge A)^6)$, $((S\wedge\neg P\wedge\neg A)^0,(S\wedge\neg P\wedge\neg A)^6)$,
$((\neg S\wedge P\wedge A)^0,(\neg S\wedge P\wedge A)^6)$, $((\neg S\wedge P\wedge\neg A)^0,(\neg S\wedge P\wedge\neg A)^6)$,
$((\neg S\wedge\neg P\wedge A)^0,(\neg S\wedge\neg P\wedge A)^6)$, $((\neg S\wedge\neg P\wedge A)^0,(\neg S\wedge\neg P\wedge A)^2)$ and
$((\neg S\wedge\neg P\wedge\neg A)^0,(\neg S\wedge\neg P\wedge\neg A)^2)$.

Next, we consider a set of circular referential relations: $\{(A^0,(B\vee(C\wedge\neg A))^1),(B^0,B^1),(C^0,C^1)\}$. Then we can prove that
(1) $\{(A^0,(B\vee(C\wedge\neg A))^1),(B^0,B^1),(C^0,C^1)\}\vdash_{\mathrm{PSC}}((B\vee(C\wedge\neg A))^0,(B\vee(C\wedge\neg A))^2)$ and
(2) $\{(A^0,(B\vee(C\wedge\neg A))^1),(B^0,B^1)\}\vdash_{\mathrm{PSC}}B\to A$ as follows:
(1):
1 $[(A^0,(B\vee(C\wedge\neg A))^1)]$ [Hypothesis]
2 $[(B^0,B^1)]$ [Hypothesis]
3 $[(C^0,C^1)]$ [Hypothesis]
4 $(\neg A^0,\neg(B\vee(C\wedge\neg A))^1)$ [1 and (C1)]
where $\neg(B\vee(C\wedge\neg A))^1\leftrightarrow(\neg B\wedge(\neg C\vee A))^1$
5 $(C^0,C^1)\wedge(\neg A^0,(\neg B\wedge(\neg C\vee A))^1)\to((C\wedge\neg A)^0,(C\wedge\neg B\wedge(\neg C\vee A))^1)$ [(C2)]
6 $((C\wedge\neg A)^0,(C\wedge\neg B\wedge(\neg C\vee A))^1)$ [3,4,5 and (Mp)]
7 $(B^0,B^1)\wedge((C\wedge\neg A)^0,(C\wedge\neg B\wedge(\neg C\vee A))^1)$
$\to((B\vee(C\wedge\neg A))^0,(B\vee(C\wedge\neg B\wedge(\neg C\vee A)))^1)$ [(C3)]
where $B\vee(C\wedge\neg B\wedge(\neg C\vee A))\leftrightarrow B\vee(C\wedge A)$
8 $((B\vee(C\wedge\neg A))^0,(B\vee(C\wedge A))^1)$ [2, 6, 7 and (Mp)]
9 $((B\vee(C\wedge A))^0,(B\vee(C\wedge\neg A))^1)$ [similar to above]
10 $((B\vee(C\wedge A))^1,(B\vee(C\wedge\neg A))^2)$ [(P4)]
11 $((B\vee(C\wedge\neg A))^0,(B\vee(C\wedge A))^1)\wedge((B\vee(C\wedge A))^1,(B\vee(C\wedge\neg A))^2)$
$\to((B\vee(C\wedge\neg A))^0,(B\vee(C\wedge\neg A))^2)$ [(E3)]
12 $((B\vee(C\wedge\neg A))^0,(B\vee(C\wedge\neg A))^2)$ [8,10,11 and (Mp)]

So, $B\vee(C\wedge\neg A)$ is paradoxical and $\tau(B\vee(C\wedge\neg A))=2$ relative to circular referential relations $\{(A^0,(B\vee(C\wedge\neg A))^1),(B^0,B^1),(C^0,C^1)\}$.

(2):
1 $[(A^0, (B \vee (C \wedge \neg A))^1)]$ [Hypothesis]
2 $[(B^0, B^1)]$ [Hypothesis]
3 $[B^0]$ [Hypothesis]
4 $(B^0, B^1) \to (B^0 \to B^1)$ [(P1)]
5 $B^0 \to B^1$ [2,4 and (Mp)]
6 B^1 [3,5 and (Mp)]
7 $B^1 \to (B \vee (C \wedge \neg A))^1$ [(A6)]
8 $(B \vee (C \wedge \neg A))^1$ [6,7 and (Mp)]
9 $(A^0, (B \vee (C \wedge \neg A))^1) \to ((B \vee (C \wedge \neg A))^1, A^0)$ [(E2)]
10 $((B \vee (C \wedge \neg A))^1, A^0)$ [1,9 and (Mp)]
11 $((B \vee (C \wedge \neg A))^1, A^0) \to ((B \vee (C \wedge \neg A))^1 \to A^0)$ [(P1)]
12 $(B \vee (C \wedge \neg A))^1 \to A^0$ [10,11 and (Mp)]
13 A^0 [8, 12 and (Mp)]
14 $(B \to A)^0$ [3, 13 and DT]

3 Cn Logic

At first we consider the following definition: (1) $A =_{df} B \vee (C \wedge \neg A)$. Here A is the definiendum and $D_A(A) = B \vee (C \wedge \neg A)$ is the definienda of A. And in this case, $D_A(A)$ has the definiendum A itself as a subformula. So, we say that the definienda $D_A(A)$ is a self-referential form. Assume that (2) $\neg B \wedge C$ and (3) A hold. By (1), (3) and (Df-Elim), we get $B \vee (C \wedge \neg A)$. And the conjunction of the result and (2) yields $(\neg B \wedge C) \wedge (B \vee (C \wedge \neg A)) \leftrightarrow \neg A$. So, we have $A \to \neg A$. Similarly, if we assume $\neg B \wedge C$ and $\neg A$, then we have $\neg A \to A$. Hence we have $A \leftrightarrow \neg A$ under the assumption of (2). To deal with such a circular definition, Gupat proposed the definitional calculus based on the natural deduction system and called $\mathbf{C}_n$. His system was defined as a predicate calculus, but here we will introduce the propositional version of $\mathbf{C}_n$, because of doing the comparison between **PSC** and $\mathbf{C}_n$ in the later section.

