Universidad del Atlántico
MATUA
Facultad de Ciencias Básicas
©Programa de Matemáticas
Vol. IV , № 2, (2017)
Revista De Matemáticas De la Universidad del Atlántico Páginas: 87–99

Desigualdades tipo Hermite-Hadamard para funciones cuya derivadas de n^{th} orden son s-convexas en el segundo sentido®

Hermite-Hadamard type inequalities for functions whose n^{th} order of derivatives are s-convex in the second sense®

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Resumen

En este documento, establecemos algunas desigualdades tipo Hermite-Hadamard para funciones cuyos n^{th} derivadas son s-convexas en el segundo sentido. Se derivan varios resultados conocidos. Las aplicaciones a medios especiales también son dado.

Palabras claves:
Desigualdad integral, Funciones s-convexas, desigualdad de Holder, desigualdad de potencia media.

Abstract

In this paper, we establish some Hermite-Hadamard type inequalities for functions whose n^{th} dirivatives are s-convex in the second sense. Several known results are derived . Applications to special means are also given.

Keywords:
integral inequality, s-convex function, Hölder inequality, power mean inequality.

1. Introduction

It is well known that convexity plays an important and central role in many areas, such as economic, finance, optimization, and game theory. Due to its diverse applications this concept has been extended and generalized in several directions.

We recall some definitions that are well known in the literature

Definition 1. *[10] A function $f : I \subset \mathbb{R} \to \mathbb{R}$ is said to be convex, if the following inequality*

$$f(tx + (1-t)y) \le tf(x) + (1-t)f(y)$$

holds for all $x, y \in I$ and $t \in [0,1]$.

Definition 2. *[5] A nonnegative function $f : I \subset [0,\infty) \to \mathbb{R}$ is said to be s-convex in the second sense for some fixed $s \in (0,1]$, if the following inequality*

$$f(tx + (1-t)y) \le t^s f(x) + (1-t)^s f(y)$$

holds for all $x, y \in I$ and $t \in [0,1]$.

Dragomir et al [8] gave the following Hermite-Hadamard inequality for s-convex

$$2^{s-1} f\left(\tfrac{a+b}{2}\right) \le \int_a^b f(t)\,dt \le \frac{f(a)+f(b)}{s+1}.$$

Cerone et al. [4] established the following identity for function

Lemma 1. *[6] Let $f : [a,b] \to \mathbb{R}$ be a mapping such that $f^{(n-1)}$ $(n \ge 1)$ is absolutely continuous on $[a,b]$. Then for any $x \in [a,b]$ one has the equality*

$$\begin{aligned}\int_a^b f(t)\,dt &= \sum_{k=0}^{n-1} \tfrac{1}{(k+1)!} \left[(x-a)^{k+1} f^{(k)}(a) + (-1)^k (b-x)^{k+1} f^{(k)}(b)\right] \\ &\quad + \tfrac{1}{n!}\int_a^b (x-t)^n f^{(n)}(t)\,dt.\end{aligned}$$

In [1] Ardiç used the above identity and established the following results for n-times differentiable convex functions

$$\begin{aligned}&\left|\int_a^b f(t)\,dt - \sum_{k=0}^{n-1} \tfrac{1}{(k+1)!} \left[(x-a)^{k+1} f^{(k)}(a) + (-1)^k (b-x)^{k+1} f^{(k)}(b)\right]\right| \\ \le\ & \tfrac{1}{n!(b-a)} \left(\left|f^{(n)}(a)\right| \left(\tfrac{(b-x)(x-a)^{n+1}}{n+1} + \tfrac{(b-x)^{n+2}}{(n+1)(n+2)} + \tfrac{(x-a)^{n+2}}{n+2}\right)\right. \\ & \left. + \left|f^{(n)}(b)\right| \left(\tfrac{(x-a)(b-x)^{n+1}}{n+1} + \tfrac{(x-a)^{n+2}}{(n+1)(n+2)} + \tfrac{(b-x)^{n+2}}{n+2}\right)\right),\end{aligned}$$

and

$$\begin{aligned}&\left|\int_a^b f(t)\,dt - \sum_{k=0}^{n-1} \tfrac{1}{(k+1)!} \left[(x-a)^{k+1} f^{(k)}(a) + (-1)^k (b-x)^{k+1} f^{(k)}(b)\right]\right| \\ \le\ & \tfrac{(b-a)^{\frac{1}{q}}}{n!} \left(\tfrac{(b-x)^{np+1}+(x-a)^{np+1}}{np+1}\right)^{\frac{1}{p}} \left(\frac{\left|f^{(n)}(a)\right|^q + \left|f^{(n)}(b)\right|^q}{2}\right)^{\frac{1}{q}}.\end{aligned}$$

Also Ardiç In [2] used the same identity and discussed the cases when $\left|f^{(n)}\right|$ and certain power of $\left|f^{(n)}\right|$ are quasiconvex.

