Universidad del Atlántico

MATUA

Revista Del Programa De Matemáticas Páginas: 62–76

Facultad de Ciencias Básicas

Vol. IV , № 2, (2017)

Una Variante de la desigualdad de Jensen-Mercer para funciones $h-$convexas y funciones de operadores $h-$convexas.

A variant of Jensen-Mercer Inequality for $h-$convex functions and Operator $h-$convex functions.

Miguel José Vivas Cortez[a], Jorge Eliecer Hernández Hernández[b]

mjvivas@puce.edu.ec and jorgehernandez@ucla.edu.ve

[a]Pontificia Universidad Católica del Ecuador
Facultad de Ciencias Naturales y Exactas
Escuela de Ciencias Físicas y Mtemáticas, Sede Quito, Ecuador
[b]Universidad Centroccidental Lisandro Alvarado
Decanato de Ciencias Económicas y Empresariales
Departamento de Técnicas Cuantitativas, Barquisimeto, Venezuela.

Resumen

En este artículo encontramos nuevas desigualdades relacionadas con la bien conocida desigualdad de Jensen-Mercer, y correspondientes aplicaciones a la Teoría de Operadores, usando funciones $h-$convexas y funciones de operadores $h-$convexas. Los resultados encontrados generalizan otros previamente formulados.

Palabras claves:
Desigualdad de Jensen-Mercer, Funciones $h-$convexas, Funciones de Operadores $h-$convexas.

Abstract

In the present paper, we have find some new inequalities related to the well known Jensen-Mercer Inequality, and its corresponding application to the theory of Operators, using $h-$convex functions and operator $h-$convex functions. These results generalize some others found in previous investigations.

Keywords:
Jensen-Mercer Inequality, $h-$convex functions, Operator $h-$convex functions.

1. Introduction

In recent years several extensions and generalizations have been considered for classical convexity, and the theory of inequalities has made essential contributions to many areas of Mathematics. In this paper we shall deal with an important and useful class of functions called operator convex functions. We introduce a new class of generalized convex functions, namely the class of *operator h-convex function.*

The theory of operator/matrix monotone functions was initiated by the celebrated paper of C. Löwner [23], which was soon followed by F. Kraus [22] on operator/matrix convex functions. After further developments due to some authors (for instance, J. Bendat and S. Sherman [5], A. Korányi [21], and U. Franz [11]), in their seminal paper [15] F. Hansen and G.K. Pedersen established a modern treatment of operator monotone and convex functions.

In the year 2003, McD Mercer [25] established an interesting variation of Jensen's inequality and later in 2009 Mercer's result was generalized to higher dimensions by M. Niezgoda [28]. Recently, Khan et al.[20] have stated an integral version of Niezgoda's result for convex functions.

In Mathematical literature we find the classical Jensen inequality, which discrete form is

$$f\left(\sum_{i=1}^{n} p_i x_i\right) \leq \sum_{i=1}^{n} p_i f(x_i) \tag{1}$$

where $x_1, ..., x_n \in \mathbb{R}$, $p_i \geq 0, i = 1, .., n$ with $\sum_{i=1}^{n} p_i = 1$ and $f : [a, b] \to R$ is a convex function.

A variant of this inequality is usually called Jensen-Mercer inequality, and its form is as follow (See [25])

$$f\left(a + b - \sum_{i=1}^{n} p_i x_i\right) \leq f(a) + f(b) - \sum_{i=1}^{n} p_i f(x_i) \tag{2}$$

where $x_1, ..., x_n \in [a, b] \subset \mathbb{R}$, $p_i \geq 0, i = 1, .., n$ with $\sum_{i=1}^{n} p_i = 1$ and $f : [a, b] \to R$ is a convex function.

It is well-known that one of the most fundamental and interesting inequalities for classical convex functions is that associated with the name of Hermite-Hadamard inequality which provides a lower and an upper estimations for the integral average of any convex functions defined on a compact interval, involving the midpoint and the endpoints of the domain. More precisely:

Theorem 1.1 (See [14]). *Let f be a convex function over $[a, b]$, $a < b$. If f is integrable over $[a, b]$, then*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{(b-a)} \int_a^b f(x)\, dx \leq \frac{f(a) + f(b)}{2}. \tag{3}$$

We find some new expressions for this last inequality, and a variant integral version in the environment of operators using $h-$convex functions.

2. Preliminaries

We know that a function $f : I \to \mathbb{R}$ is called a convex function over an interval $I \subset \mathbb{R}$ if for any $a, b \in I$ and for any $t \in [0, 1]$ we have the following inequality

$$f(ta + (1-t)b) \leq t f(a) + (1-t) f(b).$$

This concept has been generalized in several ways.