Let $\mathcal{L}_C =< \boldsymbol{FOR_C}, \neg, \wedge, \vee, \to, =_{df}, =, \top, \bot >$ be a language of the definitional calculus, where $=_{df}$ is a definition constructor like $A =_{df} D_A$ such that A is the definiendum and D_A is the definienda of A, and $=$ is an identity connective and we need this connective when replacing the identity formulas in the definienda. Then the formulas $\boldsymbol{FOR_C}$ of a language $\mathcal{L}_C$ are generated in the usual way from an infinite set $\boldsymbol{VAR_C}$ of sentential variables and constants $\top$, $\bot$ by the standard truth functional connectives $\neg$ (negation), $\wedge$ (conjunction), $\vee$ (disjunction) and $\to$ (material implication) as well as $=$ (identity) and $=_{df}$ (definitional constructor).
(1) $\boldsymbol{VAR_C} = \bigcup_{i \in \mathbf{N}} \boldsymbol{VAR}^i$, where $\boldsymbol{VAR}^i = \{p^i, q^i, r^i, \ldots\}$
(2) $\boldsymbol{VAR_C} \subseteq \boldsymbol{FOR_C}$
(3) $\forall A^i, B^i, D_A^{i-1} \in \boldsymbol{FOR_C} \Longrightarrow \neg A^i, A^i \wedge B^i, A^i \vee B^i, A^i \to B^i, A^i = B^i, A^i =_{df} D_A^{i-1} \in \boldsymbol{FOR_C}$

We will introduce several terminologies with the definition as follows:
(1) For any definition $A =_{df} D_A(A)$, if the definienda of A includes the definiendum A itself, then we say that the definition is a *circular definition*. Otherwise, is a *non-circular definition*. So, any definition that has a self-referential form is a circular definition.
(2) For a set D of several circular definitions of A, the sequence of A generated by applying two derivation rules (Df-Intro) and (Df-Elim) to D was called a *revision sequence* for A. A revision stage number i of A^i shows the current position for the process of revision.
(3) For any circular definition $A =_{df} D_A(A)$, if the truth value of A converges in some constants(true or false), then we say that A is *categorical*. Otherwise, is *paradoxical*, and whose truth value oscillates in the revision sequence.

A natural deduction system $\mathbf{C}_n$ for the language $\mathcal{L}_C$ is defined in the following way: $\forall i, j \in \mathbf{Z}$,

$$\frac{A^i \quad (A \to B)^i}{B^i}\ (\to \text{Elim}) \qquad\qquad \frac{\begin{array}{c}(k)\\ [A^i]\\ \vdots\\ B^i\end{array}}{(A \to B)^i}\ (\to \text{Intro})(k)$$

$$\frac{(A \wedge B)^i}{A^i}\ (\wedge\text{Elim1}) \qquad \frac{(A \wedge B)^i}{B^i}\ (\wedge\text{Elim2}) \qquad \frac{A^i \quad B^i}{(A \wedge B)^i}\ (\wedge\text{Intro})$$

$$\frac{(A \vee B)^i \quad \begin{array}{c}(k)\\ [A^i]\\ \vdots\\ C^i\end{array} \quad \begin{array}{c}(k)\\ [B^i]\\ \vdots\\ C^i\end{array}}{C^i}\ (\vee\text{Elim})(k) \qquad \frac{A^i}{(A \vee B)^i}\ (\vee\text{Intro1}) \qquad \frac{B^i}{(A \vee B)^i}\ (\vee\text{Intro2})$$

$$\frac{\neg\neg A^i}{A^i}\ (\neg\text{Elim}) \qquad\qquad \frac{\begin{array}{c}(k)\\ [A^i]\\ \vdots\\ \bot^i\end{array}}{(\neg A)^i}\ (\neg\text{Intro})(k)$$

$$\frac{\bot^i}{A^i}\ (\bot\text{Elim}) \qquad\qquad \frac{A^i \quad \neg A^i}{\bot^i}\ (\bot\text{Intro})$$

$$\frac{(A = B)^i \quad D_A(A)^i}{D_B(B)^i}\ (=\text{Elim}) \qquad\qquad \frac{}{(A = A)^i}\ (=\text{Intro})$$

$$\frac{A^i =_{df} D_A^{i-1} \quad A^i}{{D_A}^{i-1}}\ (\text{Df} - \text{Elim}) \qquad\qquad \frac{A^{i+1} =_{df} D_A^i \quad {D_A}^i}{A^{i+1}}\ (\text{Df} - \text{Intro})$$

$$\frac{A^i}{A^j}\ (\text{IS}) \qquad\qquad \frac{A^i}{A^{i \pm n}}\ (\text{IS}_\text{n}) \quad (\exists n \in \mathbf{N})$$

and also the identity connective $=$ is an equivalence relation, so we assume the following rules:

$$\frac{(A = B)^i}{(B = A)^i}\ (=\text{Sym}) \qquad\qquad \frac{(A = B)^i \quad (B = C)^i}{(A = C)^i}\ (=\text{Tran})$$

An occurrence of A^i in a derivation indicates the relative position of the step in a revision process. A must not contain any defined symbols in (IS), but may contain these in (IS_n). So, if D_A has not any defined symbols, i.e., A has a non-circular definition, then the indices are dispensable in all contexts by (IS) rule. For example,

(1) $[A^i]$ [Hypothesis]
(2) D_A^{i-1} [(1) and (Df-Elim)]
(3) D_A^{j-1} [(2) and (IS)]
(4) A^j [(3) and (Df-Intro)]

Thus, because the indices do not work with non-circular definitions, they can be eliminated.

Here the inference rules of $\to, \wedge, \vee, \neg, \bot$ will give the classical natural deduction system $\boldsymbol{NK}$, and if we will restrict an only non-circular definition, then each definition $A =_{df} D_A$ is collapsed into the logical equivalence $A \leftrightarrow D_A$ in $\mathbf{C}_n$. Also we call a system $\mathbf{C}_n$ except for (IS_n) rule $\mathbf{C}_0$. Thus we have $\mathbf{C}_0 = \boldsymbol{NK} + (\text{Df} - \text{Elim}) + (\text{Df} - \text{Intro}) + (=\text{Elim}) + (=\text{Intro}) + (\text{IS})$ and $\mathbf{C}_n = \mathbf{C}_0 + (\text{IS}_\text{n})$.

