Stability Regions for a Delay Cobweb Model†

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1 Introduction

Dynamic model can be constructed in discrete and continuous time scales. It is well known that in examining dynamics, continuous models are more stable than discrete systems in a sense that the form has a larger stability region in the parameter space than the latter. Sometimes two models exhibit the opposite results. For example, Theocharis (1960) shows that the N-firm Cournot adjustment model in discrete time scale is unstable if $N \geq 3$ whereas the corresponding model in continuous time scale is always stable regardless of the number of firms. It is curious to know what makes these differences. As a first step, we consider a delay differential model that is a hybrid of these two models. Our main aim of this study is to provide stability analysis with respect to time delay and the behavioral parameters of the model. In particular we construct a simple 1 D continuous model of price adjustment that is locally and globally stable. Introducing information lag to obtain the price information, we study how the qualitative behavior of dynamics changes as the model parameters including the length of delay vary.

The investigation of stability for a delay differential equation can be reduced to the root location problem for the corresponding characteristic equation. Hayes (1950) determines stability conditions with respect to the model's parameter under which the real parts of the characteristic roots are all negative. Burger (1956) modifies Hayes'conditions to improve applicability. The complete stability region in the parameters space has already been characterized by Bellman and Cooke (1963) and Boese (1993), to name a few, while Kung

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(1993) summarizes the developments of delay differential equations. The use of delay differential equations in the modeling of economic dynamics also has a long history. Haldane (1934) could be the first to examine economic dynamics in a delay differential equation. Kalecki (1935) and Goodwin (1951) investigate macroeconomic fluctuations by applying delay differential equations. It is still very active due to the fact that the delay is inevitable phenomenon in economics and also due to the recent mathematical developments for understanding the multiple delay system. In this paper, focusing on the simplest delay differential equation with constant coefficients, we analytically and geometrically determine the region of the behavioral parameters and the delay parameter for which the delay differential equation is stable or unstable.[1)]

This paper is organized as follows. Section 2 presents a continuous time nonlinear price adjustment model is presented. Section 3 is divided into two parts ; in the first part, stability analysis with respect to the length of time delay is considered and in the second part, stability analysis with respect to the behavioral parameters are examined. Section 4 concludes the paper

2 Model

Let $D(p)$ and $S(p^e)$ be the demand and supply functions of commodity. Here p is the commodity price and p^e is its expectation. For the sake of simplicity, these functions are assumed to be linear,

$$D(p) = d_1 - d_2 p,\ d_1 > 0,\ d_2 > 0 \tag{1}$$

and

$$S(p^e) = s_1 + s_2 p^e,\ s_1 > 0,\ s_2 > 0. \tag{2}$$

The equilibrium price p^* and quantity q^* satisfy the conditions of $p^* = p(t) = p^e(t)$ for all $t \geq 0$ and $q^* = D(p^*) = S(p^*)$ where

$$p^* = \frac{d_1 - s_1}{d_2 + s_2} \text{ and } q^* = \frac{d_1 s_2 + d_2 s_1}{d_2 + s_2}.$$

For positivity of the equilibrium price, we assume the following :

Assumption 1. $d_1 > s_1$

We consider price dynamics in a continuous-time framework in which relative variations

1) Matsumoto and Szidarovszky (2015) investigate the delay effects in the same Cobweb model with two different delays. Main focus is on the multiptle delay effects on dynamics but no behaviroral parameter effects are considered.

in market price at time t is adjusted to be proportional to the excess demand,

$$\frac{\dot{p}(t)}{p(t)} = K\{D[p(t)] - S[p^e(t)]\} \tag{3}$$

where $K > 0$ is the adjustment coefficient. Substituting equations (1) and (2) into (3) reduces the price adjustment equation to the following form

$$\dot{p}(t) = Kp(t)[d_1 - s_1 - d_2 p(t) - s_2 p^e(t)] \tag{4}$$

Equation (4) has two steady states, a trivial one, $p = 0$ and a positive one, $p = p^*$. We are not concerned with the trivial solution, which will be eliminated from further considerations. Concerning the expectation formation, we take into account a spirit of the naive expectation such that a past price at time $t - \tau$ is expected to be realized at time t :

Assumption 2. $p^e(t) = p(t - \tau),\ \tau \geq 0$

3 Dynamic Analysis

As a benchmark, we start with a non-delay case of $\tau = 0$. Substituting $p^e(t) = p(t)$ into the dynamic equation (4) yields an ordinary differential equation,

$$\dot{p}(t) = Kp(t)[d_1 - s_1 - (d_2 + s_2)p(t)] \tag{5}$$

The equilibrium price is the only positive steady state. Arranging the terms can reduce equation (5) to the logistic equation,

$$\dot{p}(t) = \theta p(t)\left[1 - \frac{p(t)}{p^*}\right]$$

where $\theta = K(d_1 - s_1) > 0$. Solving this equation by separating the variables, we obtain

$$p(t) = \frac{p_0 p^*}{p_0 + (p^* - p_0)e^{-\theta t}}$$

with the initial value $p_0 = p(0)$. This solution implies that the price approaches its equilibrium value as $t \to \infty$ for any initial value, so the equilibrium is globally stable.