Definition 2.1. *[See [13]]We shall say that a function $f : I \subset \mathbb{R} \to \mathbb{R}$ is a Godunova-Levin function or $f \in Q(I)$ if f is non negative and for each $x, y \in I$ and $t \in (0, 1)$ we have*

$$f(tx + (1-t)y) \leq \frac{f(x)}{t} + \frac{f(y)}{1-t}.$$

Definition 2.2. *[See [10]]We say that $f : I \to \mathbb{R}$ is a $P-$function, or that f belongs to the class $P(I)$, if f is a non-negative function and for all $x, y \in I$, $t \in [0, 1]$ we have*

$$f(tx + (1-t)y) \leq f(x) + f(y).$$

Definition 2.3. *[See [4]]Let $s \in (0, 1]$. A function $f : (0, \infty] \to (0, \infty]$ is named $s-$ convex (in the second sense), or $f \in K_s^2$ if*

$$f(\lambda x + (1-\lambda)y) \leq \lambda^s f(x) + (1-\lambda)^s f(y)$$

for each $x, y \in (0, \infty]$ and $\lambda \in [0, 1]$.

In [31], Varošanec even more widespread these concepts.

Definition 2.4. *Let $h : J \to R$ a function non negative and identically nonzero, defined on an interval $J \subset R$, with $(0, 1) \subset J$. We shall say that a function $f : I \to R$, defined on an interval $I \subset R$, is $h-$convex if f is non negative and the following inequality holds*

$$f(tx + (1-t)y) \leq h(t) f(x) + h(1-t) f(y).$$

We can see, from this definition, that

1. If $h(t) = 1$ then an $h-$ convex function f is a $P-$function.
2. If $h(t) = t^s$, $s \in (0, 1]$ then an $h-$convex function f is an $s-$function.
3. If $h(t) = t^s$, with $s = -1$ then an $h-$convex function f is a Godunova-Levin function.

In [**?**], Sarikaya et. al. established a new Hadamard-type inequality for $h-$ convex functions.

Theorem 2.5. *Let $f \in SX(h, I)$, $a, b \in I$ with $a < b$ and $f \in L_1([a, b])$. Then*

$$\frac{1}{2h(1/2)} f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x)dx \leq (f(a) + f(b)) \int_0^1 h(t)dt \tag{4}$$

With $B(H)$ we shall denote the C^*-algebra commutative of all bounded operators over a Hilbert space H with inner product $\langle , \rangle$. Let $\mathcal{A}$ be a subalgebra of $B(H)$. An operator $A \in \mathcal{A}$ is positive if $\langle Ax, x\rangle \geq 0$ for all $x \in H$. Over $\mathcal{A}$ there exists an order relation by means

$$A \leq B \text{ if } \langle Ax, x\rangle \leq \langle Bx, x\rangle$$

or

$$B \geq A \text{ if } \langle Bx, x\rangle \geq \langle Ax, x\rangle$$

for $A, B \in \mathcal{A}$ self-adjoint operators and for all $x \in H$.

The Gelfand map established a $*-$isometrically isomorphism Φ between the set $C(Sp(A))$ of all continuous functions defined over the spectrum of A,denoted by $Sp(A)$, and the C^*-algebra $C^*(A)$ generated by A and the identity operator $\mathbf{1}_H$ over H as follows:

For any $f, g \in C(Sp(A))$ and $\alpha, \beta \in \mathbb{C}$ (Complex numbers) we have

- $\Phi(\alpha f+\beta g)=\alpha\Phi(A)+\beta\Phi(B)$
- $\Phi(fg)=\Phi(A)\Phi(B)$ and $\Phi\left(\overline{f}\right)=\Phi(f)^*$
- $\|\Phi(f)\|=\|f\|:=\sup_{t\in Sp(A)}|f(t)|$
- $\Phi(f_0)=\mathbf{1}_H$ and $\Phi(f_1)=A$, where $f_0(t)=1$ y $f_1(t)=t$ for all $t\in Sp(A)$

with this notation we define

$$f(A)=\Phi(f)$$

and we call it the continuous functional calculus for a self-adjoint operator A.

If A is a self-adjoint operator and f is a continuous real valued function on $Sp(A)$ then

$$f(t)\geq 0 \text{ for all } t\in Sp(A)\Rightarrow f(A)\geq 0$$

that is to say $f(A)$ is a positive operator over H. Moreover, if both functions f,g are continuous real valued functions on $Sp(A)$ then

$$f(t)\geq g(t) \text{ for all } t\in Sp(A)\Rightarrow f(A)\geq g(A)$$

respect to the order in $B(H)$.

In [9], we can find the following definition

Definition 2.6. *A real valued continuous function f over an interval I is said operator convex if*

$$f(tA+(1-t)B)\leq tf(A)+(1-t)f(B)$$

in the operator order for $t\in[0,1]$ and for every self-adjoint operator A y B on a Hilbert space H whose spectra are contained in I.

and with this, Dragomir, in Theorem 1, in [9], proved a Hermite-Hadamard type inequality

$$f\left(\frac{A+B}{2}\right)\leq\int_0^1 f(tA+(1-t)B)dt\leq\frac{f(A)+f(B)}{2}$$

for a operator convex functions of self-adjoint positive operators in a Hilbert space H.

Ghazanfari [12] defined operator $s-$convex function in this way.

Definition 2.7. *Let I be an interval in $[0,\infty)$ y K a convex subset of $B(H)^+$. A continuous function $f:I\to\mathbb{R}$ is said to be operator $s-$convex on I for operators in K if*

$$f((1-\lambda)A+\lambda B)\leq(1-\lambda)^s f(A)+\lambda^s f(B)$$

in the operator order in $B(H)$, for all $\lambda\in[0,1]$ and for every positive operator A and B in K whose spectra are contained in I and for some fixed $s\in(0,1]$.

