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# Desigualdades del tipo Hermite-Hadamard y Fejér para funciones fuertemente armónicas convexas.

# Hermite-Hadamard and Fejér type inequalities for strongly harmonically convex functions

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**Resumen**

Introducimos la noción de funciones fuertemente armónicas convesas y presentamos algunso ejemplos y propiedades de ésta clase. También, establecemos algunas desigualdades del tipo Hermite-Hadamard and y Fejér para la clase introducida.

*Palabras claves:* Funciones fuertemente armónicas convexas, Desigualdad del tipo Hermite-Hadamrd, Desigualdad del tipo de Fejér.

**Abstract**

We introduce the notion of strongly harmonically convex function and present some examples and properties of them. We also establish some Hermite-Hadamard and Fejér type inequalities for the class of strongly harmonically convex functions which generalizes previous results.

*Keywords:* Strongly harmonically convex function, Hermite-Hadamrd, Fejér type inequalities.

## 1. Introduction

Due to its important role in mathematical economics, engineering, management science, and optimization theory, convexity of functions and sets has been studied intensively; see [1, 3, 5, 6, 7, 9, 11, 13, 14] and the references therein. Let $\mathbb{R}$ be the set of real numbers and $I \subseteq \mathbb{R}$ be a interval. A function $f : I \to \mathbb{R}$ is said to be convex in the classical sense if it satisfies the following inequality

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$$

for all $x, y \in I$ and $t \in [0, 1]$. We say that $f$ is concave if $-f$ is convex.
In recent years several extensions and generalizations have been considered for classical convexity, and the theory of inequalities has made essential contributions in many areas of mathematics. A significant subclass of convex functions is that of strongly convex functions introduced by B. T. Polyak [19]. Strongly convex functions are widely used in applied economics, as well as in nonlinear optimization and other branches of pure and applied mathematics. In this paper we present a new class of strongly convex functions, mainly the class of *strongly harmonically convex functions*. Our investigation is devoted to the classical results related to convex functions due to Charles Hermite, Jaques Hadamard [8] and Lipót Fejér [6]. The Hermite-Hadamard inequalities and Fejér inequalities have been the subject of intensive research, and many applications, generalizations and improvements of them can be found in the literature (see, for instance, [1, 5, 13, 16, 18, 20, 23] and the references therein).

Many inequalities have been established for convex functions but the most famous is the Hermite-Hadamard inequality, this asserts that the mean value of a continuous convex functions $f : [a, b] \subseteq \mathbb{R} \to \mathbb{R}$ lies between the value of $f$ at the midpoint of the interval $[a, b]$ and the arithmetic mean of the values of $f$ at the endpoints of this interval, that is,

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{1}$$

Moreover, each side of this double inequality characterizes convexity in the sense that a real-valued continuous function $f$ defined on an interval $I$ is convex if its restriction to each compact subinterval $[a, b] \subseteq I$ verifies the left hand side of (1) (equivalently, the right hand side on (1)). See [15].

In [6], Lipót Fejér established the following inequality which is the weighted generalization of Hermite-Hadamard inequality (1): If $f : [a, b] \to \mathbb{R}$ is a convex function, then the inequality

$$f\left(\frac{a+b}{2}\right)\int_a^b p(x)dx \leq \frac{1}{b-a}\int_a^b f(x)p(x)dx \leq \frac{f(a)+f(b)}{2}\int_a^b p(x)dx \tag{2}$$

holds, where $p : [a, b] \to \mathbb{R}$ is nonnegative, integrable and symmetric about $x = \dfrac{a+b}{2}$.

Various generalizations have been pointed out in many directions, for recent developments of inequalities (1) and (2) and its generalizations, see [3, 4, 5, 2, 7, 11, 17].

In [11], Imdat Iscan gave the definition of harmonically convex functions:

**Definition 1.1.** *[11] Let $I$ be an interval in $\mathbb{R} \setminus \{0\}$. A function $f : I \to \mathbb{R}$ is said to be harmonically convex on $I$ if the inequality*

$$f\left(\frac{xy}{tx+(1-t)y}\right) \leq tf(y) + (1-t)f(x), \tag{3}$$

*holds, for all $x, y \in I$ and $t \in [0, 1]$.*

If the inequality in (3) is reversed, then $f$ is said to be harmonically concave.

The following result of the Hermite-Hadamard type for harmonically convex functions holds.