Theorem 1 *If supply is instantaneous* (*i.e.*, $\tau = 0$), *then the equilibrium price of equation* (5) *is globally asymptotically stable with monotonic convergence*.

Since it is often observed in the real economy that the expectation formation is not instantaneous due to information delays, $\tau > 0$ is assumed henceforth. The dynamic equation then becomes a first-order nonlinear delay differential equation,

$$\dot{p}(t) = Kp(t)[d_1 - s_1 - d_2 p(t) - s_2 p(t-\tau)] \tag{6}$$

If $d_2 = 0$ (i.e., perfectly inelastic demand), then equation (6) can be reduced to the Hutchinson equation or delay logistic equation,

$$\dot{p}(t) = \theta p(t)\left[1 - \frac{p(t-\tau)}{\overline{p}}\right] \text{ with } \overline{p} = \frac{d_1 - s_1}{s_2} > 0.$$

Note that $\overline{p} = p^*$ when $d_2 = 0$. Concerning stability of the positive steady state, the following results are well known.2)

Theorem 2 *The positive steady state*, $\overline{p}$ *of equation* (6) *is asymptotically stable if* $0 < \tau < \pi/2\theta$ and unstable if $\tau > \pi/2\theta$; *a Hopf bifurcation occurs if* $\tau = \pi/2\theta$ *and a stable periodic solution exists if the steady state is unstable*.

We now proceed to the case where demand is imperfectly elastic (i.e., $0 < d_2 < \infty$) and see how the stability of the stationary point is affected when the values of delay and parameters are changed. If the right hand side of equation (6) is denoted by $G(p(t), p(t-\tau))$, then the linearized equation in a neighborhood of the stationary point $\boldsymbol{p}_2^* = (p^*, p^*)$ is

$$\dot{p}_\delta(t) = \left.\frac{\partial G}{\partial p(t)}\right|_{\boldsymbol{p}_2^* = (p^*, p^*)} p_\delta(t) + \left.\frac{\partial G}{\partial p(t-\tau)}\right|_{\boldsymbol{p}_2^* = (p^*, p^*)} p_\delta(t-\tau)$$

or

$$\dot{p}_\delta(t) = -kd_2 p_\delta(t) - ks_2 p_\delta(t-\tau)$$

where $k = Kp^* > 0$ and $p_\delta(t) = p(t) - p^*$. Introducing the new parameters $\alpha = kd_2 > 0$ and $\beta = ks_2 > 0$, we obtain the following form,

$$\dot{p}_\delta(t) + \alpha p_\delta(t) + \beta p_\delta(t-\tau) = 0. \tag{7}$$

Assuming an exponential solution

$$p_\delta(t) = e^{\lambda t} u$$

and substituting it into the linearized equation present the corresponding characteristic equation,

$$\lambda + \alpha + \beta e^{-\lambda\tau} = 0. \tag{8}$$

The equilibrium is asymptotically stable if all eigenvalues of equation (8) have negative real parts. Substituting $\lambda = \gamma + i\omega$ with $\omega \geq 0$ breaks down the characteristic equation into the

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2) See, for example, Kuang (1993) and Ruan (2006).

real and imaginary parts,[3]

$$\gamma+\alpha+\beta e^{-\gamma\tau}\cos\omega\tau=0,$$
$$\omega-\beta e^{-\gamma\tau}\sin\omega\tau=0. \tag{9}$$

If $\tau=0$, then $\omega=0$ via the second equation of (9), $\gamma=-(\alpha+\beta)<0$ via the first equation. This is an another way of confirming the stability of the positive equilibrium in the case of no-delay. With the positive delay, we are interested in specifying parametric combination for which the stability could be lost. Let us denote the real part of eigenvalue λ by Re λ. It continuously depends on values of the delay as well as another parameters. We seek parametric conditions such as Re $\lambda=0$. Since $\lambda=0$ is not a solution, the characteristic equation must have a pair of purely imaginary solutions if Re $\lambda=0$ holds. Supposing $\gamma=0$ simplifies system (9),

$$\alpha+\beta\cos\tau\omega=0,$$
$$\omega-\beta\sin\tau\omega=0, \tag{10}$$

We can show that all pure complex roots of equation (8) are simple. Otherwise $\lambda=i\omega$ solves both equations

$$\lambda+\alpha+\beta e^{-\lambda\tau}=0$$

and

$$1+(-\tau)\beta e^{-\lambda\tau}=0$$

implying that

$$1+\tau(\lambda+\alpha)=0$$

which cannot occurs if λ is a pure complex number.