With this definition he proved that

$$2^{s-1}f\left(\frac{A+B}{2}\right)\leq\int_0^1 f(tB+(1-t)A)dt\leq\frac{f(A)+f(B)}{s+1}$$

for an operator $s-convex$ function for operators in K.

Dragomir in [30] introduced an even more general definition of operator convex functions.

Definition 2.8. *Let J be an interval include in R with $(0,1) \subset J$. Let $h : J \to R$ be a non negative and identically nonzero function. We shall say that a continuous function $f : I \to R$, defined on an interval $I \subset R$, is an operator $h-$convex for operators in K if*

$$f\left(tA + (1-t)B\right) \leq h(t) f(A) + h(1-t) f(B)$$

for all $t \in (0,1)$ and $A, B \in K \subseteq B(H)^+$ such that $Sp(A) \subset I$ and $Sp(B) \subset I$.

With this concept he presented some results involving operators $h-$convex functions. The first of them is located as Lemma 2.3 in [30] and it involves the associated function φ. The second is the Theorem 2.4 in [30], which establishes the Hermite-Hadamard type inequality for operator $h-$convex functions.

Lemma 2.9. *If f is an operator $h-$convex function then*
$\varphi_{x,A,B}(t) = \langle (f(tA + (1-t)B)x, x) \rangle$
for $x \in H$ with $\|x\| = 1$ is an $h-$convex function over $(0,1)$

Theorem 2.10. *Let f be an operator $h-$convex function. Then*

$$\frac{1}{2h(1/2)} f\left(\frac{A+B}{2}\right) \leq \int_0^1 f(tB + (1-t)A)dt \leq (f(A) + f(B)) \int_0^1 h(t)dt \tag{5}$$

3. Main Results.

Let $0 < x_1 \leq x_2 \leq ... \leq x_n$ be real numbers and let w_k $(1 \leq k \leq n)$ be positive weights associated with these x_k and $\sum_{k=1}^{n} w_k = 1$.

From now on, let $h : J \to \mathbb{R}$ be a nonnegative function defined over an interval $J \subset \mathbb{R}$ such that $(0,1) \subset J$.

Lemma 3.1. *Let $f : I \to \mathbb{R}$ a positive and $h-$convex function, then we have*

$$f\left(x_1 + x_n - x_k\right) \leq M\left(f(x_1) + f(x_n)\right) - f(x_k)$$

for any $x_k \in I, (1 \leq k \leq n)$, and for all $\lambda \in (0,1)$,where

$$M = \sup\{h(t) : t \in (0,1)\}.$$

Proof:
Write $y_k = x_1 + x_n - x_k$. Then $x_1 + x_n = x_k + y_k$ so that the pairs x_1, x_n and x_k, y_k posses the same midpoint. Since that is the case, there exists a λ such that

$$x_k = \lambda x_1 + (1-\lambda)x_n$$

$$y_k = (1-\lambda)x_1 + \lambda x_n$$

where $0 \leq \lambda \leq 1$ and $1 \leq k \leq n$. Then, applying the h-convexity of f

$$\begin{aligned}
f(y_k) &\leq h(1-\lambda)f(x_1) + h(\lambda)f(x_n) \\
&= h(1-\lambda)f(x_1) + h(\lambda)f(x_n) + h(\lambda)f(x_1) + h(1-\lambda)f(x_n) - (h(\lambda)f(x_1) + h(1-\lambda)f(x_n)) \\
&= (h(\lambda) + h(1-\lambda))(f(x_1) + f(x_n)) - (h(\lambda)f(x_1) + h(1-\lambda)f(x_n)) \\
&\leq (h(\lambda) + h(1-\lambda))(f(x_1) + f(x_n)) - f(\lambda x_1 + (1-\lambda)x_n) \\
&\leq M(f(x_1) + f(x_n)) - f(\lambda x_1 + (1-\lambda)x_n)
\end{aligned}$$

where $M = \sup\{h(t) : t \in (0,1)\}$. And since $y_k = x_1 + x_n - x_k$, we have

$$f(x_1 + x_n - x_k) \leq M(f(x_1) + f(x_n)) - f(x_k).$$

The proof is complete.

Theorem 3.2. *Let $f : I \subset \mathbb{R}^+ \to \mathbb{R}$ a positive and $h-$convex function, and let $0 < x_1 \leq x_2 \leq ... \leq x_n$ be real numbers in I, and $w_k, (1 \leq k \leq n)$ positive numbers such that $\sum_{k=1}^n w_k = 1$. Then we have*

$$f\left(x_1 + x_n - \sum_{k=1}^n w_k x_k\right) \leq M(f(x_1) + f(x_n)) - \sum_{k=1}^n h(w_k) f(x_k)$$

where $M = \sup\{h(t) : t \in (0,1)\}$.