Avci et al. [3] gave the following results for differentiable s-convex functions

$$\begin{aligned}&\left|\frac{(b-x)f(b)+(x-a)f(a)}{b-a}-\frac{1}{b-a}\int_a^b f(t)dt\right|\\ \leq\ & \frac{1}{(s+1)(s+2)}\left[\frac{(x-a)^2+(b-x)^2}{b-a}\right]\left|f'(x)\right|+\frac{1}{s+2}\left(\frac{(x-a)^2}{b-a}\left|f'(a)\right|+\frac{(b-x)^2}{b-a}\left|f'(b)\right|\right),\end{aligned}$$

and

$$\begin{aligned}&\left|\frac{(b-x)f(b)+(x-a)f(a)}{b-a}-\frac{1}{b-a}\int_a^b f(t)dt\right|\\ \leq\ & \left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{1}{s+1}\right)^{\frac{1}{q}}\frac{(x-a)^2\left(|f'(a)\ |^q+|f'(\ x)|^q\right)^{\frac{1}{q}}+(b-x)^2\left(|f'(x)|^q+|f'(b)|^q\right)^{\frac{1}{q}}}{b-a},\end{aligned}$$

and

$$\begin{aligned}&\left|\frac{(b-x)f(b)+(x-a)f(a)}{b-a}-\frac{1}{b-a}\int_a^b f(t)dt\right|\\ \leq\ & \frac{(x-a)^2}{b-a}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left(\frac{1}{s+2}\left|f'\ (a)\ \right|^q+\frac{1}{(s+1)(s+2)}\left|f'(\ x)\right|^q\right)^{\frac{1}{q}}\\ &+\frac{(b-x)^2}{b-a}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left(\frac{1}{s+2}\left|f'\ (b)\ \right|^q+\frac{1}{(s+1)(s+2)}\left|f'(\ x)\right|^q\right)^{\frac{1}{q}}.\end{aligned}$$

We note that the results established in [3] are generalization of those established in [8].

Kavurmaci et al. [8] established the following inequalities

$$\left|\frac{f(b)+f(a)}{2}-\frac{1}{b-a}\int_a^b f(t)dt\right|\leq\frac{b-a}{12}\left(\left|f'(a)\right|+\left|f'(\tfrac{a+b}{2})\right|+\left|f'(b)\right|\right),$$

and

$$\begin{aligned}&\left|\frac{f(b)+f(a)}{2}-\frac{1}{b-a}\int_a^b f(t)dt\right|\leq\frac{b-a}{4}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{1}{2}\right)^{\frac{1}{q}}\\ &\times\left(\left(\left|f'\ (a)\ \right|^q+\left|f'(\ \tfrac{a+b}{2})\right|^q\right)^{\frac{1}{q}}+\left(\left|f'\left(\tfrac{a+b}{2}\right)\right|^q+\left|f'\ (b)\right|^q\right)^{\frac{1}{q}}\right),\end{aligned}$$

and

$$\begin{aligned}&\left|\frac{f(b)+f(a)}{2}-\frac{1}{b-a}\int_a^b f(t)dt\right|\leq\frac{b-a}{8}\left(\frac{1}{3}\right)^{\frac{1}{q}}\\ &\times\left(\left(2\left|f'\ (a)\ \right|^q+\left|f'(\ \tfrac{a+b}{2})\right|^q\right)^{\frac{1}{q}}+\left(\left|f'\left(\tfrac{a+b}{2}\right)\right|^q+2\left|f'\ (b)\right|^q\right)^{\frac{1}{q}}\right).\end{aligned}$$

Motivated by the above results, in this paper by using Lemma 1 we establish some Hermite-Hadamard type inequalities for functions whose n^{th} derivatives are s-convex in the second sense. Several known results are derived . Applications to special means are also given.

2. Main results

Theorem 1. *Let $f : [a, b] \to \mathbb{R}$ be a mapping such that the derivative $f^{(n-1)}(n \geq 1)$ is absolutely continuous on $[a, b]$. If $\left|f^{(n)}\right|$ is s-convex in the second sense for some fixed $s \in (0, 1]$, then*

$$\begin{aligned} &\left|\int_a^b f(t)\,dt - \sum_{k=0}^{n-1} \tfrac{1}{(k+1)!} \left[(x-a)^{k+1} f^{(k)}(a) + (-1)^k (b-x)^{k+1} f^{(k)}(b)\right]\right| \\ \leq\ & \tfrac{(x-a)^{n+1}\left|f^{(n)}(a)\right| + (b-x)^{n+1}\left|f^{(n)}(b)\right|}{n!(n+s+1)} + \tfrac{(x-a)^{n+1}+(b-x)^{n+1}}{n!} B(s+1, n+1)\left|f^{(n)}(x)\right| \end{aligned} \tag{2.1}$$

holds for all $x \in [a, b]$.