**Theorem 1.2.** *Let $f : I \subseteq \mathbb{R} \setminus \{0\} \to \mathbb{R}$ be a harmonically convex function and $a, b \in I$ with $a < b$. If $f \in L[a, b]$ then the following inequalities hold*

$$f\left(\frac{2ab}{a+b}\right) \leq \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx \leq \frac{f(a)+f(b)}{2}. \tag{4}$$

The main purpose of this paper is to introduce the concept of strongly harmonically convex functions and establish some results connected with the inequalities similar to the inequality (1) and (2) for these classes of functions.

## 2. Strongly harmonically convex functions

When analyzing optimization algorithms, it is sometimes easier to work with strongly convex functions, which generalize the definition of convexity.

In 1966 Polyak [19] introduced the notions of strongly convex and strongly quasi-convex functions. In 1976 Rockafellar [22] studied the strongly convex functions in connection with the proximal point algorithm. They play an important role in optimization theory and mathematical economics. Nikodem et al. have obtained some interesting properties of strongly convex functions (see [5, 10, 12]).

**Definition 2.1** (See [10, 14, 21])**.** *Let $D$ be a convex subset of $\mathbb{R}$ and let $c > 0$. A function $f : D \to \mathbb{R}$ is called strongly convex with modulus $c$ if*

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) - ct(1-t)(x-y)^2 \tag{5}$$

*for all $x, y \in D$ and $t \in [0, 1]$.*

The usual notion of convex function correspond to the case $c = 0$.

For instance, if $f$ is strongly convex, then it is bounded from below, its level sets $\{x \in I : f(x) \leq \lambda\}$ are bounded for each $\lambda$ and $f$ has a unique minimum on every closed subinterval of $I$ ([16], p. 268). Any strongly convex function defined on a real interval admits a quadratic support at every interior point of its domain.

**Theorem 2.2.** *Let $D$ be a convex subset of $\mathbb{R}$ and $c$ be a positive constant. A function $f : D \to \mathbb{R}$ is strongly convex with modulus $c$ if and only if the function $g(x) = f(x) - cx^2$ is convex.*

Next we will explore a generalization of the concept of harmonically convex functions which we will call harmonically strongly convex.

**Definition 2.3.** *Let $I$ be an interval in $\mathbb{R}\setminus\{0\}$ and let $c \in \mathbb{R}_+$. A function $f : I \to \mathbb{R}$ is said to be harmonically strongly convex with modulus $c$ on $I$, if the inequality*

$$f\left(\frac{xy}{tx+(1-t)y}\right) \leq tf(y) + (1-t)f(x) - ct(1-t)(x-y)^2, \tag{6}$$

*holds, for all $x, y \in I$ and $t \in [0, 1]$.*

The symbol $\mathrm{SHC}_{(I,c)}$ will denote the class of functions that satisfy the inequality (6).

Definition 2.3 have the following important consequence.

**Theorem 2.4.** *Let $I \subset \mathbb{R}_+$ be an interval and let $f : I \to \mathbb{R}$ be a function.*

1. *If $I \subseteq (0,+\infty)$ and $f \in SHC_{(I,c)}$ then $f$ is harmonically convex.*
2. *If $I \subseteq (0,+\infty)$ and $f \in SHC_{(I,c)}$ and nonincreasing, then $f$ is a strongly convex function with modulus $c$.*
3. *If $I \subseteq (0,+\infty)$ and $f$ is strongly convex with modulus $c$ and nondecreasing, then $f \in SHC_{(I,c)}$.*

*Demostración.* For any $x, y \in I$, and $t \in [0,1]$,

1. We will use the facts that $f \in \mathrm{SHC}_{(I,c)}$ and that the quantity $ct(1-t)(x-y)^2$ is nonnegative

$$f\left(\frac{xy}{tx+(1-t)y}\right) \leq tf(y)+(1-t)f(x)-ct(1-t)(x-y)^2 \leq tf(y)+(1-t)f(x).$$

   This last inequality ensures that $f$ is harmonically convex.

2. Since $f \in \mathrm{SHC}_{(I,c)}$ and nonincreasing, then

$$\begin{aligned} \frac{xy}{tx+(1-t)y} = \frac{1}{\frac{t}{y}+\frac{1-t}{x}} &\leq ty+(1-t)x \\ f(ty+(1-t)x) &\leq f\left(\frac{xy}{tx+(1-t)y}\right) \\ &\leq tf(y)+(1-t)f(x)-ct(1-t)(x-y)^2, \end{aligned}$$

   so the function $f$ is a strongly convex function with modulus $c$.

