Universidad del Atlántico
MATUA
Facultad de Ciencias Básicas
©Programa de Matemáticas
Vol. IV , № 2, (2017)
Revista De Matemática de la Universidad del Atlántico Páginas: 77–86

Sobre la Existencia y Unicidad de Soluciones para un Sistema de Ecuaciones No Lineal.

On the Existence and Uniqueness of the Solutions of a System of Non-Linear Differential Equations.

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Resumen

En este artículo demostraremos un teorema de existencia y unicidad para un sistema de ecuaciones lineales que incluye como consecuencia a los Modelos de Volterra

Palabras claves: Sistemas No Lineales, Ecuaciones diferenciales, Modelos de Volterra.

Abstract

In this work we present an existence and uniqueness Theorem for a very especial class of a non-linear system of differential equations which include The Volterra Models..

Keywords: non-linear systems, differential equations, Volterra Models.

1. Introduction

The study of differential equations have multiple impact in science and everyday life. It is the case that the Lotka-Volterra type of models are the most people work on, but also press-predator models and and competitive models as well [1][2].
It is also well know the extraordinary development of the theory o differential equations with finite or infinite delay.
In [1], Montes de Oca and Miguel Vivas, studied the system of differential of Lotka-Volterra type with

infinite delay.

$$\begin{cases} x_1'(t) = x_1(t)[a(t) - b(t)x_1(t) - c_1(t)\int_{-\infty}^{t} k_1(t-s)x_2(s)\,ds] & \\ x_2'(t) = x_2(t)[d(t) - f(t)x_2(t) - e_1(t)\int_{-\infty}^{t} k_2(t-s)x_1(s)\,ds] & \text{if } t \geq t_0 \\ x_1(t) = \phi_1(t) \wedge x_2(t) = \phi_2(t) & \text{if } t < t_0 \end{cases} \tag{1}$$

where the derivation at t_0 should be interpreted as the derivative from the right , that is to say; $x_i'(t_0) = x_{i_+}'(t_0)$ for $i = 1, 2$ and $a(t), b(t), c(t), d(t), e(t)$, y $f(t)$ are bounded positive from above and from below with positive constants which satisfies

$$c(t)d(t) \leq a(t)f(t)$$

$$b(t)d(t) \leq (t)a(t)$$

And also $k_i : [0, +\infty) \longrightarrow [0, +\infty)$ are continuous and positive kernels such that: $\int_0^{+\infty} k_i(s)\,ds = 1$ and the $\phi_i's$ are the initial conditions.
In this work we present an existence and uniqueness theorem for the more general case than (1), System (1) can be written as:

$$x'(t) - h(t, x(t)) - A(x(t)) \int_{-\infty}^{t} g(t, \tau, x(\tau))\,d\tau \tag{2}$$

where

$$x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}; A(x) = \begin{pmatrix} x_1 & 0 \\ 0 & x_2 \end{pmatrix}$$

$$h : (-\infty, +\infty) \times \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$

$$g : \{(t, s) \in \mathbb{R}^2 / s \leq t\} \times \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$

are continuous and are given by

$$h(t, x) = \begin{pmatrix} a(t)x_1 - b(t)(x_1)^2 \\ d(t)x_2 - e(t)(x_2)^2 \end{pmatrix} \tag{3}$$

$$g(t, s, x) = \begin{pmatrix} c(t)k_1(t-s)x_2 \\ f(t)k_2(t-s)x_1 \end{pmatrix} \tag{4}$$

It is very important to note that:

$$||A(t)|| \leq |x_1| + |x_2| = ||x|| \quad (*)$$

1.1. Preliminary Results

Lemma 1. *The function $h : (-\infty, +\infty) \times \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ defined by* (3) *satisfied a local Lipschitz condition on x, in the sense that for each $(t_0, x_0) \in (\mathbb{R}^* \cup \{0\}) \times \mathbb{R}^2$ and each $M > 0$, There exist $k > 0$ such that, if $||x - x_0|| \leq M$; $||\bar{x} - x_0|| \leq M$ and $|t - t_0| \leq M$, then;*