Proof From Lemma 1, and property of modulus, we have

$$\begin{aligned} &\left|\int_a^b f(t)\,dt - \sum_{k=0}^{n-1} \tfrac{1}{(k+1)!} \left[(x-a)^{k+1} f^{(k)}(a) + (-1)^k (b-x)^{k+1} f^{(k)}(b)\right]\right| \\ \leq\ & \tfrac{1}{n!}\int_a^b |x-t|^n \left|f^{(n)}(t)\right| dt \\ =\ & \tfrac{1}{n!}\int_a^x (x-t)^n \left|f^{(n)}(t)\right| dt + \tfrac{1}{n!}\int_x^b (t-x)^n \left|f^{(n)}(t)\right| dt \\ =\ & \tfrac{(x-a)^{n+1}}{n!}\int_0^1 (1-\lambda)^n \left|f^{(n)}((1-\lambda)a + \lambda x)\right| d\lambda \\ & + \tfrac{(b-x)^{n+1}}{n!}\int_0^1 \lambda^n \left|f^{(n)}((1-\lambda)x + \lambda b)\right| d\lambda. \end{aligned} \tag{2.2}$$

Now, using s-convexity of $\left|f^{(n)}\right|$, (2.2) gives

$$\begin{aligned}
&\left|\int_a^b f(t)\,dt - \sum_{k=0}^{n-1}\frac{1}{(k+1)!}\left[(x-a)^{k+1}f^{(k)}(a)+(-1)^k(b-x)^{k+1}f^{(k)}(b)\right]\right| \\
\leq\ & \frac{(x-a)^{n+1}}{n!}\left(\left|f^{(n)}(a)\right|\int_0^1(1-\lambda)^{n+s}\,d\lambda+\left|f^{(n)}(x)\right|\int_0^1\lambda^s(1-\lambda)^n\,d\lambda\right) \\
& \frac{(b-x)^{n+1}}{n!}\left(\left|f^{(n)}(x)\right|\int_0^1(1-\lambda)^s\lambda^n d\lambda+\left|f^{(n)}(b)\right|\int_0^1\lambda^{n+s}d\lambda\right) \\
=\ & \frac{(x-a)^{n+1}\left|f^{(n)}(a)\right|+(b-x)^{n+1}\left|f^{(n)}(b)\right|}{n!(n+s+1)}+\frac{(x-a)^{n+1}+(b-x)^{n+1}}{n!}B(s+1,n+1)\left|f^{(n)}(x)\right|.
\end{aligned}$$

This completes the proof.

Corollary 1. *In Theorem 1, if we take $x=\frac{a+b}{2}$ we obtain*

$$\begin{aligned}
&\left|\int_a^b f(t)\,dt - \sum_{k=0}^{n-1}\frac{1}{(k+1)!}\left(\frac{b-a}{2}\right)^{k+1}\left(f^{(k)}(a)+(-1)^k f^{(k)}(b)\right)\right| \\
\leq\ & \frac{(b-a)^{n+1}\left(\left|f^{(n)}(a)\right|+\left|f^{(n)}(b)\right|\right)}{n!(n+s+1)2^{n+1}}+\frac{(b-a)^{n+1}}{n!2^n}B(s+1,n+1)\left|f^{(n)}\left(\frac{a+b}{2}\right)\right|.
\end{aligned}$$

Moreover if we use the s-convexity of $\left|f^{(n)}\right|$ we get

$$\begin{aligned}
&\left|\int_a^b f(t)\,dt - \sum_{k=0}^{n-1}\frac{1}{(k+1)!}\left(\frac{b-a}{2}\right)^{k+1}\left(f^{(k)}(a)+(-1)^k f^{(k)}(b)\right)\right| \\
\leq\ & \left(\frac{(b-a)^{n+1}}{n!(n+s+1)2^{n+1}}+\frac{(b-a)^{n+1}}{n!(s+1)2^{n-1+s}}B(s+1,n+1)\right)\left(\left|f^{(n)}(a)\right|+\left|f^{(n)}(b)\right|\right).
\end{aligned}$$

Corollary 2. *In Theorem 1, if we take $s=1$ we have*

$$\begin{aligned}
&\left|\int_a^b f(t)\,dt - \sum_{k=0}^{n-1}\frac{1}{(k+1)!}\left[(x-a)^{k+1}f^{(k)}(a)+(-1)^k(b-x)^{k+1}f^{(k)}(b)\right]\right| \\
\leq\ & \frac{(x-a)^{n+1}\left|f^{(n)}(a)\right|+(b-x)^{n+1}\left|f^{(n)}(b)\right|}{n!(n+2)}+\frac{(x-a)^{n+1}+(b-x)^{n+1}}{(n+2)!}\left|f^{(n)}(x)\right|.
\end{aligned}$$

Moreover if we choose $x=\frac{a+b}{2}$ we get

$$\begin{aligned}
&\left|\int_a^b f(t)\,dt - \sum_{k=0}^{n-1}\frac{1}{(k+1)!}\left(\frac{b-a}{2}\right)^{k+1}\left(f^{(k)}(a)+(-1)^k f^{(k)}(b)\right)\right| \\
\leq\ & \frac{(b-a)^{n+1}}{(n+2)!2^{n+1}}\left((n+1)\left(\left|f^{(n)}(a)\right|+\left|f^{(n)}(b)\right|\right)+2\left|f^{(n)}\left(\frac{a+b}{2}\right)\right|\right).
\end{aligned}$$