3. Note that

$$\frac{xy}{tx+(1-t)y} = \frac{1}{\frac{t}{y}+\frac{1-t}{x}} \leq ty+(1-t)x.$$

   Therefore

$$\begin{aligned} f\left(\frac{xy}{tx+(1-t)y}\right) &\leq f(ty+(1-t)x), \\ &\leq tf(y)+(1-t)f(x)-ct(1-t)(x-y)^2. \end{aligned}$$

Hence Theorem 2.4 is true. □

**Example 2.5.**
1. *The constant function is harmonically convex but does not belong to the class $SHC_{(I,c)}$, for all $c > 0$.*
2. *The function $f : (0,+\infty) \to \mathbb{R}$, defined by $f(x) = -x^2$, is not a harmonic convex function (since $f$ is not convex and decreasing), then $f \notin SHC_{(I,c)}$.*

## 3. Basic properties

In this section we present some properties that meets this new class of functions introduced in the previous section. Some examples are also given.

**Theorem 3.1.** *Let $I \subset \mathbb{R}_+$ be a real interval and let $f : I \to \mathbb{R}$ be a function. If the function $g : I \to \mathbb{R}$ defined by $g(x) = f(x) - cx^2$ is harmonically convex, then $f \in SHC_{(I,c)}$.*

*Demostración.* Let $x, y \in I$ and $t \in [0, 1]$, we have

$$\begin{aligned}
& f\left(\frac{xy}{tx + (1-t)y}\right) \\
= \; & g\left(\frac{xy}{tx + (1-t)y}\right) + c\left(\frac{xy}{tx + (1-t)y}\right)^2 \\
\leq \; & tg(y) + (1-t)g(x) + c(ty + (1-t)x)^2 \\
= \; & tg(y) + (1-t)g(x) + c[t^2y^2 + 2t(1-t)xy + (1-t)^2x^2] \\
= \; & tg(y) + (1-t)g(x) + c[t(1-1+t)y^2 + 2t(1-t)xy + (1-t)(1-t)x^2] \\
= \; & t[g(y) + cy^2] + (1-t)[g(x) + cx^2] - ct(1-t)[x^2 - 2xy + y^2] \\
= \; & tf(y) + (1-t)f(x) - ct(1-t)(x-y)^2. \qquad (7)
\end{aligned}$$

Therefore, the proof is completed. □

**Example 3.2.** *Given $a, c \in \mathbb{R}_+$ and $b, d \in \mathbb{R}$*

1. *The function $f : \mathbb{R}_+ \to \mathbb{R}$ defined by $f(x) := \log(x) + cx^2 \in SHC_{(I,c)}$ (Since $g(x) = f(x) - cx^2 = \log(x)$ is a harmonically convex function).*
2. *The function $f : \mathbb{R}_+ \to \mathbb{R}$ defined by $f(x) := e^x + cx^2 \in SHC_{(I,c)}$, since $g(x) = f(x) - cx^2 = e^x$ is a harmonically convex function ($g$ is increasing and convex).*
3. *Let $f$ be a function defined by $f(x) := (a+c)x^2 + bx + d$ on $I = \left[-\frac{b}{2(a+c)}, +\infty\right) \cap \left[-\frac{b}{2a}, +\infty\right) \cap \mathbb{R}_+$ belongs to $SHC_{(I,c)}$, since $g(x) = f(x) - cx^2 = ax^2 + bx + d$ is a harmonically convex function.*

**Theorem 3.3.** *Let $I \subset \mathbb{R}_+$ be an interval and let $f : I \to \mathbb{R}$ be a function. If $f \in SHC_{(I,c)}$ and is decreasing on $I$ then the function $g : I \to \mathbb{R}$, defined by $g(x) := f(x) - cx^2$, is a decreasing and convex function.*

*Demostración.* Let $x, y \in I$ and $t \in [0, 1]$. By Theorem (2.4) $f$ is strongly convex function with modulus $c$. Thus

$$\begin{aligned}
& g(tx + (1-t)y) \\
= \; & f(tx + (1-t)y) - c(tx + (1-t)y)^2 \\
\leq \; & tf(x) + (1-t)f(y) - ct(1-t)(x-y)^2 - c(tx + (1-t)y)^2 \\
= \; & tf(x) + (1-t)f(y) - c[(t-t^2)(x^2 - 2xy + y^2) + t^2x^2 + 2t(1-t)xy + (1-t)^2y^2] \\
= \; & tf(x) + (1-t)f(y) - c[tx^2 - t^2x^2 + ty^2 - t^2y^2 + t^2x^2 + y^2 - 2ty^2 + t^2y^2] \\
= \; & tf(x) + (1-t)f(y) - c[tx^2 + y^2 - ty^2] \\
= \; & t[f(x) - cx^2] + (1-t)[f(y) - cy^2] \\
= \; & tg(x) + (1-t)g(y).
\end{aligned}$$

Hence, $g$ is a convex function.