Definition 3.1 (Derivability) *Let D be a set of definition in a language $\mathcal{L}_C$, A a formula and $\mathbf{C}_n$(or $\mathbf{C}_0$) a system in $\mathcal{L}_C$. Then we say that:*
(1) A is derivable *from D in $\mathbf{C}_0$, we write $D\vdash_{\mathrm{C0}}A$ iff a derivation of A^0 can be constructed in $\mathbf{C}_0$ from D. If $D=\emptyset$, we write $\vdash_{\mathrm{C0}}A$, and we say that A is a* theorem *of $\mathbf{C}_0$ relative to D.*
(2) A^j is derivable in $\mathbf{C}_n$ on the basis of D from a set X of indexed formulas, we write $X\vdash_{\mathrm{Cn}}A^j$ iff there is a derivation of A^j in $\mathbf{C}_n$ on the basis of D from some indexed formulas $B_1^{i1}, B_2^{i2}, \ldots, B_m^{im}$ that belong to X.
(3) A is derivable in $\mathbf{C}_n$ on the basis of D from a set X of indexed formulas, we write $X\vdash_{\mathrm{Cn}}A$ iff there are formulas $B_1, B_2, \ldots, B_m \in X$ such that $B_1^0, B_2^0, \ldots, B_m^0\vdash_{\mathrm{Cn}}A^0$. If $X=\emptyset$, we write $\vdash_{\mathrm{Cn}}A$, and we say that A is a theorem *of $\mathbf{C}_n$ relative to D.*

Proposition 3.2 *Let $D=\{A =_{df} D_A, B =_{df} D_B\}$ be circular definitions of A and B, then we can prove the following as drived rules:*
(1) $\neg A^{i+1}\vdash_{\mathrm{C0}}\neg D_A^i$ (¬Df-Elim)
(2) $\neg D_A^i\vdash_{\mathrm{C0}}\neg A^{i+1}$ (¬Df-Intro)
(3) $(A\%B)^{i+1}\vdash_{\mathrm{C0}}(D_A\%D_B)^i$ where $\% \in \{\wedge, \vee, \rightarrow, =\}$
(4) $(D_A\%D_B)^i\vdash_{\mathrm{C0}}(A\%B)^{i+1}$ where $\% \in \{\wedge, \vee, \rightarrow, =\}$
(5) $(D_A \leftrightarrow E_A)^i \Longrightarrow A^{i+1}\vdash_{\mathrm{C0}}E_A^i$
(6) $(A \leftrightarrow C)^{i+1} \Longrightarrow C^{i+1}\vdash_{\mathrm{C0}}D_A^i$

Proof. (1) and (2):

$$\dfrac{\dfrac{\dfrac{\overset{(Hypothesis)}{[\neg A^{i+1}]} \qquad \dfrac{\overset{(k)}{[D_A^i]}}{A^{i+1}}\,(\mathrm{Df-Intro})}{\bot^{i+1}}\,(\bot\mathrm{Intro})}{\bot^{i}}\,(\mathrm{IS})}{\neg D_A^i}\,(\neg\mathrm{Intro})(k) \qquad\qquad \dfrac{\dfrac{\dfrac{\overset{(Hypothesis)}{[\neg D_A^{i}]} \qquad \dfrac{\overset{(k)}{[A^{i+1}]}}{D_A^{i}}\,(\mathrm{Df-Elim})}{\bot^{i}}\,(\bot\mathrm{Intro})}{\bot^{i+1}}\,(\mathrm{IS})}{\neg A^{i+1}}\,(\neg\mathrm{Intro})(k)$$

(3):(i) $(A\wedge B)^{i+1}\vdash_{\mathrm{C0}}(D_A\wedge D_B)^i$

$$\dfrac{\dfrac{\dfrac{\overset{(Hypothesis)}{[(A\wedge B)^{i+1}]}}{A^{i+1}}\,\wedge(\mathrm{Elim1})}{D_A^i}\,(\mathrm{Df-Elim}) \qquad \dfrac{\dfrac{\overset{(Hypothesis)}{[(A\wedge B)^{i+1}]}}{B^{i+1}}\,\wedge(\mathrm{Elim2})}{D_B^i}\,(\mathrm{Df-Elim})}{(D_A\wedge D_B)^i}\,(\wedge\mathrm{Intro})$$

(ii) $(A\vee B)^{i+1}\vdash_{\mathrm{C0}}(D_A\vee D_B)^i$

$$\dfrac{\overset{(Hypothesis)}{[(A\vee B)^{i+1}]} \qquad \dfrac{\dfrac{\overset{(k)}{[A^{i+1}]}}{D_A^i}\,(\mathrm{Df-Elim})}{(D_A\vee D_B)^i}\,\vee(\mathrm{Intro1}) \qquad \dfrac{\dfrac{\overset{(k)}{[B^{i+1}]}}{D_B^i}\,(\mathrm{Df-Elim})}{(D_A\vee D_B)^i}\,\vee(\mathrm{Intro2})}{(D_A\vee D_B)^i}\,(\vee\mathrm{Elim})(k)$$

(iii) $(A\rightarrow B)^{i+1}\vdash_{\mathrm{C0}}(D_A\rightarrow D_B)^i$ and (iv) $(A=B)^{i+1}\vdash_{\mathrm{C0}}(D_A=D_B)^i$

$$\dfrac{\dfrac{\dfrac{\dfrac{\overset{(k)}{[D_A^i]}}{A^{i+1}}\,(\mathrm{Df-Intro}) \qquad \overset{(Hypothesis)}{[(A\rightarrow B)^{i+1}]}}{B^{i+1}}\,(\rightarrow\mathrm{Elim})}{D_B^i}\,(\mathrm{Df-Elim})}{(D_A\rightarrow D_B)^i}\,(\rightarrow\mathrm{Intro})(k) \qquad\qquad \dfrac{\overset{(Hypothesis)}{[(A=B)^{i+1}]} \qquad \dfrac{}{(D_B=D_B)^i}\,(=\mathrm{Intro})}{(D_A=D_B)^i}\,(=\mathrm{Elim})$$