Corollary 3. *In Theorem 1, if we take $s = 1$, we get*

$$\begin{aligned}
&\left|\int_a^b f(t)\,dt - \sum_{k=0}^{n-1} \tfrac{1}{(k+1)!}\ \left[(x-a)^{k+1} f^{(k)}(a) + (-1)^k (b-x)^{k+1} f^{(k)}(b)\right]\right| \\
\leq\ & \tfrac{(x-a)^{n+1}}{n!(n+2)} \left|f^{(n)}(a)\right| + \tfrac{(x-a)^{n+1}+(b-x)^{n+1}}{(n+2)!} \left|f^{(n)}(x)\right| + \tfrac{(b-x)^{n+1}}{n!(n+2)} \left|f^{(n)}(b)\right|.
\end{aligned}$$

Moreover if we choose $x = \frac{a+b}{2}$ we obtain

$$\begin{aligned}
&\left|\int_a^b f(t)\,dt - \sum_{k=0}^{n-1} \tfrac{1}{(k+1)!}\ \left(\tfrac{b-a}{2}\right)^{k+1} \left(f^{(k)}(a) + (-1)^k f^{(k)}(b)\right)\right| \\
\leq\ & \tfrac{(b-a)^{n+1}}{2^{n+1}(n+2)!} \left((n+1)\left|f^{(n)}(a)\right| + 2\left|f^{(n)}\left(\tfrac{a+b}{2}\right)\right| + (n+1)\left|f^{(n)}(b)\right|\right) \\
\leq\ & \tfrac{(b-a)^{n+1}}{2^{n+1}(n+1)!} \left(\left|f^{(n)}(a)\right| + \left|f^{(n)}(b)\right|\right).
\end{aligned}$$

Remark 1. *Theorem 1 will be reduces to Theorem 5 from [3] if we choose $n = 1$, also Theorem 2.1 from [4]. And Theorem 4 from [8] if we take $n = s = 1$, Moreover if we put $x = \frac{a+b}{2}$ we obtain Corollary 2 from [8] and by using the convexity of $|f'|$ we get Theorem 2.2 from [7].*

Theorem 2. *Let $f : [a,b] \to \mathbb{R}$ be a mapping such that the derivative $f^{(n-1)}(n \geq 1)$ is absolutely continuous on $[a,b]$. If $\left|f^{(n)}\right|^q$ is s-convex in the second sense for some fixed $s \in (0,1]$, where $q \geq 1$, then the following inequality*

$$\begin{aligned}
&\left|\int_a^b f(t)\,dt - \sum_{k=0}^{n-1} \tfrac{1}{(k+1)!}\ \left[(x-a)^{k+1} f^{(k)}(a) + (-1)^k (b-x)^{k+1} f^{(k)}(b)\right]\right| \\
\leq\ & \tfrac{(x-a)^{n+1}}{n!(n+1)^{1-\frac{1}{q}}} \left(\tfrac{\left|f^{(n)}(a)\right|^q}{n+s+1} + B(s+1, n+1)\left|f^{(n)}(x)\right|^q\right)^{\frac{1}{q}} \\
&+ \tfrac{(b-x)^{n+1}}{n!(n+1)^{1-\frac{1}{q}}} \left(B(n+1, s+1)\left|f^{(n)}(x)\right|^q + \tfrac{\left|f^{(n)}(b)\right|^q}{n+s+1}\right)^{\frac{1}{q}} \qquad (2.3)
\end{aligned}$$

holds for all $x \in [a,b]$.

Proof From Lemma 1, property of modulus, and power mean inequality, we have

$$\begin{aligned}
&\left|\int_a^b f(t)\,dt - \sum_{k=0}^{n-1} \tfrac{1}{(k+1)!}\ \left[(x-a)^{k+1} f^{(k)}(a) + (-1)^k (b-x)^{k+1} f^{(k)}(b)\right]\right| \\
\leq\ & \tfrac{(x-a)^{n+1}}{n!}\int_0^1 (1-\lambda)^n \left|f^{(n)}((1-\lambda)a+\lambda x)\right| d\lambda \\
&+\tfrac{(b-x)^{n+1}}{n!}\int_0^1 \lambda^n \left|f^{(n)}((1-\lambda)x+\lambda b)\right| d\lambda \qquad (2.4)\\
\leq\ & \tfrac{(x-a)^{n+1}}{n!}\left(\int_0^1 (1-\lambda)^n\,d\lambda\right)^{1-\frac{1}{q}}\left(\int_0^1 (1-\lambda)^n \left|f^{(n)}((1-\lambda)a+\lambda x)\right|^q d\lambda\right)^{\frac{1}{q}} \\
&+\tfrac{(b-x)^{n+1}}{n!}\left(\int_0^1 \lambda^n d\lambda\right)^{1-\frac{1}{q}}\left(\int_0^1 \lambda^n \left|f^{(n)}((1-\lambda)x+\lambda b)\right|^q d\lambda\right)^{\frac{1}{q}} \\
=\ & \tfrac{(x-a)^{n+1}}{n!(n+1)^{1-\frac{1}{q}}}\left(\int_0^1 (1-\lambda)^n \left|f^{(n)}((1-\lambda)a+\lambda x)\right|^q d\lambda\right)^{\frac{1}{q}} \\
&+\tfrac{(b-x)^{n+1}}{n!(n+1)^{1-\frac{1}{q}}}\left(\int_0^1 \lambda^n \left|f^{(n)}((1-\lambda)x+\lambda b)\right|^q d\lambda\right)^{\frac{1}{q}}. \qquad (2.5)
\end{aligned}$$