Proof:
Since $\sum w_k = 1$ we have

$$f\left(x_1 + x_n - \sum_{k=1}^n w_k x_k\right) = f\left(\sum_{k=1}^n w_k(x_1 + x_n - x_k)\right)$$

and applying the $h-$convexity of f

$$f\left(x_1 + x_n - \sum_{k=1}^n w_k x_k\right) \leq \sum_{k=1}^n h(w_k) f(x_1 + x_n - x_k)$$

and using Lemma 3.1 we have

$$\begin{aligned} f\left(x_1 + x_n - \sum_{k=1}^n w_k x_k\right) &\leq \sum_{k=1}^n h(w_k) f(x_1 + x_n - x_k) \\ &\leq \sum_{k=1}^n h(w_k)\left[M(f(x_1) + f(x_n)) - f(x_k)\right] \\ &\leq M(f(x_1) + f(x_n)) - \sum_{k=1}^n h(w_k) f(x_k) \end{aligned}$$

where $M = \sup\{h(t) : t \in (0,1)\}$.

Remark 3.3. *If $h(t) = t$ we have*

$$M = \sup\{h(t) : t \in (0,1)\} = 1$$

so the inequality in Theorem 3.2 takes the form

$$f\left(x_1 + x_n - \sum_{k=1}^n w_k x_k\right) \leq f(x_1) + f(x_n) - \sum_{k=1}^n w_k f(x_k)$$

and it coincides with the result showed by Mercer in Theorem 1.2 in [25].

Remark 3.4. *If* $h(t) = t^s, (0 < s \leq 1)$ *we have*

$$M = \sup\{t^s : t \in (0,1)\} = 1$$

an so, for $s-$*convex functions we have*

$$f\left(x_1 + x_n - \sum_{k=1}^{n} w_k x_k\right) \leq f(x_1) + f(x_n) - \sum_{k=1}^{n} w_k^s f(x_k).$$

Corollary 3.5. *Let* $A_1, A_2, ..., A_n \in \mathcal{B}(H)$ *be self-adjoint operators with spectra in* $[a,b]$ *for some scalars* $a < b$ *and* $w_k, (1 \leq k \leq n)$ *positive numbers such that* $\sum_{k=1}^{n} w_k = 1$. *Then*

$$f\left(A_1 + A_n - \sum_{k=1}^{n} w_k A_k\right) \leq M\left(f(A_1) + f(A_n)\right) - \sum_{k=1}^{n} h(w_k) f(A_k)$$

where $M = \sup\{h(t) : t \in (0,1)\}$.

Proof:
Following the scheme of proof corresponding to Theorem 3.2, we find the desired result.

In what follows we assume that H and K are Hilbert spaces, $\mathcal{B}(H)$ and $\mathcal{B}(K)$ are C^*-algebras of all bounded operators on the appropriate Hilbert space and $P[\mathcal{B}(H), \mathcal{B}(K)]$ is the set of all positive linear maps from $\mathcal{B}(H)$ to $\mathcal{B}(K)$. We denote by $C([a,b])$ the set of all real valued continuous functions on an interval $[a,b]$.

Theorem 3.6. *Let* h *be a super-additive function and* $A_1, A_2, .., A_n \in \mathcal{B}(H)$ *be self-adjoint operators with spectra in* $[a,b]$ *for some scalars* $a < b$ *and* $\Phi_1, \Phi_2, .., \Phi_n \in P[\mathcal{B}(H), \mathcal{B}(K)]$ *positive linear maps with* $\sum_{i=1}^{n} \Phi_i(I_H) = I_K$. *If* $f \in C([a,b])$ *is an operator* $h-$*convex on* $[a,b]$ *then*

$$f\left(aI_K + bI_K - \sum_{i=1}^{n} \Phi_i(A_i)\right) \leq f(a)h(I_K) + f(b)h(I_K) - \sum_{i=1}^{n} \Phi_i\left(f(A_i)\right).$$

Proof:
Since f is continuous and operator $h-$convex then the function $g : [a,b] \to R$ defined by $g(t) = f(a+b-t), t \in [a,b]$ is continuous and $h-$convex too. Note that

$$\frac{t-a}{b-a} \in [0,1]\ ,\ \frac{b-t}{b-a} \in [0,1]$$

and

$$\frac{t-a}{b-a} + \frac{b-t}{b-a} = 1$$

for all $t \in [a,b]$. Furthermore

$$\begin{aligned} \frac{t-a}{b-a}b + \frac{b-t}{b-a}a &= \frac{1}{b-a}(bt - ab + ab - ta) \\ &= \frac{1}{b-a}(b-a)t \\ &= t \end{aligned}$$

for all $t \in [a,b]$.

Since f is continuous and h−convex then the function $g:[a,b] \to R$ defined by $g(t) = f(a+b-t), t \in [a,b]$ is continuous and h−convex too. Hence

$$f(t) = f\left(\frac{t-a}{b-a}b + \frac{b-t}{b-a}a\right) \leq h\left(\frac{t-a}{b-a}\right)f(b) + h\left(\frac{b-t}{b-a}\right)f(a)$$

$$g(t) = g\left(\frac{t-a}{b-a}b + \frac{b-t}{b-a}a\right) \leq h\left(\frac{t-a}{b-a}\right)g(b) + h\left(\frac{b-t}{b-a}\right)g(a)$$

Since $aI_H \leq A_i \leq bI_H, (i = 1,2,..,n)$ and $\sum_{i=1}^n \Phi_i(I_H) = I_K$, it follows that $aI_K \leq \sum_{i=1}^n \Phi_i(A_i) \leq bI_K$.