(4): we can prove the similar way to (3).
(5) and (6):

$$\dfrac{\dfrac{\overset{(Hypothesis)}{[A^{i+1}]}}{D^{i}_{A}}\,(\mathrm{Df-Elim}) \qquad \dfrac{\overset{(Hypothesis)}{[(D_A \leftrightarrow E_A)^{i}]}}{(D_A \to E_A)^{i}}\,(\wedge\mathrm{Elim1})}{E^{i}_{A}}\,(\to\mathrm{Elim}) \qquad \dfrac{\dfrac{\overset{(Hypothesis)}{[C^{i+1}]} \qquad \dfrac{\overset{(Hypothesis)}{[(C \leftrightarrow A)^{i+1}]}}{(C \to A)^{i+1}}\,(\wedge\mathrm{Elim2})}{A^{i+1}}\,(\to\mathrm{Elim})}{D^{i}_{A}}\,(\mathrm{Df-Elim})$$

□

For example, let $D_1 = \{A =_{df} B \vee (C \wedge \neg A)\}$ be a circular definition of A. Then we can prove that $D_1 \vdash_{\mathrm{C0}} B \to A$ as follows:

1 $[B^0]$ [Hypothesis]
2 B^{-1} [1 and (IS)]
3 $B^{-1} \vee (C \wedge \neg A)^{-1}$ [2 and ($\vee$Intro1)]
4 $(B \vee (C \wedge \neg A))^{-1}$ [syntactical equivalence]
5 A^0 [4 and (Df-Intro)]
6 $B^0 \to A^0$ [1,5 and ($\to$Intro)]
7 $(B \to A)^0$ [syntactical equivalence]

Next let $D_2 = \{A =_{df} \neg A\}$ be a circular definition of A. Then we can prove that $D_2 \vdash_{\mathrm{C3}} \bot$ as follows:

$$\dfrac{\dfrac{\dfrac{\dfrac{\overset{(l)}{[\neg A^0]} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\overset{(l)}{[\neg A^0]}}{A^1}\,(\mathrm{Df-Intro}) \qquad \dfrac{\overset{(k)}{[A^2]}}{\neg A^1}\,(\mathrm{Df-Elim})}{\bot^1}\,(\bot\mathrm{Intro})}{\bot^2}\,(\mathrm{IS})}{\neg A^2}\,(\neg\mathrm{Intro})(k)}{A^3}\,(\mathrm{Df-Intro})}{A^0}\,(\mathrm{IS}_3)}{\bot^0}\,\bot(\mathrm{Intro})}{\neg\neg A^0}\,\neg(\mathrm{Intro})(l)}{\neg A^1}\,(\mathrm{Df-Intro})}{A^2}\,(\mathrm{Df-Intro}) \qquad \dfrac{\dfrac{\dfrac{\overset{(drived:\,see\ \ leftside)}{\neg\neg A^0}}{\neg\neg A^3}\,(\mathrm{IS}_3)}{A^3}\,(\neg\mathrm{Elim})}{\neg A^2}\,(Df-Elim)}{\dfrac{\bot^2}{\bot^0}\,(\mathrm{IS})}\,(\bot\mathrm{Intro})$$

Moreover, we can prove $D_2 \vdash_{\mathrm{C_{2m+1}}} \bot$, but $D_2 \nvdash_{\mathrm{C_{2m}}} \bot$ for $\forall m \in \mathbf{N}$ such that $m \geq 1$.

4 Some syntactic comparisons between PSC and Cn

At first we will introduce a general method of showing syntactical equivalence between various logics owing to mainly K. Segerberg's book [4]. For two logics which are formulated in very different object languages, we can intuitively say that these logics are the same or at least equivalent if they are equally strong, or they come to the same thing. We can also say this fact if the languages in which they are formulated are intertranslatable, namely if what can be also expressed in one language can be expressed in other one. And moreover, whenever a formula in one logic is valid, then its counterpart in the other is also valid. We will define the above notion of equivalent of logics more precisely in the following.

Suppose that $\mathbf{L_1}$ and $\mathbf{L_2}$ are two logics in the language $\mathcal{L}_1$ and $\mathcal{L}_2$ such that $\mathbf{L_1} = (\mathcal{L}_1, C_1)$ and $\mathbf{L_2} = (\mathcal{L}_2, C_2)$ where C_1 and C_2 are structural consequence operators, i.e., $C_{\mathrm{i}}(X) = \{A | X \vdash_{\mathrm{L_i}} A\}$ $(i = 1, 2)$ on $\mathcal{L}_1$ and $\mathcal{L}_2$, and the sets of formulas of which are L_1 and L_2, respectively. Furthermore assume that the languages $\mathcal{L}_1$ and $\mathcal{L}_2$ have equivalence connectives $\leftrightarrow_1$ and $\leftrightarrow_2$, respectively. Then we define syntactically equivalent of two logics $\mathbf{L_1}$ and $\mathbf{L_2}$ as follows.

Definition 4.1 *(i)* $\mathbf{L_1}$ *and* $\mathbf{L_2}$ *are* syntactically equivalent with respect to t_1 and t_2 *if and only if* $t_1 : \mathrm{L}_1 \to \mathrm{L}_2$ *and* $t_2 : \mathrm{L}_2 \to \mathrm{L}_1$ *are functions such that the following conditions are satisfied:*
(1) for all $\alpha \in \mathrm{L}_1$, $(t_2(t_1(\alpha)) \leftrightarrow_1 \alpha) \in \mathbf{L_1}$,
(2) for all $A \in \mathrm{L}_2$, $(t_1(t_2(A)) \leftrightarrow_2 A) \in \mathbf{L_2}$,
(3) for all $\alpha \in \mathrm{L}_1$, $\alpha \in \mathbf{L_1}$ iff $t_1(\alpha) \in \mathbf{L_2}$,
(4) for all $A \in \mathrm{L}_2$, $A \in \mathbf{L_2}$ iff $t_2(A) \in \mathbf{L_1}$.
(ii) $\mathbf{L_1}$ *and* $\mathbf{L_2}$ *are called* syntactically equivalent *if there exist functions* t_1 *and* t_2 *with respect to which they are syntactically equivalent.*

The definition of the above syntactic equivalence can be understood intuitively as follows. Two functions t_1 and t_2 are to be understood as translations of one language into the other. Conditions (1) and (2) are to denote a way of checking that two translations do their jobs that at least they are inverse operations of one another. Conditions (3) and (4) are meant to guarantee that both translations preserve logical relationships.