Since $\left|f^{(n)}\right|^q$ is s-convex function , we deduce

$$\begin{aligned}
&\left|\int_a^b f(t)\,dt - \sum_{k=0}^{n-1} \tfrac{1}{(k+1)!}\ \left[(x-a)^{k+1} f^{(k)}(a) + (-1)^k (b-x)^{k+1} f^{(k)}(b)\right]\right| \\
\leq\ & \tfrac{(x-a)^{n+1}}{n!(n+1)^{1-\frac{1}{q}}}\left(\int_0^1 (1-\lambda)^n \left|f^{(n)}((1-\lambda)a+\lambda x)\right|^q d\lambda\right)^{\frac{1}{q}} \\
&+\tfrac{(b-x)^{n+1}}{n!(n+1)^{1-\frac{1}{q}}}\left(\int_0^1 \lambda^n \left|f^{(n)}((1-\lambda)x+\lambda b)\right|^q d\lambda\right)^{\frac{1}{q}} \\
\leq\ & \tfrac{(x-a)^{n+1}}{n!(n+1)^{1-\frac{1}{q}}}\left(\left|f^{(n)}(a)\right|^q \int_0^1 (1-\lambda)^{n+s}\,d\lambda + \left|f^{(n)}(x)\right|^q \int_0^1 \lambda^s (1-\lambda)^n\,d\lambda\right)^{\frac{1}{q}}
\end{aligned}$$

$$+\frac{(b-x)^{n+1}}{n!(n+1)^{1-\frac{1}{q}}}\left(\left|f^{(n)}(x)\right|^q\int_0^1\lambda^n(1-\lambda)^s\,d\lambda+\left|f^{(n)}(b)\right|^q\int_0^1\lambda^{n+s}d\lambda\right)^{\frac{1}{q}}$$

$$=\frac{(x-a)^{n+1}}{n!(n+1)^{1-\frac{1}{q}}}\left(\frac{\left|f^{(n)}(a)\right|^q}{n+s+1}+B(s+1,n+1)\left|f^{(n)}(x)\right|^q\right)^{\frac{1}{q}}$$

$$+\frac{(b-x)^{n+1}}{n!(n+1)^{1-\frac{1}{q}}}\left(B(n+1,s+1)\left|f^{(n)}(x)\right|^q+\frac{\left|f^{(n)}(b)\right|^q}{n+s+1}\right)^{\frac{1}{q}},$$

which is the desired result.

Corollary 4. *In Theorem 2, if we choose $x=\frac{a+b}{2}$ we obtain*

$$\left|\int_a^b f(t)\,dt-\sum_{k=0}^{n-1}\frac{1}{(k+1)!}\left(\frac{b-a}{2}\right)^{k+1}\left(f^{(k)}(a)+(-1)^k f^{(k)}(b)\right)\right|$$

$$\le\frac{(b-a)^{n+1}}{2^{n+1}n!(n+1)^{1-\frac{1}{q}}}\left(\left(\frac{\left|f^{(n)}(a)\right|^q}{n+s+1}+B(s+1,n+1)\left|f^{(n)}\left(\frac{a+b}{2}\right)\right|^q\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(B(n+1,s+1)\left|f^{(n)}\left(\frac{a+b}{2}\right)\right|^q+\frac{\left|f^{(n)}(b)\right|^q}{n+s+1}\right)^{\frac{1}{q}}\right)$$

$$\le\frac{(b-a)^{n+1}}{2^{n+1}n!(n+1)^{1-\frac{1}{q}}}\left(\left(\frac{1+2^{1-s}}{(n+s+1)s+1}\left|f^{(n)}(a)\right|^q+\frac{2^{1-s}B(s+1,n+1)}{s+1}\left|f^{(n)}(b)\right|^q\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\frac{2^{1-s}B(s+1,n+1)}{s+1}\left|f^{(n)}\left(\frac{a+b}{2}\right)\right|^q+\frac{1+2^{1-s}}{(n+s+1)s+1}\left|f^{(n)}(b)\right|^q\right)^{\frac{1}{q}}\right).$$