To prove that $g$ is a decreasing function, let $x, y \in I$ such that $x \leq y$. Since $f$ is decreasing $f(x) \geq f(y)$ or equivalent $f(y) - f(x) \leq 0$. Thus

$$g(y) - g(x) \quad = \quad f(y) - cy^2 - [f(x) - cx^2] = [f(y) - f(x)] - c[y^2 - x^2] \leq 0.$$

□

**Proposition 3.4.** *Let $I$ be an interval in $\mathbb{R} \setminus \{0\}$. Let $f, g : I \to \mathbb{R}$ be two functions and let $c_1, c_2, k \in \mathbb{R}_+$,*

1. *If $f \in SHC_{(I,c_1)}$ and $g \in SHC_{(I,c_2)}$ then $f + g \in SHC_{(I,c_1+c_2)}$, $f + g \in SHC_{(I,c_1)}$ and $f + g \in SHC_{(I,c_2)}$.*
2. *If $f \in SHC_{(I,c)}$ then $kf \in SHC_{(I,kc)}$. In addition, if $k \geq 1$, then $kf \in SHC_{(I,c)}$.*

*Demostración.* 1. Let $x, y \in I$ and $t \in [0, 1]$. Then

$$\begin{aligned} & (f+g)\left(\frac{xy}{tx+(1-t)y}\right) \\ = \quad & f\left(\frac{xy}{tx+(1-t)y}\right) + g\left(\frac{xy}{tx+(1-t)y}\right) \\ \leq \quad & tf(y) + (1-t)f(x) - c_1 t(1-t)(x-y)^2 \\ & + tg(y) + (1-t)g(x) - c_2 t(1-t)(x-y)^2 \\ \leq \quad & t(f+g)(y) + (1-t)(f+g)(x) - (c_1+c_2)t(1-t)(x-y)^2. \end{aligned} \tag{8}$$

This shows that $f + g \in \mathrm{SHC}_{(I,c_1+c_2)}$. Now, since

$$-(c_1 + c_2)t(1-t)(x-y)^2 \quad \leq \quad -c_1 t(1-t)(x-y)^2 \qquad \text{and}$$
$$-(c_1 + c_2)t(1-t)(x-y)^2 \quad \leq \quad -c_2 t(1-t)(x-y)^2,$$

we obtain from (8) that $f + g \in \mathrm{SHC}_{(I,c_1)}$ and $f + g \in \mathrm{SHC}_{(I,c_2)}$.

2. Follows easily from the definitions.

□

**Proposition 3.5.** *Let $I \subset \mathbb{R}_+$ be an interval. If $f_1 \in SHC_{(I,c_1)}$ and $f_2 \in SHC_{(I,c_2)}$, then $f := \text{máx}\{f_1, f_2\} \in SHC_{(I,c)}$ where $c = \text{mín}\{c_1, c_2\}$.*

*Demostración.* Let $x, y \in I$ and $t \in [0, 1]$. Then

$$\begin{aligned} f_1\left(\frac{xy}{tx+(1-t)y}\right) \quad & \leq \quad tf_1(y) + (1-t)f_1(x) - c_1 t(1-t)(x-y)^2 \\ & \leq \quad tf(y) + (1-t)f(x) - ct(1-t)(x-y)^2, \end{aligned}$$

and

$$\begin{aligned} f_2\left(\frac{xy}{tx+(1-t)y}\right) \quad & \leq \quad tf_2(y) + (1-t)f_2(x) - c_2 t(1-t)(x-y)^2 \\ & \leq \quad tf(y) + (1-t)f(x) - ct(1-t)(x-y)^2, \end{aligned}$$

from which we obtain that

$$\begin{aligned} f\left(\frac{xy}{tx+(1-t)y}\right) &= \text{máx}\left\{f_1\left(\frac{xy}{tx+(1-t)y}\right), f_2\left(\frac{xy}{tx+(1-t)y}\right)\right\} \\ &\leq tf(y)+(1-t)f(x)-ct(1-t)(x-y)^2. \end{aligned}$$

□

**Proposition 3.6.** *Let $I \subset \mathbb{R}_+$ be an interval. If $f_n : I \to \mathbb{R}$ is a sequence of harmonically strongly convex functions with modulus c, converging pointwise to a function $f$ on I, then $f \in SHC_{(I,c)}$.*