We can use the word 'extension' as refer to either languages or logics. Suppose that $\mathcal{L}_1 = \langle \mathrm{VAR}_1, \mathrm{BOP}_1, \mathrm{AdOP}_1, \mathrm{RNK}_1 \rangle$ and $\mathcal{L}_2 = \langle \mathrm{VAR}_2, \mathrm{BOP}_2, \mathrm{AdOP}_2, \mathrm{RNK}_2 \rangle$ are languages, where VAR_1 and VAR_2 are denumerably infinite variables, BOP_1 and BOP_2 Boolean operators, AdOP_1 and AdOP_2 additional non-Boolean operators, and RNK_1 and RNK_2 ranks, respectively. Then we have the following definitions.

Definition 4.2 *(i)* $\mathcal{L}_1$ *is a* sublanguage *of* $\mathcal{L}_2$ *or* $\mathcal{L}_2$ *is an* extension *of* $\mathcal{L}_1$ *if the following conditions are satisfied:*
(1) $\mathrm{VAR}_1 \subseteq \mathrm{VAR}_2$,
(2) $\mathrm{BOP}_1 \subseteq \mathrm{BOP}_2$,
(3) $\mathrm{AdOP}_1 \subseteq \mathrm{AdOP}_2$,
(4) RNK_1 *and* RNK_2 *agree on* $\mathrm{BOP}_1 \cup \mathrm{AdOP}_1$.
(ii) If $\mathbf{L_1} = (\mathcal{L}_1, C_1)$ *and* $\mathbf{L_2} = (\mathcal{L}_2, C_2)$ *are logics on* $\mathcal{L}_1$ *and* $\mathcal{L}_2$ *respectively, and in addition to* (1)-(4)*, also*
(5) $C_1 \subseteq C_2$,
then we say that $\mathbf{L_1}$ *is a* sublogic *of* $\mathbf{L_2}$ *or that* $\mathbf{L_2}$ *is an* extension *of* $\mathbf{L_1}$.
(iii) Furthermore, an extension $\mathbf{L_2}$ *of* $\mathbf{L_1}$ *is* conservative over $\mathbf{L_1}$ *if*
(6) $\mathbf{L_1} = \mathbf{L_2} \cap (\wp(\mathrm{L}_1) \times \wp(\mathrm{L}_1))$.
(iv) An extension $\mathbf{L_2}$ *of* $\mathbf{L_1}$ *is* definitional over $\mathbf{L_1}$ *if it is satisfied in addition to* (1)-(6)*, also*
(7) $\mathrm{VAR}_1 = \mathrm{VAR}_2$.

Theorem 4.3 *If* $\mathbf{L_1}$ *and* $\mathbf{L_2}$ *are logics such that* $\mathbf{L_2}$ *is a conservative definitional extension of* $\mathbf{L_1}$*, then* $\mathbf{L_1}$ *and* $\mathbf{L_2}$ *are syntactically equivalent.*

Next, we will consider translations between **PSC** and $\mathbf{C}_0$. We already introduced a language of Gupta's definitional calculus and its natural deduction system $\mathbf{C}_0$ in Section 3, so at first we will define two translations t_{C} and t_{P} between $\mathbf{C}_0$-language $\mathcal{L}_C$ and **PSC**-language $\mathcal{L}_P$ in order to investigate whether two logics $\mathbf{C}_0$ and **PSC** are syntactically equivalent or not with respect to these maps in the sense of Definition 4.1.

Definition 4.4 *The mapping* $t_{\mathrm{C}} : \boldsymbol{FOR_C} \to \boldsymbol{FOR_P}$*, called a* C -translation*, is defined inductively as follows:*
(1) $t_{\mathrm{C}}(p) := p, \ \ p \in \boldsymbol{VAR_C}$,
(2) $t_{\mathrm{C}}((\neg\alpha)^i) := \neg t_{\mathrm{C}}(\alpha^i)$,
(3) $t_{\mathrm{C}}((\alpha \wedge \beta)^i) := t_{\mathrm{C}}(\alpha^i) \wedge t_{\mathrm{C}}(\beta^i)$,
(4) $t_{\mathrm{C}}((\alpha \vee \beta)^i) := t_{\mathrm{C}}(\alpha^i) \vee t_{\mathrm{C}}(\beta^i)$,
(5) $t_{\mathrm{C}}((\alpha \to \beta)^i) := t_{\mathrm{C}}(\alpha^i) \to t_{\mathrm{C}}(\beta^i)$,
(6) $t_{\mathrm{C}}((\alpha = \beta)^i) := (t_{\mathrm{C}}(\alpha^i), t_{\mathrm{C}}(\beta^i))$,
(7) $t_{\mathrm{C}}(\alpha^i =_{df} \beta^j) := (t_{\mathrm{C}}(\alpha^i), t_{\mathrm{C}}(\beta^j))$.

Definition 4.5 *The mapping* $t_{\mathrm{P}} : \boldsymbol{FOR_P} \to \boldsymbol{FOR_C}$*, called a* PSC-translation*, is defined inductively as follows:*
(1) $t_{\mathrm{P}}(p) := p, \;\; p \in \boldsymbol{VAR_P}$*,*
(2) $t_{\mathrm{P}}((\neg A)^i) := \neg t_{\mathrm{P}}(A^i)$*,*
(3) $t_{\mathrm{P}}((A \wedge B))^i := t_{\mathrm{P}}(A^i) \wedge t_{\mathrm{P}}(B^i)$*,*
(4) $t_{\mathrm{P}}((A \vee B)^i) := t_{\mathrm{P}}(A^i) \vee t_{\mathrm{P}}(B^i)$*,*
(5) $t_{\mathrm{P}}((A \to B)^i) := t_{\mathrm{P}}(A^i) \to t_{\mathrm{P}}(B^i)$*,*
(6) $t_{\mathrm{P}}(A^i \to B^j) := t_{\mathrm{P}}(A^i) \to t_{\mathrm{P}}(B^j)$*,*
(7) $t_{\mathrm{P}}((A, B)^i) := t_{\mathrm{P}}(A^i) = t_{\mathrm{P}}(B^i)$*,*
(8) $t_{\mathrm{P}}((A^i, B^j)) := t_{\mathrm{P}}(A^i) =_{df} t_{\mathrm{P}}(B^j)$*.*

For two maps t_{C} and t_{P} we can prove the following two propositions.