Now, using the functional calculus, we have

$$g\left(\sum_{i=1}^n \Phi_i(A_i)\right) \leq h\left(\frac{\sum_{i=1}^n \Phi_i(A_i) - aI_K}{b-a}\right)g(b) + h\left(\frac{bI_K - \sum_{i=1}^n \Phi_i(A_i)}{b-a}\right)g(a)$$

using the definition of g we can rewrite

$$\begin{aligned}
f\left(aI_K + bI_K - \sum_{i=1}^n \Phi_i(A_i)\right) \leq\ & h\left(\frac{\sum_{i=1}^n \Phi_i(A_i) - aI_K}{b-a}\right)f(a) + h\left(\frac{bI_K - \sum_{i=1}^n \Phi_i(A_i)}{b-a}\right)f(b) \\
=\ & h\left(\frac{\sum_{i=1}^n \Phi_i(A_i) - aI_K}{b-a}\right)f(a) + h\left(\frac{bI_K - \sum_{i=1}^n \Phi_i(A_i)}{b-a}\right)f(b) \\
& + h\left(\frac{bI_k - \sum_{i=1}^n \Phi_i(A_i)}{b-a}\right)f(a) + h\left(\frac{\sum_{i=1}^n \Phi_i(A_i) + aI_k}{b-a}\right)f(b) \\
& - \left(h\left(\frac{bI_k - \sum_{i=1}^n \Phi_i(A_i)}{b-a}\right)f(a) + h\left(\frac{\sum_{i=1}^n \Phi_i(A_i) + aI_k}{b-a}\right)f(b)\right) \\
=\ & \left(h\left(\frac{\sum_{i=1}^n \Phi_i(A_i) - aI_K}{b-a}\right) + h\left(\frac{bI_k - \sum_{i=1}^n \Phi_i(A_i)}{b-a}\right)\right)f(a) \\
& + \left(h\left(\frac{\sum_{i=1}^n \Phi_i(A_i) - aI_K}{b-a}\right) + h\left(\frac{bI_k - \sum_{i=1}^n \Phi_i(A_i)}{b-a}\right)\right)f(b) \\
& - \left(h\left(\frac{bI_k - \sum_{i=1}^n \Phi_i(A_i)}{b-a}\right)f(a) + h\left(\frac{\sum_{i=1}^n \Phi_i(A_i) + aI_k}{b-a}\right)f(b)\right)
\end{aligned}$$

using the the superadditivity property of the function of h, we get

$$\begin{aligned}
f\left(aI_K + bI_K - \sum_{i=1}^n \Phi_i(A_i)\right) \leq\ & h\left(\frac{bI_k - aI_K}{b-a}\right)(f(a) + f(b)) \\
& - \left(h\left(\frac{bI_k - \sum_{i=1}^n \Phi_i(A_i)}{b-a}\right)f(a) + h\left(\frac{\sum_{i=1}^n \Phi_i(A_i) + aI_k}{b-a}\right)f(b)\right) \\
=\ & h(I_k)(f(a) + f(b)) \\
& - \left(h\left(\frac{bI_k - \sum_{i=1}^n \Phi_i(A_i)}{b-a}\right)f(a) + h\left(\frac{\sum_{i=1}^n \Phi_i(A_i) + aI_k}{b-a}\right)f(b)\right)
\end{aligned}$$

On the other hand

$$f(A_j) \leq h\left(\frac{A_j - aI_H}{b-a}\right)f(b) + h\left(\frac{bI_H - A_j}{b-a}\right)f(a)$$

Applying positive linear maps Φ_j and summing, it follows that

$$\sum_{i=1}^{n}\Phi_i(f(A_i)) \le h\left(\frac{bI_k - \sum_{i=1}^{n}\Phi_i(A_i)}{b-a}\right)f(a) + h\left(\frac{\sum_{i=1}^{n}\Phi_i(A_i) + aI_k}{b-a}\right)f(b)$$

Now the inequality can be written like

$$f\left(aI_K + bI_K - \sum_{i=1}^{n}\Phi_i(A_i)\right) \le h(I_k)f(a) + h(I_k)f(b) - \sum_{i=1}^{n}\Phi_i(f(A_i)).$$

The proof is complete.

Theorem 3.7. *Let h be an integrable function and f a h−convex function on* $[m, M]$. *Then*

$$\begin{aligned} f\left(M+m+\frac{x+y}{2}\right) &\le f(M)+f(m)-\int_0^1 f(tx+(1-t)y)dt \\ &\le f(M)+f(m)-2h(1/2)f\left(\frac{x+y}{2}\right) \end{aligned} \tag{6}$$

and

$$\begin{aligned} f\left(M+m+\frac{x+y}{2}\right) &\le \frac{1}{y-x}\int_x^y f(M+m-t)dt \\ &\le f(M)+f(m)-(f(x)+f(y))\int_0^1 h(t)dt \end{aligned} \tag{7}$$

for all $x, y \in [m, M]$

Proof:
From Jensen-Mercer inequality for h−convex function (Theorem 3.2), we have

$$f\left(M+m-\frac{a+b}{2}\right) \le f(M)+f(m)-h(1/2)(f(a)+f(b)) \tag{8}$$

for all $a, b \in [m, M]$. Let $t \in [0, 1]$ and $x, y \in [m, M]$. Let $a = tx + (1-t)y$ and $b = (1-t)x + ty$, and replacing in 8 we have

$$\begin{aligned} f\left(M+m-\frac{x+y}{2}\right) &\le f(M)+f(m) \\ &\quad - h(1/2)(f(tx+(1-t)y)+f((1-t)x+ty)) \end{aligned} \tag{9}$$