*Demostración.* Indeed, let $x, y \in I$ and $t \in [0, 1]$. Then

$$\begin{aligned} f\left(\frac{xy}{tx+(1-t)y}\right) &= \lim_{n\to\infty} f_n\left(\frac{xy}{tx+(1-t)y}\right) \\ &\leq \lim_{n\to\infty}[tf_n(y)+(1-t)f_n(x)-ct(1-t)(x-y)^2] \\ &= tf(y)+(1-t)f(x)-ct(1-t)(x-y)^2. \end{aligned}$$

This completes the demonstration. □

We finish this section by presenting an instance of how to produce harmonically strongly convex functions by means of compositions.

**Definition 3.7** (See [9, 18]). *Two functions $f, g$ are said to be similarly ordered on I if,*

$$[f(x)-f(y)][g(x)-g(y)] \geq 0, \tag{9}$$

*for all $x, y \in I$.*

**Proposition 3.8.** *Let $I, I_1 \subset \mathbb{R}_+$ be two intervals. Let $f : I \to \mathbb{R}$ be a harmonically convex function and let $g : I_1 \to \mathbb{R}$ be a strongly convex function with module c such that $f(I) \subset I_1$.*

*If g is nondecreasing and the functions $f - id$ and $f + id$ are similarly ordered on I, where id is the identity function, then $g \circ f \in SHC_{(I,c)}$.*

*Demostración.* Since $f : I \to \mathbb{R}$ is a harmonically strongly convex we have, for any $x, y \in I$ and $t \in [0, 1]$,

$$f\left(\frac{xy}{tx+(1-t)y}\right) \leq tf(y)+(1-t)f(x).$$

Since $g$ is a nondecreasing function and strongly convex function with module $c$

$$\begin{aligned} (g\circ f)\left(\frac{xy}{tx+(1-t)y}\right) &\leq g(tf(y)+(1-t)f(x)), \\ &\leq tg(f(y))+(1-t)g(f(x))-ct(1-t)(f(x)-f(y))^2. \end{aligned} \tag{10}$$

On the other hand, $f - id$ and $f + id$ are similarly ordered on $I$,

$$
\begin{aligned}
(f(x) - x - (f(y) - y))(f(x) + x - (f(y) + y)) &\geq 0 \\
(f(x) - x - f(y) + y)(f(x) + x - f(y) - y) &\geq 0 \\
(f(x))^2 - 2f(x)f(y) + (f(y))^2 - x^2 + 2xy - y^2 &\geq 0 \\
(f(x) - f(y))^2 - (x - y)^2 &\geq 0 \\
(f(x) - f(y))^2 &\geq (x - y)^2,
\end{aligned}
$$

and replacing this last inequality in (10), we get

$$
(g \circ f)\left(\frac{xy}{tx + (1-t)y}\right) \leq t(g \circ f)(y) + (1-t)(g \circ f)(x) - ct(1-t)(x-y)^2.
$$

□

## 4. Main results

This section is dedicated to present the main results of this article, namely, to establish some Fejér - Hermite - Hadamard type inequalities for the class $\mathrm{SHC}_{(I,c)}$.

**Theorem 4.1** (Hermite - Hadamard type inequalities)**.** *Let $f : I \subseteq (0, +\infty) \to \mathbb{R}$ be a function in $SHC_{(I,c)}$ and let $a, b \in I$ with $a < b$. If $f \in L[a, b]$ then the following inequalities hold*

$$
\begin{aligned}
f\left(\frac{2ab}{a+b}\right) + \frac{cab}{2}\left(1 - \frac{2ab\ln(b/a)}{b^2 - a^2}\right) &\leq \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx \\
&\leq \frac{f(a) + f(b)}{2} - \frac{c}{6}(a-b)^2.
\end{aligned} \tag{11}
$$

*Demostración.* Since $f \in \mathrm{SHC}_{(I,c)}$, inequality (6) $\left(\text{with} t = \frac{1}{2}\right)$ implies

$$
f\left(\frac{2xy}{x+y}\right) \leq \frac{f(y) + f(x)}{2} - \frac{c}{4}(x-y)^2, \text{ for all } x, y \in I. \tag{12}
$$