$$||h(t, x) - h(t, \bar{x})|| \leq k||x - \bar{x}||$$

where

$$||(x_1, x_2)|| = |x_1| + |x_2|$$

Proof:
Let us observe that

$$k = \text{máx}\{a_M + 2Mb_M, d_M + 2Me_M\}$$

satisfies the previous conditions; where

$$a_L \leq a(t) \leq a_M \;\; , \;\; b_L \leq b(t) \leq b_M$$

$$d_L \leq d(t) \leq d_M \;\; \textit{and} \;\; e_L \leq e(t) \leq e_M.$$

Lemma 2. *The function g defined in* (4) *satisfies a local Lipschitz condition on x, in the following sense: For each $M > 0$ There exist $k > 0$ such that:*
if $-M \leq s \leq t \leq M$, $\|x\| \leq M$, $\|\bar{x}\| \leq M$ Then;

$$\|g(t,s,x) - g(t,s,\bar{x})\| \leq k\|x - \bar{x}\|$$

Proof:

$$\begin{aligned} \|g(t,s,x) - g(t,s,\bar{x})\| &= \left\| \begin{pmatrix} c(t)k_1(t-s)(x_2 - \bar{x_2}) \\ f(t)k_2(t-s)(x_1 - \bar{x_1}) \end{pmatrix} \right\| \\ &\leq k(|x_1 - \bar{x_1}| + |x_2 - \bar{x_2}|) \\ &= k\|x - \bar{x}\| \end{aligned}$$

$$k = \text{máx}\{c_M \max_{s\in[0,M]} k_1(s), c_M \max_{s\in[0,M]} k_2(s)\}$$

Lemma 3. *let $t_0 \geq 0$, then the set*

$$B(t_0) = \{\Phi = (\phi_1, \phi_2)\phi_1, \phi_2 \in FA_{t_0}\}$$

where

$$FA_{t_0} = \{\phi : (-\infty, t_0) \longrightarrow \mathbb{R}^+ \cup \{0\})/\phi \textit{ is continuous, bounded above and } \phi(t_0) > 0\}$$

is a convex subset of the space of all continuous functions $\Phi : (-\infty, t_0] \longrightarrow \mathbb{R}^2$ and satisfies that if $\Phi \in B(t_0)$, then

$$G(t, t_0, \Phi) = \int_{-\infty}^{t_0} g(t, t_s, \Phi(s))\, ds$$

defines a continuous function in $[t_0, +\infty)$ where

$$g(t, s, (\phi_1, \phi_2)) = \begin{pmatrix} c(t)k_1(t-s)\phi_2 \\ f(t)k_2(t-s)\phi_1 \end{pmatrix}$$

Proof:
let $\alpha\lambda \in [0, 1)$ with $\alpha + \lambda = 1$ and $(\phi_1, \phi_2), (\bar{\phi_1}, \bar{\phi_2}) \in B(t_0)$. Then

$$\alpha(\phi_1, \phi_2) + \lambda(\bar{\phi_1}, \bar{\phi_2}) = (\alpha\phi_1 + \lambda\bar{\phi_1}, \alpha\phi_2 + \lambda\bar{\phi_2})$$

Also, $\phi_1, \bar{\phi_1}, \phi_2, \bar{\phi_2} \in FA_{t_0}$ and α, λ are non-negative and they do not vanish simultaneously,them $\alpha\phi_1, \lambda\bar{\phi_1}, \alpha\phi_2, \lambda\bar{\phi_2}$ are continuous on $[t_0, +\infty)$, non-negative, strictly positive on to and bounded.
Thus $\alpha\phi_1 + \lambda\bar{\phi_1}, \alpha\phi_2 + \lambda\bar{\phi_2}$ belong to FA_{t_0}. from where $(\alpha\phi_1 + \lambda\bar{\phi_1}, \alpha\phi_2 + \lambda\bar{\phi_2}) \in B(t_0)$ and therefore $B(t_0)$ is convex.