Proposition 4.6 *For any formula* φ *in* $\boldsymbol{FOR_C}$*,* $\varphi \in \mathbf{C}_0$ *implies* $t_{\mathrm{C}}(\varphi) \in \mathbf{PSC}$*.*

Proof. By induction on the length of derivation in $\mathbf{C}_0$.
(i) Base step: We have to check the provability of an axiom $\varphi := (\alpha^i = \alpha^i)$ and a definition $\varphi := (\alpha^{i+1} =_{df} \beta^i)$ of $\mathbf{C}_0$ in $\mathbf{PSC}$ after a t_{C}-translation. In the first case, we have $t_{\mathrm{C}}(\varphi) := t_{\mathrm{C}}(\alpha^i = \alpha^i) = (t_{\mathrm{C}}(\alpha^i), t_{\mathrm{C}}(\alpha^i)) = (A^i, A^i) \in \mathbf{PSC}$ by (E1). In the second case,
$t_{\mathrm{C}}(\varphi) := t_{\mathrm{C}}(\alpha^{i+1} =_{df} \beta^i) = (t_{\mathrm{C}}(\alpha^{i+1}), t_{\mathrm{C}}(\beta^i)) = (A^{i+1}, B^i) \in \mathbf{PSC}$ by a pair sentence hypothesis.
(ii) Induction step: We have to check the admissibility of every inference rules for $\{\to, \wedge, \vee, \neg, \bot, =, Df, IS\}$ in $\mathbf{PSC}$ after a t_{C}-translation. But the inference rules for Boolean connectives are a routine work and so we omitted. Here we only show the cases of $\{=, Df, IS\}$.
(1) (= Elim): Assume that $(\alpha = \beta)^i$ and $\varphi^i_\alpha(\alpha)$. Then we have $t_{\mathrm{C}}((\alpha = \beta)^i) = (t_{\mathrm{C}}(\alpha^i), t_{\mathrm{C}}(\beta^i)) = (A^i, B^i)$ and $t_{\mathrm{C}}(\varphi^i_\alpha(\alpha)) = t_{\mathrm{C}}(\varphi^i_\alpha)(t_{\mathrm{C}}(\alpha^i)) = D^i_A(A^i)$. By (C1)–(C5), $(A^i, B^i) = (A, B)^i = (A \equiv B)^i$ is a congruence relation, so we get $t_{\mathrm{C}}(\varphi^i_\alpha(\beta)) = t_{\mathrm{C}}(\varphi^i_\alpha)(t_{\mathrm{C}}(\beta^i)) = D^i_A(B^i)$ from $D^i_A(A^i)$. (2) (Df − Elim): Assume that $\alpha^i =_{df} \varphi^{i-1}_\alpha$ and α^i. Then we have $t_{\mathrm{C}}(\alpha^i =_{df} \varphi^{i-1}_\alpha) = (t_{\mathrm{C}}(\alpha^i), t_{\mathrm{C}}(\varphi^{i-1}_\alpha)) = (A^i, D^{i-1}_A)$ and $t_{\mathrm{C}}(\alpha^i) = A^i$. Here we have the following derivation:

$$\dfrac{\begin{matrix} Hypo \\ [A^i] \end{matrix} \qquad \dfrac{\begin{matrix} Hypo \\ [(A^i, D^{i-1}_A)] \end{matrix} \quad \begin{matrix} (P1) \\ (A^i, D^{i-1}_A) \to (A^i \to D^{i-1}_A) \end{matrix}}{A^i \to D^{i-1}_A}\,(\mathrm{Mp})}{D^{i-1}_A}\,(\mathrm{Mp})$$

(3) (Df − Intro): Assume that $\alpha^{i+1} =_{df} \varphi^i_\alpha$ and φ^i_α. Then we have $t_{\mathrm{C}}(\alpha^{i+1} =_{df} \varphi^i_\alpha) = (t_{\mathrm{C}}(\alpha^{i+1}), t_{\mathrm{C}}(\varphi^i_\alpha)) = (A^{i+1}, D^i_A)$ and $t_{\mathrm{C}}(\varphi^i_\alpha) = D^i_A$. Here we have the following derivation:

$$\dfrac{\begin{matrix} Hypo \\ [D^i_A] \end{matrix} \qquad \dfrac{\dfrac{\begin{matrix} Hypo \\ [(A^{i+1}, D^i_A)] \end{matrix} \quad \begin{matrix} (E2) \\ (A^{i+1}, D^i_A) \to (D^i_A, A^{i+1}) \end{matrix}}{(D^i_A, A^{i+1})}\,(\mathrm{Mp}) \quad \begin{matrix} (P1) \\ (D^i_A, A^{i+1}) \to (D^i_A \to A^{i+1}) \end{matrix}}{D^i_A \to A^{i+1}}\,(\mathrm{Mp})}{A^{i+1}}\,(\mathrm{Mp})$$

(4) (IS): Assume that α^i. Then we have $t_{\mathrm{C}}(\alpha^i) = A^i$. Here we have the following derivation:

$$\dfrac{\begin{matrix} Hypo \\ [A^i] \end{matrix} \qquad \dfrac{\begin{matrix} Hypo \\ [(A^i, A^j)] \end{matrix} \quad \begin{matrix} (P1) \\ (A^i, A^j) \to (A^i \to A^j) \end{matrix}}{A^i \to A^j}\,(\mathrm{Mp})}{A^j}\,(\mathrm{Mp})$$

□

Corollary 4.7 *For any formula* φ *in* $\boldsymbol{FOR_C}$*,* $\varphi \in \mathbf{C}_n$ *implies* $t_{\mathrm{C}}(\varphi) \in \mathbf{PSC}_n$*.*