Integrating (9) over $[0, 1]$ we get

$$f\left(M+m-\frac{x+y}{2}\right) \le f(M)+f(m)-h(1/2)\int_0^1 (f(tx+(1-t)y)+f((1-t)x+ty))\,dt. \tag{10}$$

Due to

$$\int_0^1 f(tx+(1-t)y)dt = \int_0^1 f((1-t)x+ty)dt = \frac{1}{y-x}\int_x^y f(t)dt \tag{11}$$

then, the inequality (10) takes the form

$$f\left(M+m-\frac{x+y}{2}\right)\le f(M)+f(m)-2h\,(1/2)\int_0^1 f((1-t)x+ty)dt$$

and using the Hermite-Hadamard inequality for $h-$convex function (4)

$$\begin{aligned} f\left(M+m-\frac{x+y}{2}\right) &\le f(M)+f(m)-2h\,(1/2)\int_0^1 f((1-t)x+ty)dt \\ &\le f(M)+f(m)-2h\,(1/2)\,f\left(\frac{x+y}{2}\right). \end{aligned}$$

Now, to obtain the inequality 7 we will use the Hermite-Hadamard inequality.

$$\begin{aligned} \int_0^1 f(M+m-((1-t)x+ty))dt &= \int_0^1 f(t(M+m-x)+(1-t)(M+m-y))dt \\ &\ge f\left(\frac{M+m-x+M+m-y}{2}\right) \\ &= f\left(M+m-\frac{x+y}{2}\right) \end{aligned}$$

In the other hand, using the Jensen Mercer inequality for $h-$convex function (Theorem 3.2)

$$f(M+m-((1-t)x+ty))\le f(M)+f(m)-(h(t)f(x)+h(1-t)f(y))$$

Integrating over $[0,1]$ and having account that

$$\int_0^1 h(t)dt=\int_0^1 h(1-t)dt$$

we get

$$\begin{aligned} \int_0^1 f(M+m-((1-t)x+ty))dt &\le f(M)+f(m)-\int_0^1 (h(t)f(x)+h(1-t)f(y))\,dt \\ &= f(M)+f(m)-(f(x)+f(y))\int_0^1 h(t)dt \end{aligned}$$

In consequence, from (11),(12) and (12) we get

$$\begin{aligned} f\left(M+m+\frac{x+y}{2}\right) &\le \frac{1}{y-x}\int_x^y f(M+m-t)dt \\ &\le f(M)+f(m)-(f(x)+f(y))\int_0^1 h(t)dt. \end{aligned}$$

Remark 3.8. *If $h(t)=t, t\in[0,1]$ we have, from inequality (6),*

$$\begin{aligned} f\left(M+m+\frac{x+y}{2}\right) &\le f(M)+f(m)-\int_0^1 f(tx+(1-t)y)dt \\ &\le f(M)+f(m)-f\left(\frac{x+y}{2}\right), \end{aligned}$$

and from inequality (7), we get

$$\begin{aligned} f\left(M+m+\frac{x+y}{2}\right) &\le \frac{1}{y-x}\int_x^y f(M+m-t)dt \\ &\le f(M)+f(m)-\frac{1}{2}\left(f(x)+f(y)\right), \end{aligned}$$

which correspond to convex functions. If $h(t)=t^s, t\in[0,1], (0<s\le 1)$ *we have, from inequality (6),*

$$\begin{aligned} f\left(M+m+\frac{x+y}{2}\right) &\le f(M)+f(m)-\frac{1}{2^{s+1}}\int_0^1 f(tx+(1-t)y)dt \\ &\le f(M)+f(m)-f\left(\frac{x+y}{2}\right), \end{aligned}$$

and from inequality (7), we get

$$\begin{aligned} f\left(M+m+\frac{x+y}{2}\right) &\le \frac{1}{y-x}\int_x^y f(M+m-t)dt \\ &\le f(M)+f(m)-\frac{1}{s+1}\left(f(x)+f(y)\right), \end{aligned}$$

which correspond to $s-$*convex functions in second sense. Finally, if* $h(t)=1, t\in[0,1]$, *we have, from inequality (6),*

$$\begin{aligned} f\left(M+m+\frac{x+y}{2}\right) &\le f(M)+f(m)-\int_0^1 f(tx+(1-t)y)dt \\ &\le f(M)+f(m)-f\left(\frac{x+y}{2}\right), \end{aligned}$$

and from inequality (7), we get

$$\begin{aligned} f\left(M+m+\frac{x+y}{2}\right) &\le \frac{1}{y-x}\int_x^y f(M+m-t)dt \\ &\le f(M)+f(m)-\left(f(x)+f(y)\right), \end{aligned}$$