On the other hand,
i.) $G(t, t_0, \Phi)$ is well defined since

$$\begin{aligned}\int_{-\infty}^{t_0} c(t)k_1(t-s)\phi_2(s)\,ds &\leq c_M\phi_{2M}\int_{-\infty}^{t_0} k_1(t-s)\,ds\\ &= c_M\phi_{2M}\int_{t-t_0}^{+\infty} k_1(\sigma)\,d\sigma\\ &\leq c_M\phi_{2M}\int_{0}^{+\infty} k_1(\sigma)\,d\sigma\\ &=\leq c_M\phi_{2M} < +\infty\end{aligned}$$

similarly

$$\int_{-\infty}^{t_0} f(t)k_2(t-s)\phi_1(s)\,ds \leq f_M\phi_{1M} < +\infty$$

ii.) $G(t, t_0, \Phi)$ in continuous at each $t_0 \leq \bar{t}$, being that if we consider the interval $[t_0, d]$ such that $t_0 \leq \bar{t} \leq d$,k then each component

$$G_1(t, t_0, \Phi) = G_1(t, t_0, (\phi_1, \phi_2)) = \int_{-\infty}^{t_0} c(t)k_1(t-s)\phi_2(s)\,ds$$

and

$$G_2(t, t_0, \Phi) = \int_{-\infty}^{t_0} f(t)k_2(t-s)\phi_1(s)\,ds$$

is continuous in $\bar{t}$. to see this look at Theorem 14-21,Page 421 [4]. Now, the integral

$$\int_{-\infty}^{t_0} c(t)k_1(t-s)\phi_2(s)\,ds$$

can be written as

$$\int_{-t_0}^{+\infty} c(t)k_1(t+s)\phi_2(-s)\,ds$$

Now

$$\begin{aligned}\int_{-t_0}^{+\infty} c(t)\phi_2(s)k_1(t+s)\,ds &\leq c_M\Phi_{2M}\int_{-t_0}^{+\infty} k_1(t+s)\,ds\\ &\leq c_M\Phi_{2M}\int_{t-t_0}^{+\infty} k_1(\sigma)\,d\sigma\\ &\leq c_M\Phi_{2M}\bar{k_1}\end{aligned}$$

Also, given $\varepsilon > 0$ there exists $R > 0$ such that

$$\left|\int_{-t_0}^{b} k_1(\sigma)\,d\sigma - \int_{-t_0}^{+\infty} k_1(\sigma)\,d\sigma\right| < \frac{\varepsilon}{c_M\Phi_{2M}+1}$$

$\forall b \geq R$ and $t \in [t_0, d]$ let $b > R - t_0$, then $t_0 \leq t \leq d$ implies that $t_0 + b \leq t + b \leq d + b$ which also implies that $R \leq t + b \leq d + b$ son, given $\varepsilon > 0$ there exists $R > 0$ such that for $b > R - t_0$

$$\begin{aligned}\int_b^{+\infty} c(t)\phi_2(s)k_1(t+s)\,ds &\leq c_M\Phi_{2M}\int_b^{+\infty} k_1(t+s)\,ds\\ &= c_M\Phi_{2M}\int_{t+b}^{+\infty} k_1(\sigma)\,d\sigma\\ &\leq c_M\Phi_{2M}\varepsilon < \varepsilon\end{aligned}$$

for every $t \in [t_0, d]$ Therefore, $\int_b^{+\infty} c(t)\phi_2(s)k_1(t+s)\,ds$ converges uniformly on $[t_0, d]$. In consequence $G_1(t, t_0, \Phi)$ is continuous in $\bar{t}$ (we say even more in $[t_0, d]$). By the same token $G_2(t, t_0, \Phi)$ is continuous in $\bar{t}$. Therefore $G(t, t_0, \Phi)$ is continuous for every $\bar{t} \geq t_0$.

2. Existence and Uniqueness Theorem

An existence and theorem or equation (2) is given.
In fact, given a real number and an initial function $\Phi \in B(t_0)$ we look for a continuous solution $x(t) = x(t, t_0, \Phi)$ that satisfies (2) for every t $\in [t_0, t_0 + \beta)$ for some $\beta > 0$ and $x(t) = \phi(t)$ for all $t \leq t_0$.
let us observe that if $x(t)$ is a solution, that $x(t)$ is also a solution of the integral equation.