Moreover if we use the s-convexity of $\left|f^{(n)}\right|^p$ we get

$$\left|\int_a^b f(t)\,dt-\sum_{k=0}^{n-1}\frac{1}{(k+1)!}\left(\frac{b-a}{2}\right)^{k+1}\left(f^{(k)}(a)+(-1)^k f^{(k)}(b)\right)\right|$$

$$\le\frac{(b-a)^{n+1}}{2^{n+1}n!(n+1)^{1-\frac{1}{p}}}\left(\left(\left(\frac{2^{1-s}B(n+1,s+1)}{s+1}+\frac{1}{n+s+1}\right)\left|f^{(n)}(a)\right|^p+\frac{2^{1-s}B(n+1,s+1)}{s+1}\left|f^{(n)}(b)\right|^p\right)^{\frac{1}{p}}\right.$$

$$\left.+\left(\frac{2^{1-s}B(n+1,s+1)}{s+1}\left|f^{(n)}(a)\right|^p+\left(\frac{2^{1-s}B(n+1,s+1)}{s+1}+\frac{1}{n+s+1}\right)\left|f^{(n)}(b)\right|^p\right)^{\frac{1}{p}}\right).$$

Corollary 5. *In Theorem 2, if we take $s=1$, we get*

$$\left|\int_a^b f(t)\,dt-\sum_{k=0}^{n-1}\frac{1}{(k+1)!}\left[(x-a)^{k+1}f^{(k)}(a)+(-1)^k(b-x)^{k+1}f^{(k)}(b)\right]\right|$$

$$\le\frac{(x-a)^{n+1}}{(n+1)!(n+2)^{\frac{1}{q}}}\left((n+1)\left|f^{(n)}(a)\right|^q+\left|f^{(n)}(x)\right|^q\right)^{\frac{1}{q}}$$

$$+\frac{(b-x)^{n+1}}{(n+1)!(n+2)^{\frac{1}{q}}}\left(\left|f^{(n)}(x)\right|^q+(n+1)\left|f^{(n)}(b)\right|^q\right)^{\frac{1}{q}}.$$

Moreover if we take $x = \frac{a+b}{2}$, we get

$$\left|\int_a^b f(t)\,dt - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \left(\frac{b-a}{2}\right)^{k+1} \left(f^{(k)}(a) + (-1)^k f^{(k)}(b)\right)\right|$$
$$\leq \frac{(b-a)^{n+1}}{2^{n+1}(n+1)!(n+2)^{\frac{1}{q}}} \left(\left((n+1)\left|f^{(n)}(a)\right|^q + \left|f^{(n)}\left(\frac{a+b}{2}\right)\right|^q\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\left|f^{(n)}\left(\frac{a+b}{2}\right)\right|^q + (n+1)\left|f^{(n)}(b)\right|^q\right)^{\frac{1}{q}}\right)$$
$$\leq \frac{(b-a)^{n+1}}{2^{n+1+\frac{1}{p}}(n+1)!(n+2)^{\frac{1}{q}}} \left(\left((n+2)\left|f^{(n)}(a)\right|^q + \left|f^{(n)}(b)\right|^q\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\left|f^{(n)}(a)\right|^q + (n+2)\left|f^{(n)}(b)\right|^q\right)^{\frac{1}{q}}\right).$$

Corollary 6. *In Theorem 2, if we take $n = s = 1$ and $x = \frac{a+b}{2}$, and using the convexity of $|f'|^p$we get*

$$\left|\frac{1}{b-a}\int_a^b f(t)\,dt - \frac{f(a)+f(b)}{2}\right|$$
$$\leq \frac{b-a}{8}\left(\left(\frac{5|f'(a)|^p+|f'(b)|^p}{6}\right)^{\frac{1}{p}} + \left(\frac{|f'(a)|^p+5|f'(b)|^p}{6}\right)^{\frac{1}{p}}\right).$$

Remark 2. *Theorem 2 will be reduces to Theorem 7 from [3] if we choose $n = 1$, also Theorem 2.7 from [4]. And Theorem 7 from [8] if we take $n = s = 1$, Moreover if we put $x = \frac{a+b}{2}$ we obtain Corollary 4 from [8].*

Theorem 3. *Under the same hypotheses of Theorem 2, we have the following inequality*

$$\left|\int_a^b f(t)\,dt - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \left[(x-a)^{k+1} f^{(k)}(a) + (-1)^k (b-x)^{k+1} f^{(k)}(b)\right]\right|$$
$$\leq \frac{1}{n!}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{1}{1+s}\right)^{\frac{1}{q}}\left((x-a)^{n+1}\left(\left|f^{(n)}(a)\right|^q + \left|f^{(n)}(x)\right|^q\right)^{\frac{1}{q}}\right.$$
$$\left.+(b-x)^{n+1}\left(\left|f^{(n)}(x)\right|^q + \left|f^{(n)}(b)\right|^q\right)^{\frac{1}{q}}\right) \tag{2.6}$$

holds for all $x \in [a, b]$ where $\frac{1}{p} + \frac{1}{q} = 1$.