which correspond to $P-$*convex functions.*

Theorem 3.9. *Let* h *be an integrable function and* f *a* $h-$*convex function on* $[m,M]$, *then*

$$\int_0^1 f\left(M+m-(t\Phi(A)+(1-t)\Phi(B)\right)dt \le f(M)+f(m)-(\Phi(f(A))+\Phi(f(B))\int_0^1 h(t)dt \qquad (12)$$

and

$$f\left(M+m-\frac{\Phi(A)+\Phi(B)}{2}\right) \le 2h(1/2)\int_0^1 f\left(M+m-t\Phi(A)-(1-t)\Phi(B)\right)dt \qquad (13)$$

for all self-adjoint operators A, B *with spectra in* $[m,M]$ *and a unitary positive linear map . Furthermore, if* f *is operator* $h-$*convex, then*

$$\begin{aligned} f\left(M+m-\frac{\Phi(A)+\Phi(B)}{2}\right) &\le \int_0^1 f\left(M+m-(t\Phi(A)+(1-t)\Phi(B))\right)dt \qquad (14) \\ &\le f(M)+f(m)-(\Phi(f(A))+\Phi(f(B))\int_0^1 h(t)dt \end{aligned}$$

Proof:
Using the Jensen Mercer inequality for operator $h-$convex function we have

$$\begin{aligned}\int_0^1 f\left(M+m-(t\Phi(A)+(1-t)\Phi(B)\right)dt &\leq \int_0^1 f(M)+f(m)-h(t)\Phi(f(A))-h(1-t)\Phi(f(B))dt \\ &= f(M)+f(m)-(\Phi(f(A))+\Phi(f(B))\int_0^1 h(t)dt\end{aligned}$$

this corresponds to 12. Using Jensen Mercer inequality

$$\begin{aligned}& f\left(M+m-\frac{\Phi(A)+\Phi(B)}{2}\right) \\ &= f\left(\frac{M+m-t\Phi(A)-(1-t)\Phi(B)+M+m-(1-t)\Phi(A)-t\Phi(B)}{2}\right) \\ &\leq h(1/2)\left[f\left(M+m-t\Phi(A)-(1-t)\Phi(B)\right)+f\left(M+m-(1-t)\Phi(A)-t\Phi(B)\right)\right]\end{aligned}$$

Integrating over $[0,1]$

$$f\left(M+m-\frac{\Phi(A)+\Phi(B)}{2}\right)\leq 2h(1/2)\int_0^1 f\left(M+m-t\Phi(A)-(1-t)\Phi(B)\right)dt$$

which corresponds to the left side of 14. The second is clear.

Remark 3.10. *If $h(t)=t, t\in[0,1]$ we have, from inequality (14),*

$$\begin{aligned}f\left(M+m-\frac{\Phi(A)+\Phi(B)}{2}\right) &\leq \int_0^1 f\left(M+m-(t\Phi(A)+(1-t)\Phi(B))\right)dt \\ &\leq f(M)+f(m)-\frac{1}{2}(\Phi(f(A))+\Phi(f(B))\end{aligned}$$

which corresponds to convex functions. If $h(t)=t^s, t\in[0,1], (0<s\leq 1)$ we have, from inequality (14) ,

$$\begin{aligned}f\left(M+m-\frac{\Phi(A)+\Phi(B)}{2}\right) &\leq \int_0^1 f\left(M+m-(t\Phi(A)+(1-t)\Phi(B))\right)dt \\ &\leq f(M)+f(m)-\frac{1}{s+1}(\Phi(f(A))+\Phi(f(B))\end{aligned}$$

which corresponds to $s-$convex functions in second sense. Finally, if $h(t)=1, t\in[0,1]$, we have, from inequality (14),

$$\begin{aligned}f\left(M+m-\frac{\Phi(A)+\Phi(B)}{2}\right) &\leq \int_0^1 f\left(M+m-(t\Phi(A)+(1-t)\Phi(B))\right)dt \\ &\leq f(M)+f(m)-(\Phi(f(A))+\Phi(f(B)).\end{aligned}$$

4. Conclusions

In the present work we have found results that relate the classic inequality of Jensen-Mercer with the h-convex functions, and from these, inequalities of the same type are deduced in the framework of self-adjoint operators in Hilbert spaces. Particularized results are also shown for convex, $s-$convex functions in the second sense and $P-$convex. This article is expected to serve as a motivation to generate other investigations that involve other types of generalized convexity.

Acknowledgment

The authors want to thank to Council for the Development of Science, Humanities and Technology (Concejo para el Desarrollo de las Ciencias, Humanidades y Tecnología - CDCHT) from Universidad Centroccidental Lisandro Alvarado (Venezuela), for the technical support provided in the preparation of this work.

Referencias

[1] J.L. Aujla, H. L. Vasudeva, *Convex and monotone operator functions*, Annales Polonici Mathematici 62 (1995) 1–11.

[2] V. Bacak., R. Turkmen, *New Inequalities for operator convex functions*, Journal of Inequalities and Applications 2013 (2013): 190.

[3] H.H. Bauschke, P.L. Combettes, *Convex Analysis and Monotone Operator Theory in Hilbert Spaces*, Springer New York Dordrecht Heidelberg London, 2010.