$$x(t) = \begin{cases}\phi(t) & \text{if } t \leq t_0\\ f(t) & \text{if } t \in [t_0, t_0 + \beta]\end{cases} \tag{I}$$

where

$$f(t) = \Phi(t_0) + \int_{t_0}^{t}[h(s, x(s)) - A(x(s))G(s, t_0, \Phi)]\,ds - \int_{t_0}^{t}\int_{t_0}^{s}[A(x(s))g(s, \tau, x(\tau))]d\tau ds$$

Conversely, every function $x(t)$ which satisfies (I) is necessarily a solution of the system (2) with initial function Φ. So the problem of existence of solutions of equation (2) is equivalent to the problem of existence of (I).
The right hand side of (I) define a continuous function in $(-\infty, t_0 + \beta)$ for every $\Phi \in B(t_0)$, $t \geq 0$ and

$$x(t) = \begin{cases}\phi(t) & \text{if } t \leq t_0\\ \omega(x_1, x_2) & \text{if } t_0 \leq t \leq t_0 + \beta\end{cases}$$

even more, it is continuously differentiable on $[t_0, t_0 + \beta]$.
From that point of view, (I) allows us to define an operator P which send the continuous function

$$x(t) = \begin{cases}\phi(t) & \text{if } t \leq t_0\\ \omega(x_1, x_2) & \text{if } t_0 \leq t \leq t_0 + \beta\end{cases}$$

to the continuous function given by the right hand side of (I).
In particular, if $x(t) = x(t, t_0, \Phi)$ is a solution of I, then,

$$(Px)(t) = x(t)$$

Then, the solutions to (I), or it is equivalent in the original problem (2) with initial condition (t_0, Φ), appear to be the fixed points of the operator P.
Now, to determinate the existence and uniqueness of a fixed point we will use the principle of the contraction applications.
For that, we need to define P on a subset S of the continuous function from $(-\infty, t_0 + \beta] \longrightarrow \mathbb{R}^2$ to which we impose certain conditions such that it is a complete metric space, P is a map from S to itself and P and P be a contraction.
Given problem (2) with the initial condition $\Phi \in B(t_0)$, $t_0 \geq 0$, and positive constants M and β, we consider the set S those function x which satisfies the following conditions:

a.) $x \in C[(-\infty, t_0 + \beta], \mathbb{R}^2]$

b.) $x(t) = \phi(t)$ if $t \leq t_0$

c.) $||x(t_1) - x(t_2)|| \leq M|t_1 - t_2|$ for $t_1, t_2 \in [t_0, t_0 + \beta]$

d.) $||x(t) - \Phi(t_0)|| \leq 1$ for $t \in [t_0, t_0 + \beta]$

and then on S we define the function.

$$\rho(x_1, x_2) = \max_{t \in [t_0, t_0 + \beta)} ||x_1(t) - x_2(t)||$$

So (S, ρ) is a complete metric space, and if M and β are selected appropriately, P will be a contraction from S to itself.

Lemma 4. *There exist M and β such that $\Phi \in B(t_0)$, $t_0 \leq 0$ such that the operator P act from (S, ρ) to itself and is a contraction*

Proof:
Let β_1, be a positive real number less than 1, now because of the continuity of the functions h, g, G and the compactness of the interval $[t_0, t_0 + \beta]$ and the sets

$$C_1 = \{(t, x) \in \mathbb{R} \times \mathbb{R}^2 / ||x(t) - \Phi(t_0)|| \leq 1 \, , t \in [t_0, t_0 + \beta_1]\}$$

$$C_2 = \{(t, s, x) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^2 / ||x(t) - \Phi(t_0)|| \leq 1 \, , t_0 \leq s \leq t \leq t_0 + \beta_1\}$$

there exists positive constants M_1,M_2 and M_3 such that

$$||h(t, x)|| \leq M_1 \quad \text{for all} \quad (t, x) \in C_1 \tag{5}$$

$$||G(t, t_0, \Phi)|| \leq M_2 \quad \text{for all} \quad t \in [t_0, t_0 + \beta] \tag{6}$$

$$||g(s, \tau, x)|| \leq M_3 \quad \text{for all} \quad (s, \tau, x) \in C_3 \tag{7}$$