Proof From Lemma 1, property of modulus, and Hölder inequality, we have

$$\left|\int_a^b f(t)\,dt - \sum_{k=0}^{n-1}\frac{1}{(k+1)!}\left[(x-a)^{k+1} f^{(k)}(a) + (-1)^k (b-x)^{k+1} f^{(k)}(b)\right]\right|$$

$$\leq \frac{(x-a)^{n+1}}{n!}\int_0^1 (1-\lambda)^n \left|f^{(n)}((1-\lambda)a+\lambda x)\right| d\lambda$$

$$+\frac{(b-x)^{n+1}}{n!}\int_0^1 \lambda^n \left|f^{(n)}((1-\lambda)x+\lambda b)\right| d\lambda.$$

$$\leq \frac{(x-a)^{n+1}}{n!}\left(\int_0^1 (1-\lambda)^p\,d\lambda\right)^{\frac{1}{p}}\left(\int_0^1 \left|f^{(n)}((1-\lambda)a+\lambda x)\right|^q d\lambda\right)^{\frac{1}{q}}$$

$$+\frac{(b-x)^{n+1}}{n!}\left(\int_0^1 \lambda^p d\lambda\right)^{\frac{1}{p}}\left(\int_0^1 \left|f^{(n)}((1-\lambda)x+\lambda b)\right|^q d\lambda\right)^{\frac{1}{q}}$$

$$= \frac{(x-a)^{n+1}}{n!}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\int_0^1 \left|f^{(n)}((1-\lambda)a+\lambda x)\right|^q d\lambda\right)^{\frac{1}{q}}$$

$$+\frac{(b-x)^{n+1}}{n!}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\int_0^1 \left|f^{(n)}((1-\lambda)x+\lambda b)\right|^q d\lambda\right)^{\frac{1}{q}}. \tag{2.7}$$

Since $\left|f^{(n)}\right|^q$ is s-convex, (2.7) gives

$$\left|\int_a^b f(t)\,dt - \sum_{k=0}^{n-1}\frac{1}{(k+1)!}\left[(x-a)^{k+1} f^{(k)}(a) + (-1)^k (b-x)^{k+1} f^{(k)}(b)\right]\right|$$

$$\leq \left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{(x-a)^{n+1}}{n!}\left(\int_0^1 (1-\lambda)^s \left|f^{(n)}(a)\right|^q + \lambda^s \left|f^{(n)}(x)\right|^q d\lambda\right)^{\frac{1}{q}}\right.$$

$$\left. + \frac{(b-x)^{n+1}}{n!}\left(\int_0^1 (1-\lambda)^s \left|f^{(n)}(x)\right|^q + \lambda^s \left|f^{(n)}(b)\right|^q d\lambda\right)^{\frac{1}{q}}\right)$$

$$= \frac{1}{n!}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left((x-a)^{n+1}\left(\frac{\left|f^{(n)}(a)\right|^q + \left|f^{(n)}(x)\right|^q}{1+s}\right)^{\frac{1}{q}}\right.$$

$$\left. + (b-x)^{n+1}\left(\frac{\left|f^{(n)}(x)\right|^q + \left|f^{(n)}(b)\right|^q}{1+s}\right)^{\frac{1}{q}}\right).$$

The proof is thus achieved.

Corollary 7. *In Theorem 3, if we choose $x = \frac{a+b}{2}$ we obtain*

$$\leq \frac{1}{n!}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{1}{1+s}\right)^{\frac{1}{q}}\left((x-a)^{n+1}\left(\left|f^{(n)}(a)\right|^q+\left|f^{(n)}(x)\right|^q\right)^{\frac{1}{q}}\right.$$
$$\left.+(b-x)^{n+1}\left(\left|f^{(n)}(x)\right|^q+\left|f^{(n)}(b)\right|^q\right)^{\frac{1}{q}}\right)$$
$$\left|\int_a^b f(t)\,dt-\sum_{k=0}^{n-1}\frac{1}{(k+1)!}\left(\frac{b-a}{2}\right)^{k+1}\left(f^{(k)}(a)+(-1)^k f^{(k)}(b)\right)\right|$$
$$\leq \frac{(b-a)^{n+1}}{n!2^{n+1}}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{1}{1+s}\right)^{\frac{1}{q}}\left(\left(\left|f^{(n)}(a)\right|^q+\left|f^{(n)}\left(\frac{a+b}{2}\right)\right|^q\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\left|f^{(n)}\left(\frac{a+b}{2}\right)\right|^q+\left|f^{(n)}(b)\right|^q\right)^{\frac{1}{q}}\right)$$
$$\leq \frac{(b-a)^{n+1}}{n!2^{n+1}}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{1}{1+s}\right)^{\frac{1}{q}}\left(\left(\left(1+\frac{2^{1-s}}{s+1}\right)\left|f^{(n)}(a)\right|^q+\frac{2^{1-s}}{s+1}\left|f^{(n)}(b)\right|^q\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\frac{2^{1-s}}{s+1}\left|f^{(n)}(a)\right|^q+\left(1+\frac{2^{1-s}}{s+1}\right)\left|f^{(n)}(b)\right|^q\right)^{\frac{1}{q}}\right).$$