Now, if $x = \dfrac{ab}{ta + (1-t)b}$, $y = \dfrac{ab}{tb + (1-t)a}$, $t \in [0, 1]$, (12) turns into

$$
\begin{aligned}
f\left(\frac{\frac{2ab}{ta+(1-t)b} \cdot \frac{ab}{tb+(1-t)a}}{\frac{ab}{ta+(1-t)b} + \frac{ab}{tb+(1-t)a}}\right) &\leq \frac{1}{2}f\left(\frac{ab}{ta+(1-t)b}\right) + \frac{1}{2}f\left(\frac{ab}{tb+(1-t)a}\right) \\
&\quad -\frac{c}{4}\left(\frac{ab}{ta+(1-t)b} - \frac{ab}{tb+(1-t)a}\right)^2,
\end{aligned}
$$

or equvalently

$$
\begin{aligned}
f\left(\frac{2ab}{a+b}\right) &\leq \frac{1}{2}f\left(\frac{ab}{ta+(1-t)b}\right) + \frac{1}{2}f\left(\frac{ab}{tb+(1-t)a}\right) \\
&\quad -\frac{c}{4}\left(\frac{ab}{ta+(1-t)b} - \frac{ab}{tb+(1-t)a}\right)^2.
\end{aligned} \tag{13}
$$

Integrating the above inequality on $[0,1]$, we obtain

$$\begin{aligned} f\left(\frac{2ab}{a+b}\right) &\leq \frac{1}{2}\left[\int_0^1 f\left(\frac{ab}{ta+(1-t)b}\right)dt + \int_0^1 f\left(\frac{ab}{tb+(1-t)a}\right)dt\right] \\ &\quad -\frac{c}{4}\int_0^1 \left(\frac{ab}{ta+(1-t)b} - \frac{ab}{tb+(1-t)a}\right)^2 dt \\ &= \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx - \frac{c}{4}2ab\left[1-\frac{2ab}{b^2-a^2}\ln\left(\frac{b}{a}\right)\right], \end{aligned} \tag{14}$$

thus obtaining the left hand side of inequality (11).
On the other hand, setting $x=a$ and $y=b$ in (6) we get that

$$f\left(\frac{ab}{ta+(1-t)b}\right) \leq tf(b)+(1-t)f(a)-ct(1-t)(a-b)^2.$$

Integrating again on $[0,1]$ we obtain that

$$\begin{aligned} &\int_0^1 f\left(\frac{ab}{ta+(1-t)b}\right)dt \\ \leq\ & f(b)\int_0^1 tdt + f(a)\int_0^1 (1-t)dt - c(a-b)^2\int_0^1 t(1-t)dt \\ =\ & \frac{f(a)+f(b)}{2} - \frac{c}{6}(a-b)^2. \end{aligned}$$

Thus we get the right side part of the inequality (11), i.e.

$$\int_0^1 f\left(\frac{ab}{ta+(1-t)b}\right) = \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx \leq \frac{f(a)+f(b)}{2} - \frac{c}{6}(a-b)^2.$$

□

**Remark 4.2.** *Note that if* $c=0$ *in* (11) *it is obtained the Hermite Hadamard type inequalities for harmonically convex function ( see [11]).*

**Theorem 4.3** (Fejér type inequalities)**.** *Let* $f: I \subset \mathbb{R}_+ \to \mathbb{R}$ *be a function belonging to* $SHC_{(I,c)}$*, let* $a,b \in I$ *with* $a<b$*. If* $f \in L[a,b]$*, then*

$$f\left(\frac{2ab}{a+b}\right)\int_a^b \frac{p(x)}{x^2}dx + \frac{c}{2}\left[\int_a^b p(x)dx - \frac{2ab}{a+b}\int_a^b \frac{p(x)}{x}dx\right] \tag{15}$$

$$\leq \int_a^b \frac{f(x)}{x^2}p(x)dx$$

$$\leq \frac{f(a)+f(b)}{2}\int_a^b \frac{p(x)}{x^2}dx, \tag{16}$$

*where* $p:[a,b]\to\mathbb{R}$ *is a nonnegative and integrable functions that satisfies the condition*

$$p\left(\frac{ab}{x}\right) = p\left(\frac{ab}{a+b-x}\right). \tag{17}$$

*Demostración.* Note that for $x = tb + (1-t)a$, (15) becomes

$$p\left(\frac{ab}{tb+(1-t)a}\right) = p\left(\frac{ab}{ta+(1-t)b}\right). \tag{18}$$

Since $f \in \mathrm{SHC}_{(I,c)}$ function on $[a,b]$

$$f\left(\frac{2xy}{x+y}\right) \le \frac{f(y)+f(x)}{2} - \frac{c}{4}(x-y)^2, \quad x, y \in [a,b]. \tag{19}$$