We choose $M \in \mathbb{R}^+$ such that

$$M_1 + (M_2 M_3)(1 + ||a(\Phi(t_0)||) \leq M \tag{8}$$

If we apply the local Lipschitz condition for $h(t, x)$ at the point $(t_0, \Phi(t_0))$ to the set

$$C_3 = \{(t, x) \in \mathbb{R} \times \mathbb{R}^2 / ||x(t) - \Phi(t_0)|| \leq 1 \, , |t - t_0| \leq 1\}$$

we obtain that there exist $L_1 \in \mathbb{R}^+$ such that if (t, x) and $(t, \bar{x}) \in C_3$ then

$$||h(t, x) - h(t, \bar{x})|| \leq L_1 |x - \bar{x}| \tag{9}$$

Similarly, using the Lipschitz condition for g in the set

$$C_4 = \{(t, s, x) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^2 / \|x\| \leq T \,, -T \leq s \leq t \leq T\}$$

where $T = \text{máx}\{t_0 + \beta, 1 + \|\Phi(t_0)\|\}$, there exist an $L_2 \in \mathbb{R}^+$ such that if $(t, s, x), (t, s, \bar{x}) \in C_4$ then

$$\|g(t, s, x) - g(t, s, \bar{x})\| \leq L_2 |x - \bar{x}| \tag{10}$$

Let $L = \text{máx}\{L_1, L_2\}$ and pick $\beta \in \mathbb{R}^+$ such that

$$\beta < \beta_1 \quad \text{and} \quad \beta_1 < \frac{1}{2M + 2L + L\|\Phi(t_0)\|} \tag{11}$$

The condition $0 < \beta < \beta_1 < 1$ implies that $\beta^2 < \beta$. The second condition implies that $\beta < \frac{1}{M}$
If we consider the space (S, ρ) with M by (8) and β given by (11) then for $x \in S$ we have that.

$$\|x(s) - \Phi(t_0)\| \leq 1$$

for every $s \in [t_0, t_0 + \beta]$
therefore

$$(s, x(s)) \in C_1 \quad \text{for every} \quad s \in [t_0, t_0 + \beta] \tag{12}$$

It is also easy to verify that if $t_0 \leq \tau \leq s \leq t \leq t_0 + \beta$ then $(s, \tau, x(\tau)) \in C_2$ and the

$$\beta_1 \|g(s, \tau, x(\tau)\| \leq M_3 \quad \text{for} \quad t_0 \leq \tau \leq s \leq t \leq t_0 + \beta \tag{13}$$

Also, if $x, \bar{x} \in S$ then $(s, x(s))$ and $(s, \bar{x}(s)) \in C_3$ for every $s \in [t_0, t_0 + \beta]$ and
$(s, \tau, x(\tau)), (s, \tau, \bar{x}(\tau)) \in C_4$ if $t_0 \leq \tau \leq s \leq t \leq t_0 + \beta$.
So we get that

$$\|h(s, x(s)) - h(s, \bar{x}(s))\| \leq L|x(s) - \bar{x}(s)| \leq L\rho(x, \bar{x}) \;\; s \in [t_0, t_0 + \beta] \tag{14}$$

And every $t_0 \leq \tau \leq s \leq t \leq t_0 + \beta$

$$\|g(s, \tau, x(\tau)) - g(s, \tau, \bar{x}(\tau))\| \leq L|x(s) - \bar{x}(s)| \leq L\rho(x, \bar{x}). \tag{15}$$

The technique we will use in the next proof is similar to that of Theorem 3.3.5 Page 193 [3].

2.1. The Main Theorem

Theorem 1. *(Existence and Uniqueness) Let $t_0 \geq 0$ and $\Phi \in B(t_0)$. Then there exist a unique solution $x(t) = x(t, t_0, \Phi)$ of* (2) *defined in the interval $[t_0, t_0 + \beta]$ for some $\beta > 0$ and $x(t, t_0, \phi) = \Phi(t)$ for $x \leq t_0$.*
Proof:
Consider the metric space (S, ρ) given above, with M as in (8) *and β as in* (11).
Let the operator P, define for $x \in S$ by