Corollary 8. *In Theorem 3, if we take $s = 1$, we get*

$$\left|\int_a^b f(t)\,dt-\sum_{k=0}^{n-1}\frac{1}{(k+1)!}\left[(x-a)^{k+1}f^{(k)}(a)+(-1)^k(b-x)^{k+1}f^{(k)}(b)\right]\right|$$
$$\leq \frac{1}{n!}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{1}{2}\right)^{\frac{1}{q}}\left((x-a)^{n+1}\left(\left|f^{(n)}(a)\right|^q+\left|f^{(n)}(x)\right|^q\right)^{\frac{1}{q}}\right.$$
$$\left.+(b-x)^{n+1}\left(\left|f^{(n)}(x)\right|^q+\left|f^{(n)}(b)\right|^q\right)^{\frac{1}{q}}\right).$$

Moreover if we put $x = \frac{a+b}{2}$, we get

$$\left|\int_a^b f(t)\,dt-\sum_{k=0}^{n-1}\frac{1}{(k+1)!}\left(\frac{b-a}{2}\right)^{k+1}\left(f^{(k)}(a)+(-1)^k f^{(k)}(b)\right)\right|$$
$$\leq \frac{(b-a)^{n+1}}{n!2^{n+1+\frac{1}{q}}}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\left(\left|f^{(n)}(a)\right|^q+\left|f^{(n)}\left(\frac{a+b}{2}\right)\right|^q\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\left|f^{(n)}\left(\frac{a+b}{2}\right)\right|^q+\left|f^{(n)}(b)\right|^q\right)^{\frac{1}{q}}\right)$$
$$\leq \frac{(b-a)^{n+1}}{n!2^{n+1+\frac{2}{q}}}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\left(3\left|f^{(n)}(a)\right|^q+\left|f^{(n)}\left(\frac{a+b}{2}\right)\right|^q\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\left|f^{(n)}(a)\right|^q+3\left|f^{(n)}(b)\right|^q\right)^{\frac{1}{q}}\right).$$

Remark 3. *Theorem 3 will be reduces to Theorem 6 from [3] if we choose* $n = 1$, *also Theorem 2.4 from [4]. And Theorem 5 from [8] if we take* $n = s = 1$, *Moreover if we put* $x = \frac{a+b}{2}$ *we obtain Corollary 3 from [8].*

3. Applications to special means

We shall consider the following special means
The arithmetic mean: $A(a,b) = \frac{a+b}{2}$
The p-logarithmic mean: $L_p(a,b) = \begin{cases} \left(\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right)^{\frac{1}{p}} \text{ if } a \neq b \\ a \text{ if } a = b \end{cases}$, $p \in \mathbb{R} \setminus \{-1, 0\}$, $a, b > 0$.

Proposition 1. *Let* $a, b \in \mathbb{R}$, *with* $0 < a < b$, *and* $0 < s \leq 1$, *we have*

$$\left| L_{n+s}^{n+s}(a,b) - \sum_{k=0}^{n-1} \frac{(b-a)^k \left(\prod_{p=0}^{k-1}(n+s-p)\right)}{(k+1)!2^k} \left(A\left(a^{n+s-k}, b^{n+s-k}\right) + \left(\frac{(-1)^k-1}{2}\right) b^{n+s-k}\right) \right|$$
$$\leq \frac{(b-a)^n \left(\prod_{p=0}^{n-1}(n+s-p)\right)}{n!2^n} \left(\frac{1}{n+s+1} A(a^s, b^s) + B(s+1, n+1) A^s(a,b)\right).$$

Proof The proof is immediate from Theorem 1 with $x = \frac{a+b}{2}$, applied to the function $f(t) = t^{n+s}$ with $n \in \mathbb{N}$. Clearly we have, $f^{(k)}(t) = \left(\prod_{p=0}^{k-1}(n+s-p)\right) t^{n+s-k}$ and $f^{(n)}(t) = \left(\prod_{p=0}^{n-1}(n+s-p)\right) t^s$ which is a s-convex function.

Proposition 2. *Let* $a, b \in \mathbb{R}$, *with* $0 < a < b$, $0 < s \leq 1$, *and* $q > 1$ *such that* $\frac{1}{p} + \frac{1}{q} = 1$, *we have*

$$\left| L_{n+s}^{n+s}(a,b) - \sum_{k=0}^{n-1} \frac{(b-a)^k \left(\prod_{p=0}^{k-1}(n+s-p)\right)}{(k+1)!2^k} \left(A\left(a^{n+s-k}, b^{n+s-k}\right) + \left(\frac{(-1)^k-1}{2}\right) b^{n+s-k}\right) \right|$$
$$\leq \frac{(b-a)^n \left(\prod_{p=0}^{n-1}(n+s-p)\right)}{n!2^{n+1+\frac{1}{q}}} \left(\frac{1}{p+1}\right)^{\frac{1}{p}} \left((a^{sq} + A^{sq}(a,b))^{\frac{1}{q}} + (A^{sq}(a,b) + b^{sq})^{\frac{1}{q}}\right).$$

Proof The proof is immediate from Theorem 1 with $x = \frac{a+b}{2}$, applied to the function $f(t) = t^{n+s}$ with $n \in \mathbb{N}$.

4. Competing of interest

The authors declare that there is no conflict of interests regarding the publication of this paper. Also our research is not involving any human participants and/or animals.

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