[4] W.W. Breckner, *Stetigkeitsaussagen für eine Klasse verallgemeinerter konvexer funktionen in topologischen linearen Räumen*, Pub. Inst. Math., 23 (1978) 13–20.

[5] J. Bendat and S. Sherman, *Monotone and Convex Operator Functions*, Trans. Amer. Math. Soc. 79 (1955) 58–71

[6] L. Bougoffa. *New Inequalities for convex functions*. Journal of Inequalities in Pure and Applied Mathematics. Vol, /(4), Nro. 148. 2006

[7] P. Chansangiam. *A Survey on Operator Monotonicity, Operator convexity and Operator Means.* International Journal of Analysis. Vol. 2015. Article Id 649839. 8 pp.

[8] M.J. Cloud, B.C. Drachman. *Inequalities: With Applications to Engineering*. Springer-Verlag, New York, Inc. 1998

[9] S.S. Dragomir. *Hermite Hadamard's type inequalities for operator convex functions*. Journal of Mathematical Inequalities. 4(4), 2010, 587 - 591

[10] S.S. Dragomir, J. Pečarić, L.E Persson. *Some inequalities of Hadamard type*. Soochow J. Math. 21(1995) 335 - 341

[11] U. Franz, F. Hial, E. Ricard. *Higher Order Exyension of Löwner's Theory: Operator k-Tone Functions*, arXiv: 1105.3881v4. 2014

[12] A.G. Ghazanfari. *The Hermite Hadamard type inequalities for operator s-convex functions*. Journal of Advanced Research in Pure Mathematics. Vol. 6, Issue. 3, 2014, pp. 52-61

[13] E.Godunova, V. Levin. *Neravenstva dlja funkcii širokogo klassa, soderžaščego vypuklye, monotonnye,i nekotorye drugie vidy funkcii, in : Vyčislitel.* Mat. i. Mat. Fiz. Mežvuzov. Sb. Nauč. Trudov. MGPI. Moskva. 1985. pp 138-142

[14] J.S Hadamard. *Etude sur les propiètés des fonctions entieres et en particulier dúne fontion considerer per Riemann*, J. Math. Pure and Appl. 58 (1893) 171- 215

[15] F. Hansen and J.K. Pedersen. *Jensen's Inequality for Operators and Löowner Theorem*, Math. Ann. 258 (1982), pp 229-241

[16] G.H. Hardy, J.E. Littlewood, G. Polya. *Inequalities.* Cambidge University Press. London. 1934

[17] *İ*. Işcan, *Hermite-Hadamard-Fejer type inequalities for convex functions via fractional integrals*, Stud. Univ. Babeş Bolyai Math. 60(2015) 3 355–366.

[18] I. Işcan, M. Kunt, *Hermite-Hadamard-Fejér type inequalities for quasi-geometrically convex functions via fractional integrals, Hindawi Publishing Corporation*, Journal of Mathematics 2016 (2016) 7.

[19] J.L.W Jensen, *Sur les fonctions convexes et le inequalitiés entre les valeurs moyennes*, Acta Math. 32(1906) 175 – 193.

[20] A. R. Khan, J. Pečarić, M. Praljak, *A Note on Generalized Mercer's*, Bulletin of the Malaysian Mathematical Sciences Society 40 (2017) 2 881–889.

[21] A. Korànyi, *On a Theorem of Löwner and its Connection with of Resolvent of Transformation*, Acta Sci. Math. (Szeged) 17 (1956) 63–70

[22] P. Kraus, *Über Konvexe Matrixfunktionen*, Math. Z. 41 (1936) 18–42.

[23] K. Löwner, *Über Monotone Matrixfunktionen*, Math. Z. 38 (1934) 177–216.

[24] A. Matković, J. Pečarić, I. Perić, *A variant of Jensen's inequality of Mercer's type for operators with applications*, Linear Algebra Appl. 418 (2006) 2, 551–564.

[25] A. McD. Mercer, *A variant of Jensen's inequality*, J. Inequal. Pure Appl. Math. 4 (2003) Article 73

[26] M. Minculete, F.C. Mitroi, *Féjer-type inequalities* Aust. J. Math. Anal. Appl. 9 (2012) 8 pp.

[27] M. Kian, M.S. Moslehian, *Refinements of the operator Jensen-Mercer inequality, Electron*, J. Linear Algebra 26 (2013) 742–753.

[28] M. Niezgoda, *A generalization of Mercer's result on convex functions*, Nonlinear Analysis 71 (2009) 2771-2779.

[29] J. Park, *Inequalities of Hermite-Hadamard-Fejér type for convex functions via fractional integrals*, International Journal of Mathematical Analysis, 8 (2014) 59 2927–2937.

[30] A. Taghavi, V. Darvish, M. Nazari, S.S. Dragomir, *Some Inequalities Associated with the Hemite-Hadamard Inequalities for Operator h-convex Functions*, RGMIA Research Report Collection, 18(2015).

[31] S. Varošanec, *On h convexity* , J. Math. Anal. Appl. 326 (2007) 303 – 311

[32] X. Zhan, *Matrix Inequalities*, Lectures Notes in Mathematics, Springer-Verlag Berlin Heidelberg, 2002.