$$(Px)(t) = \begin{cases} \Phi(t) & \text{if } t \geq t_0 \\ \Phi(t_0) + \int_{t_0}^{t} [h(s, x(s)) - A(x(s))G(s, t_0, \Phi)]\, ds & \\ - \int_{t_0}^{t} \int_{t_0}^{s} [A(x(s))g(s, \tau, x(\tau))] d\tau ds & \text{if } t \in [t_0, t_0 + \beta] \end{cases}$$

from (5), (6), (8), (12) *and* (13) P *is a mapping from* S *to* S *and*

$$\begin{aligned}
\|Px(t)-\Phi(t_0)\| \leq & \int_{t_0}^{t}\|h(s,x(s))\|\,ds+\int_{t_0}^{t}\|A(x(s))\|\|G(s,t_0,\Phi)\|\,ds \\
& +\int_{t_0}^{t}\|A(x(s))\|\int_{t_0}^{s}\|g(s,\tau,x(\tau))\|d\tau ds \\
\leq & M_1\int_{t_0}^{t}ds+\int_{t_0}^{t}\|x(s)\|M_2\,ds+\int_{t_0}^{t}\|x(s)\|\int_{t_0}^{s}\frac{M_3}{\beta_1}d\tau ds \\
\leq & M_1(t-t_0)+(1+\|\Phi(t_0)\|)M_2(t-t_0) \\
& +(1+\|\Phi(t_0)\|)\frac{M_3}{2}\frac{(t-t_0)^2}{\beta_1} \\
\leq & M_1\beta+(1+\|\Phi(t_0)\|)M_2\beta+(1+\|\Phi(t_0)\|)\frac{M_3}{2}\frac{\beta^2}{\beta_1}
\end{aligned}$$

Then

$$\begin{aligned}
\|Px(t)-\Phi(t_0)\| \leq & M_1\beta+(1+\|\Phi(t_0)\|)M_2\beta+(1+\|\Phi(t_0)\|)M_3\beta \\
= & (M_1+(1+\|\Phi(t_0)\|)(M_2+M_3))\beta \leq M\beta < 1
\end{aligned}$$

Similar, for $t_1, t_2 \in [t_0, t_0+\beta]$ *with* $t_1 < t_2$ *we have that*

$$\begin{aligned}
\|(Px)(t_1)-(Px)(t_2)\| \leq & \int_{t_1}^{t_2}\|h(s,x(s))\|\,ds+\int_{t_1}^{t_2}\|A(x(s))\|\|G(s,t_0,\Phi)\|\,ds \\
& +\int_{t_1}^{t_2}\|A(x(s))\|\int_{t_0}^{s}\|g(s,\tau,x(\tau))\|d\tau ds \\
\leq & M_1\int_{t_1}^{t_2}ds+(1+\|\Phi(t_0)\|)M_2\int_{t_0}^{t}ds \\
& +\int_{t_0}^{t}(1+\|\Phi(t_0)\|)\int_{t_0}^{s}\frac{M_3}{\beta}d\tau ds \\
\leq & M_1(t_2-t_1)+(1+\|\Phi(t_0)\|)M_2(t_2-t_1) \\
& +(1+\|\Phi(t_0)\|)\frac{M_3}{\beta}\int_{t_1}^{t_2}(s-t_0)\,ds \\
\leq & M(t_2-t_1)+(1+\|\Phi(t_0)\|)M(t_2-t_1) \\
& +(1+\|\Phi(t_0)\|)\frac{M_3}{\beta}(t_2-t_1) \\
= & (M_1+(1+\|\Phi(t_0)\|)(M_2+M_3))(t_2-t_1) \leq M(t_2-t_1)
\end{aligned}$$

So Px *satisfies the conditions (a),(b),(c), and (d), that is to say that* $(Px) \in S$ *Finally, let* $x, \bar{x} \in S$ *and* $t \in [t_0, t_0+\beta]$ *,now,from* (5), (6), (12), (13) *and* (14) *if we let*
$Q = \|(Px)(t)-(P\bar{x})(t)\|$ *we have that*

$$\begin{aligned}
Q \leq & \int_{t_0}^{t}\|h(s,x(s))-h(s,\bar{x}(s))\|\,ds+\int_{t_0}^{t}\|A(x(s)-\bar{x}(s))\|\|G(s,t_0,\Phi)\|\,ds \\
& +\int_{t_0}^{t}\int_{t_0}^{s}\|A(\bar{x}(s))g(s,\tau,\bar{x}(\tau))-A(x(s))g(s,\tau,x(\tau))\|d\tau ds
\end{aligned}$$