Proof. (5) ($\mathrm{IS_n}$): Assume that α^i. Then we have $t_{\mathrm{C}}(\alpha^i) = A^i$. Here we have the following derivation:

$$\dfrac{\underset{[A^i]}{Hypo} \quad \dfrac{\overset{(P5)}{(A^i, A^{i\pm n})} \quad \overset{(P1)}{(A^i, A^{i\pm n}) \to (A^i \to A^{i\pm n})}}{A^i \to A^{i\pm n}} \text{ (Mp)}}{A^{i\pm n}} \text{ (Mp)}$$

□

Proposition 4.8 *For any formula A in $\boldsymbol{FOR_P}$, $A \in \mathbf{PSC}$ generally does not imply $t_{\mathrm{P}}(A) \in \mathbf{C}_0$.*

Proof. We will investigate what is the matter to hold that $A \in \mathbf{PSC}$ implies $t_{\mathrm{P}}(A) \in \mathbf{C}_0$ by induction on the length of derivation in **PSC**.
(i) Base step: We have to check the provability of every axioms of **PSC** in $\mathbf{C}_0$ after a t_{P}-translation. But the classical axioms (A1)–(A10) are a trivial, so we omitted. Here we only check the cases of (E1)–(E3), (C1)–(C5) and (P1)–(P4).
(E1): $t_{\mathrm{P}}((A^i, A^i)) = (t_{\mathrm{P}}(A^i) = t_{\mathrm{P}}(A^i)) = (\alpha^i = \alpha^i) \in \mathbf{C}_0$ by (= Intro). Next, $t_{\mathrm{P}}((A^i, A^j)) = (t_{\mathrm{P}}(A^i) =_{df} t_{\mathrm{P}}(A^j)) = (\alpha^i =_{df} \alpha^j) \in \mathbf{C}_0$ for $i \neq j$ by (= IS). (E2): $t_{\mathrm{P}}((A^i, B^j) \to (B^j, A^i)) = t_{\mathrm{P}}((A^i, B^j)) \to t_{\mathrm{P}}((B^j, A^i)) = (t_{\mathrm{P}}(A^i) =_{df} t_{\mathrm{P}}(B^j)) \to (t_{\mathrm{P}}(B^j) =_{df} t_{\mathrm{P}}(A^i)) = (\alpha^i =_{df} \beta^j) \to (\beta^j =_{df} \alpha^i) \notin \mathbf{C}_0$. (E3): $t_{\mathrm{P}}((A^i, B^j) \wedge (B^j, C^k) \to (A^i, C^k)) = t_{\mathrm{P}}((A^i, B^j)) \wedge t_{\mathrm{P}}((B^j, C^k)) \to t_{\mathrm{P}}((A^i, C^k)) = (\alpha^i =_{df} \beta^j) \wedge (\beta^j =_{df} \gamma^k) \to (\alpha^i =_{df} \gamma^k) \notin \mathbf{C}_0$. (C1): $t_{\mathrm{P}}((A^i, B^j) \to (\neg A^i, \neg B^j))$ $= (t_{\mathrm{P}}(A^i) =_{df} t_{\mathrm{P}}(B^j)) \to (t_{\mathrm{P}}(\neg A^i) =_{df} t_{\mathrm{P}}(\neg B^j)) = (\alpha^i =_{df} \beta^j) \to (\neg\alpha^i =_{df} \neg\beta^j) \in \mathbf{C}_0$ by Proposition 3.2 (1) and (2). (C2)–(C5): let % be a logical connective of $\{\wedge, \vee, \to\}$ or identity $=$. Then we have $t_{\mathrm{P}}((A^i, B^j) \wedge (C^i, D^j) \to ((A\%C)^i, (B\%D)^j)) = (\alpha^i =_{df} \beta^j) \wedge (\gamma^i =_{df} \delta^j) \to (\alpha\%\gamma)^i =_{df} (\beta\%\delta)^j \in \mathbf{C}_0$ by Proposition 3.2 (3) and (4). (P1): $t_{\mathrm{P}}((A^i, B^j) \to (A^i \to B^j)) = (\alpha^i =_{df} \beta^j) \to (\alpha^i \to \beta^j) \in \mathbf{C}_0$ by (Df − Elim). (P2): $t_{\mathrm{P}}((A^i, B^j) \wedge (B \leftrightarrow C)^j \to (A^i, C^j)) = (\alpha^i =_{df} \beta^j) \wedge (\beta \leftrightarrow \gamma)^j \to (\alpha^i =_{df} \gamma^j) \in \mathbf{C}_0$ by Proposition 3.2 (5). (P3): $t_{\mathrm{P}}((A^i, B^j) \wedge (A \leftrightarrow C)^i \to (C^i, B^j)) = (\alpha^i =_{df} \beta^j) \wedge (\alpha \leftrightarrow \gamma)^i \to (\gamma^i =_{df} \beta^j) \in \mathbf{C}_0$ by Proposition 3.2 (6).
(P4): $t_{\mathrm{P}}((A^i, B^j) \to (A^{i\pm n}, B^{j\pm n})) = (\alpha^i =_{df} \beta^j) \to (\alpha^{i\pm n} =_{df} \beta^{j\pm n}) \notin \mathbf{C}_0$.
(ii) Induction step: We have to check the admissibility of (Mp) in $\mathbf{C}_0$ after a t_{P}-translation.
(Mp): Assume that A^i, $(A \to B)^i$ are provable in **PSC**. Then by I.H. $t_{\mathrm{P}}(A^i), t_{\mathrm{P}}((A \to B)^i)$, i.e., $\alpha^i, \alpha^i \to \beta^i$ hold in $\mathbf{C}_0$. So, we have the derivation of $t_{\mathrm{P}}(B^i) = \beta^i$ in $\mathbf{C}_0$ by ($\to$ Elim).

□

So, from the above results, the following conditions must be satisfied in $\mathbf{C}_0$ in order to hold that $A \in \mathbf{PSC}$ implies $t_{\mathrm{P}}(A) \in \mathbf{C}_0$:
(1) $(\alpha^i =_{df} \beta^j) \to (\beta^j =_{df} \alpha^i)$
(2) $(\alpha^i =_{df} \beta^j) \wedge (\beta^j =_{df} \gamma^k) \to (\alpha^i =_{df} \gamma^k)$
(3) $(\alpha^i =_{df} \beta^j) \to (\alpha^{i\pm n} =_{df} \beta^{j\pm n}) \quad (\exists n \geq 0)$

Here (1) means the exchangeability of indices between A^{i+1} and D^i_A. (2) means the transitivity of indices both A^{i+1} and D^i_A. (3) means the relativity of indices both A^{i+1} and D^i_A.