Setting $x = \dfrac{ab}{tb+(1-t)a}$ and $y = \dfrac{ab}{ta+(1-t)b}$ in (19), we get

$$\begin{aligned}
& f\left(\frac{2ab}{a+b}\right) \\
\le\ & \frac{f\left(\frac{ab}{ta+(1-t)b}\right)+f\left(\frac{ab}{tb+(1-t)a}\right)}{2} - \frac{c}{4}\left(\frac{ab}{tb+(1-t)a} - \frac{ab}{ta+(1-t)b}\right)^2 \\
\le\ & \frac{tf(b)+(1-t)f(a)+tf(a)+(1-t)f(b)}{2} - \frac{c}{4}\left(\frac{ab}{tb+(1-t)a} - \frac{ab}{ta+(1-t)b}\right)^2 \\
=\ & \frac{f(a)+f(b)}{2} - \frac{c}{4}\left(\frac{ab}{tb+(1-t)a} - \frac{ab}{ta+(1-t)b}\right)^2.
\end{aligned}$$

Thus, as $p$ satisfies (18), we obtain

$$\begin{aligned}
& f\left(\frac{2ab}{a+b}\right)p\left(\frac{ab}{tb+(1-t)a}\right) \\
& \le \frac{1}{2}\left[f\left(\frac{ab}{ta+(1-t)b}\right)p\left(\frac{ab}{ta+(1-t)b}\right)+f\left(\frac{ab}{tb+(1-t)a}\right)p\left(\frac{ab}{tb+(1-t)a}\right)\right] \\
& -\frac{c}{4}\left(\frac{ab}{tb+(1-t)a} - \frac{ab}{ta+(1-t)b}\right)^2 p\left(\frac{ab}{tb+(1-t)a}\right) \\
& \le \frac{f(a)+f(b)}{2}p\left(\frac{ab}{tb+(1-t)a}\right) - \frac{c}{4}\left(\frac{ab}{tb+(1-t)a} - \frac{ab}{ta+(1-t)b}\right)^2 p\left(\frac{ab}{tb+(1-t)a}\right).
\end{aligned}$$

Integrating both sides of the above inequalities with respect to $t$ over $[0,1]$, we obtain

$$\begin{aligned}
& f\left(\frac{2ab}{a+b}\right)\int_0^1 p\left(\frac{ab}{tb+(1-t)a}\right)dt \\
& \le \frac{1}{2}\int_0^1\left[f\left(\frac{ab}{ta+(1-t)b}\right)p\left(\frac{ab}{ta+(1-t)b}\right)+f\left(\frac{ab}{tb+(1-t)a}\right)p\left(\frac{ab}{tb+(1-t)a}\right)\right]dt \\
& -\frac{c}{4}\int_0^1\left(\frac{ab}{tb+(1-t)a} - \frac{ab}{ta+(1-t)b}\right)^2 p\left(\frac{ab}{tb+(1-t)a}\right)dt \\
& \le \frac{f(a)+f(b)}{2}\int_0^1 p\left(\frac{ab}{tb+(1-t)a}\right)dt \\
& -\frac{c}{4}\int_0^1\left(\frac{ab}{tb+(1-t)a} - \frac{ab}{ta+(1-t)b}\right)^2 p\left(\frac{ab}{tb+(1-t)a}\right)dt,
\end{aligned}$$

and, by making the necessary change of variables, we finally obtain the following equivalent inequality

$$\begin{aligned}
& f\left(\frac{2ab}{a+b}\right)\frac{ab}{b-a}\int_a^b \frac{p(x)}{x^2}dx \\
\leq\ & \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}p(x)\,dx - \frac{cab}{2(b-a)}\left[\int_a^b p(x)dx - \frac{2ab}{b+a}\int_a^b \frac{p(x)}{x}dx\right] \\
\leq\ & \frac{f(a)+f(b)}{2}\frac{ab}{b-a}\int_a^b \frac{p(x)}{x^2}dx - \frac{cab}{2(b-a)}\left[\int_a^b p(x)dx - \frac{2ab}{b+a}\int_a^b \frac{p(x)}{x}dx\right].
\end{aligned}$$

Which implies (15).