Then

$$
\begin{aligned}
Q \leq & L_1 \int_{t_0}^{t} \|x(s) - \bar{x}(s)\| \, ds + \int_{t_0}^{t} \|x(s) - \bar{x}(s)\| M_2 \, ds \\
& + \int_{t_0}^{t} \int_{t_0}^{s} \|A(\bar{x}(s))g(s,\tau,\bar{x}(\tau)) - A(\bar{x}(s))g(s,\tau,x(\tau)) \\
& + A(\bar{x}(s))g(s,\tau,x(\tau) - A(x(s))g(s,\tau,x(\tau)\| d\tau ds \\
\leq & (L_1 + M_2) \int_{t_0}^{t} \|x(s)) - \bar{x}(s)\| \, ds + \int_{t_0}^{t} \int_{t_0}^{s} \|A(\bar{x}(s))(g(s,\tau,\bar{x}(\tau)) - g(s,\tau,x(\tau))) \\
& + (A(\bar{x}(s)) - A(x(s))g(s,\tau,x(\tau))\| d\tau ds \\
\leq & (L_1 + M_2) \int_{t_0}^{t} \|x(s)) - \bar{x}(s)\| \, ds + \int_{t_0}^{t} \int_{t_0}^{s} \|\bar{x}(s)\| L_2 \|\bar{x}(\tau)) - x(\tau)\| d\tau ds \\
& + \int_{t_0}^{t} \int_{t_0}^{s} \|x(s)) - \bar{x}(s)\| \|g(s,\tau,x(\tau))\| d\tau ds \\
\leq & (L_1 + M_2)(t - t_0)\rho(x,\bar{x}) + \int_{t_0}^{t} (1 + \|\Phi(t_0)\|)\rho(x,\bar{x})(s - t_0) \, ds \\
& + \int_{t_0}^{t} \int_{t_0}^{s} \rho(x,\bar{x}) \frac{M_3}{\beta_1} d\tau ds \\
\leq & (L_1 + M_2)\beta\rho(x,\bar{x}) + \beta(1 + \|\Phi(t_0)\|)\rho(x,\bar{x})(t - t_0) + \rho(x,\bar{x}) \frac{M_3}{\beta_1} \int_{t_0}^{t} (s - t_0) ds \\
\leq & (L_1 + M_2)\beta\rho(x,\bar{x}) + \beta^2(1 + \|\Phi(t_0)\|)\rho(x,\bar{x}) + \rho(x,\bar{x}) \frac{M_3}{\beta_1} \beta^2 \\
\leq & ((L_1 + M_2)\beta + \beta(1 + \|\Phi(t_0)\|) + \beta M_3)\rho(x,\bar{x}) \leq k\rho(x,\bar{x})
\end{aligned}
$$

which means that

$$\|(Px)(t) - (P\bar{x})(t)\| \leq k\rho(x,\bar{x})$$

for $t \in [t_0, t_0 + \beta]$ *and* $x, \bar{x} \in S$
So $\rho(Px, P\bar{x}) \leq \rho(x,\bar{x})$ *for some* $\beta > 0$ *chosen in such a way that*

$$k = ((L_1 + M_1) + (1 + \|\Phi(t_0)\|) + M_3)\beta < 1$$

Proving that there exist $\beta > 0$ *such that* Px *is a contraction* $[t_0, t_0 + \beta]$ *and in consequence there exist a unique* $x \in S$ *with* $Px = x$
Since the fixed points of P *are the solutions of* (2) *the conclusion follows.*

Conclusions

(1) This theorem of existence and uniqueness illustrate a classical method to assure the existence of solutions of a differential equation, and in this case with infinite delay , but this technique can be extended to some others type o differential equations.

(2) The existence of the solution in a local result in the interval $[t_0, t_0 + \beta]$ for some $\beta > 0$. Now using traditional methods we can extend the solution to $[t_0, +\infty)$.

Acknowledgement

We would like to thanks Dr. Jorge Vielma for his suggestions that improve the paper.

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