If we extend the systems $\mathbf{C}_0$ and $\mathbf{C}_n$ by adding the above conditions (1)–(3) for a definition constructor $=_{df}$ and call the result systems $\mathbf{C}'_0$ and $\mathbf{C}'_n$, then we have the following corollary.

Corollary 4.9 *For any formula A in $\boldsymbol{FOR_P}$,*
(1) $A \in \mathbf{PSC}$ implies $t_{\mathrm{P}}(A) \in \mathbf{C}'_0$.
(2) $A \in \mathbf{PSC}_n$ implies $t_{\mathrm{P}}(A) \in \mathbf{C}'_n$.

Proof. Trivial from the above discussion.

Therefore we can prove the following two theorems.

Theorem 4.10 *(1) For any formula φ in $\boldsymbol{FOR_C}$, $t_{\mathrm{P}}(t_{\mathrm{C}}(\varphi)) \leftrightarrow \varphi \in \mathbf{C}'_0$.*
(2) For any formula A in $\boldsymbol{FOR_P}$, $t_{\mathrm{C}}(t_{\mathrm{P}}(A)) \leftrightarrow A \in \mathbf{PSC}$.

Proof. Both cases are almost trivial and will be omitted. □

Theorem 4.11 *(1) For any formula φ in $\boldsymbol{FOR_C}$, $\varphi \in \mathbf{C}'_0$ if and only if $t_{\mathrm{C}}(\varphi) \in \mathbf{PSC}$.*
(2) For any formula A in $\boldsymbol{FOR_P}$, $A \in \mathbf{PSC}$ if and only if $t_{\mathrm{P}}(A) \in \mathbf{C}'_0$.

Proof. By using Proposition 4.6, 4.7 and Theorem 4.10. □

Corollary 4.12 *(1) For any formula φ in $\boldsymbol{FOR_C}$, $\varphi \in \mathbf{C}'_n$ if and only if $t_{\mathrm{C}}(\varphi) \in \mathbf{PSC}_n$.*
(2) For any formula A in $\boldsymbol{FOR_P}$, $A \in \mathbf{PSC}_n$ if and only if $t_{\mathrm{P}}(A) \in \mathbf{C}'_n$.

Hence we may conclude that two logics $\mathbf{C}'_0$ and $\mathbf{PSC}$ (, and also similary $\mathbf{C}'_n$ and $\mathbf{PSC}_n$) are syntactically equivalent by Definition 4.1, Theorem 4.10 and Theorem 4.11.

5 Conclusion

In this paper we have compared two logical systems $\mathbf{PSC}$ and $\mathbf{C}_n$ with a syntactical point of view, and we had the following results:
(1) For any formula φ in $\boldsymbol{FOR_C}$, $\varphi \in \mathbf{C}_0$ implies $t_{\mathrm{C}}(\varphi) \in \mathbf{PSC}$, but the converse direction, i.e., for any formula A in $\boldsymbol{FOR_P}$, $A \in \mathbf{PSC}$ implies $t_{\mathrm{P}}(A) \in \mathbf{C}_0$, generally does not hold under the two translations $t_{\mathrm{C}} : \boldsymbol{FOR_C} \to \boldsymbol{FOR_P}$ and $t_{\mathrm{P}} : \boldsymbol{FOR_P} \to \boldsymbol{FOR_C}$. So, we say that $\mathbf{C}_0$ is a sublogic of $\mathbf{PSC}$, or $\mathbf{PSC}$ is an extension of $\mathbf{C}_0$ by Definition 4.2. Similary, $\mathbf{PSC}_n$ is an extension of $\mathbf{C}_n$.
(2) If we extend the systems $\mathbf{C}_0$ and $\mathbf{C}_n$ by adding the following conditions (i)–(iii) for a definition constructor $=_{df}$ and call the result systems $\mathbf{C}'_0$ and $\mathbf{C}'_n$:
(i) $(\alpha^i =_{df} \beta^j) \to (\beta^j =_{df} \alpha^i)$ (exchangeability of indices)
(ii) $(\alpha^i =_{df} \beta^j) \wedge (\beta^j =_{df} \gamma^k) \to (\alpha^i =_{df} \gamma^k)$ (transitivity of indices)
(iii) $(\alpha^i =_{df} \beta^j) \to (\alpha^{i\pm n} =_{df} \beta^{j\pm n}) \quad (\exists n \geq 0)$ (relativity of indices)
Then, the converse direction also holds, i.e., for any formula A in $\boldsymbol{FOR_P}$, $A \in \mathbf{PSC}$ implies $t_{\mathrm{P}}(A) \in \mathbf{C}'_0$, and also $A \in \mathbf{PSC}_n$ implies $t_{\mathrm{P}}(A) \in \mathbf{C}'_n$. As the result, two logics $\mathbf{C}'_0$ and $\mathbf{PSC}$ (, and also similary $\mathbf{C}'_n$ and $\mathbf{PSC}_n$) are syntactically equivalent by Definition 4.1, Theorem 4.10 and Theorem 4.11.

Next, we will consider the differences between $\mathbf{PSC}$ and $\mathbf{C}_0$. At first, the pair-sentence constructor $((\_)^i, (\_)^j)$, where i, j are some stage numbers, has an equivalence relation:(E1)–(E3) and a relativity of indices:(P4), so we can calculate a referential cycle number of each pair sentence in $\mathbf{PSC}$, but not in $\mathbf{C}_0$. Secondly, $\mathbf{PSC}$ can deal with multiple pair sentences, i.e., $\{(S^0, \neg P^1), (P^0, \neg A^1), (A^0, S^1)\}$, but it is difficult to deal with such multiple definitions $\{S =_{df} \neg P, P =_{df} \neg A, A =_{df} A\}$ in $\mathbf{C}_0$ or $\mathbf{C}_n$.

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