□

**Remark 4.4.** *Note that if we let $p(x) \equiv 1$ in Theorem 4.3, we get the left-hand side of the inequality 4.1.*

**Theorem 4.5.** *Under the same assumptions of the Theorem 4.3, we have*

$$\begin{aligned}
& f\left(\frac{2ab}{a+b}\right)\int_a^b \frac{p(x)}{x^2}dx + \frac{c}{2}\left[\int_a^b p(x)dx - \frac{2ab}{a+b}\int_a^b \frac{p(x)}{x}dx\right] \\
\leq\ & \int_a^b \frac{f(x)}{x^2}p(x)dx \\
\leq\ & \frac{a\left[f(a)+f(b)\right]}{b-a}\int_a^b \left(\frac{b}{x}-1\right)\frac{p(x)}{x^2}dx - cab\int_a^b \left(\frac{b}{x}-1\right)\left(1-\frac{a}{x}\right)\frac{p(x)}{x^2}dx. \qquad (20)
\end{aligned}$$

*Demostración.* The left-hand side of these inequalities coincides with the Theorem 4.3, hence we only need to show the right hand side inequality. Indeed, since

$$f\left(\frac{ab}{ta+(1-t)b}\right) \leq tf(b) + (1-t)f(a) - ct(1-t)(a-b)^2,$$

so we get

$$\begin{aligned}
& f\left(\frac{ab}{ta+(1-t)b}\right)p\left(\frac{ab}{ta+(1-t)b}\right) \\
\leq\ & \left[tf(b) + (1-t)f(a) - ct(1-t)(a-b)^2\right]p\left(\frac{ab}{ta+(1-t)b}\right).
\end{aligned}$$

Integrating both sides of the above inequalities with respect to $t$ over $[0, 1]$, we obtain

$$\begin{aligned}
& \int_0^1 f\left(\frac{ab}{ta+(1-t)b}\right)p\left(\frac{ab}{ta+(1-t)b}\right)dt \\
\leq\ & \int_0^1 \left[tf(b) + (1-t)f(a) - ct(1-t)(a-b)^2\right]p\left(\frac{ab}{ta+(1-t)b}\right)dt.
\end{aligned}$$

Therefore we have that

$$
\begin{aligned}
& \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}p(x)dx \\
\leq\ & \frac{a^2bf(b)}{(b-a)^2}\int_a^b \left(\frac{b}{x}-1\right)\frac{p(x)}{x^2}dx + \frac{a^2bf(a)}{(b-a)^2}\int_a^b \left(\frac{b}{x}-1\right)\frac{p(x)}{x^2}dx \\
& -\frac{c(ab)^2}{(b-a)}\int_a^b \left(\frac{b}{x}-1\right)\left(1-\frac{a}{x}\right)\frac{p(x)}{x^2}dx \\
=\ & \frac{a^2b(f(a)+f(b))}{(b-a)^2}\int_a^b \left(\frac{b}{x}-1\right)\frac{p(x)}{x^2}dx \\
& -\frac{c(ab)^2}{(b-a)}\int_a^b \left(\frac{b}{x}-1\right)\left(1-\frac{a}{x}\right)\frac{p(x)}{x^2}dx.
\end{aligned}
$$

From which we obtain the desired inequality (20).

□

**Remark 4.6.** *If $p(x) \equiv 1$ in the Theorem 4.5, we obtain inequalities of Hermite- Hadamard type equal to that of Theorem 4.1.*

In effect, the left-hand side Hermite-Hadamard type inequalities is obtained.

On the other hand, if we replace $p(x) = 1$ for all $x \in [a, b]$ in the right side of the inequality of Theorem 4.5, we will obtain that

$$
\int_a^b \frac{f(x)}{x^2}dx \leq \frac{a(f(a)+f(b))}{b-a}\int_a^b \left(\frac{b}{x}-1\right)\frac{1}{x^2}dx - cab\int_a^b \left(\frac{b}{x}-1\right)\left(1-\frac{a}{x}\right)\frac{1}{x^2}dx.
$$

From this last inequality we get that

$$
\begin{aligned}
& \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx \\
\leq\ & \frac{a^2b(f(a)+f(b))}{(b-a)^2}\int_a^b \left(\frac{b}{x}-1\right)\frac{1}{x^2}dx - \frac{ca^2b^2}{b-a}\int_a^b \left(\frac{b}{x}-1\right)\left(1-\frac{a}{x}\right)\frac{1}{x^2}dx \\
=\ & \frac{f(a)+f(b)}{2} - \frac{c}{6}\left[b-a\right]^2,
\end{aligned}
$$

thus completing the proof.

## 5. Comments

The main contributions of this paper has been the introduction of a new class of function of generalized convexity, we have shown that these classes contain some previously known classes as special cases as well as Hermite-Hadamard type inequalities for these functions. We expect that the ideas and techniques used in this paper may inspire interested readers to explore some new applications of these newly introduced functions in various fields of pure and applied sciences.

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