We will obtain a threshold value of the delay for which stability might get lost. Before proceeding, we check the roles of the parameters α and β on dynamics. Returning to equation (7) and assuming $\beta=0$, we have a solution

$$p_\delta(t)=p_\delta(0)e^{-\alpha t}$$

where $p_\delta(0)$ is an initial value. It monotonically converges to the zero solution if $\alpha>0$. Hence a positive α is a stabilizing factor. On the other hand, assuming $\alpha=0$ and solving the two equations in (10) for τ yields a threshold value of the delay

3) We will have the same results even if $\omega\leq 0$ is assumed.

$$\tau = \frac{\pi}{2\beta}.$$

A positive β is a destabilizing factor in a sense that a larger β makes the threshold value of τ smaller. Roughly speaking it is more likely that the model with a positive delay is more destabilized as β is larger.

Moving α and ω to the right hand side of the equations in system (10) and adding the squares of the resultant equations yield

$$\omega^2 = \beta^2 - \alpha^2.$$

This implies that ω is not an independent variable anymore. If $\alpha > \beta$, then there is no $\omega > 0$ and thus equation (8) has no purely imaginary solutions implying that no stability switch occurs. The stability of the equilibrium point is preserved for any value of τ. If $\alpha = \beta$, then we have $\omega = 0$. However $\lambda = 0$ does not solve equation (8). Hence the stability is also preserved. A delay that does not affect stability is called *harmless*. So $\alpha \geq \beta$ intuitively implies that the stabilizing effect dominates over the destabilizing effect. This is a case in which any delay is harmless.

Theorem 3 *If $\beta \leq \alpha$, then the positive steady state of dynamic equation(6) is locally asymptotic stable for any positive value of τ.*

On the other hand, if $\beta > \alpha$, then we can obtain $\overline{\omega} > 0$ such as

$$\overline{\omega} = \sqrt{(\beta + \alpha)(\beta - \alpha)}. \tag{11}$$

There is a unique τ, $0 < \tau\overline{\omega} \leq 2\pi$ such that $\tau\overline{\omega}$ makes two equations in (10) hold. Thus solving the first equation of system (10) yields the threshold value of τ,

$$\overline{\tau}_m(\alpha, \beta) = \frac{1}{\overline{\omega}}\left[\cos^{-1}\left(-\frac{\alpha}{\beta}\right) + 2m\pi\right] (m = 0, 1, 2, \ldots). \tag{12}$$

and solving the second equation gives

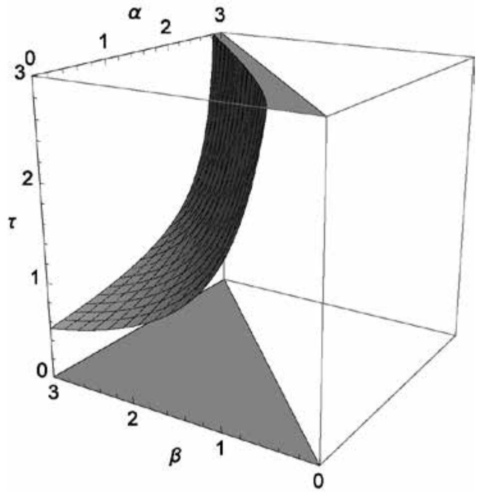
$$\overline{\tau}_n(\alpha, \beta) = \frac{1}{\overline{\omega}}\left[\pi - \sin^{-1}\left(\frac{\overline{\omega}}{\beta}\right) + 2n\pi\right] (n = 0, 1, 2, \ldots).$$

Further, solving the first equation for β and substituting it into the second equation, then solving the resultant equation for τ yield the third expression for the critical value of the delay,

$$\overline{\tau}_k(\alpha, \beta) = \frac{1}{\overline{\omega}}\left[\pi - \tan^{-1}\left(\frac{\overline{\omega}}{\alpha}\right) + 2k\pi\right] (k = 0, 1, 2, \ldots).$$

Notice that $\overline{\tau}_m(\alpha, \beta)$, $\overline{\tau}_n(\alpha, \beta)$ and $\overline{\tau}_k(\alpha, \beta)$ give rise to the same value if $m = n = k$,

Figure 1 Stability changing surface in the (α, β, τ) space



although these have different forms. The $\tau = \bar{\tau}_0(\alpha, \beta)$ surface is illustrated in the (α, β, τ) space in Figure 1 with $\alpha \geq 0, \beta \geq 0$ and $\tau \geq 0$. The cube is divided into two triangular prisms by the plane standing on the $\alpha = \beta$ line. Since $\alpha \geq \beta$ holds in the right-hand triangular prism, the system is stable due to Theorem 3. The left-hand triangular prism is further divided into two parts by the convex-shaped $\tau = \bar{\tau}_0(\alpha, \beta)$ surface. It will be shown that the system is stable in the region below the surface (i.e., $\tau < \bar{\tau}_0(\alpha, \beta)$) and unstable above (i.e., $\tau > \bar{\tau}_0(\alpha, \beta)$).

3.1 Stability switching curve

We concentrate on finding the stability regions in the (α, τ) plane and the (β, τ) plane. To this end, the $\tau = \bar{\tau}_0(\alpha, \beta)$ surface is projected to the 2 D plane to find the parameter effects on dynamics caused by a change in the parameter value. In Figures 2(A) and (B), the $\tau = \bar{\tau}_0(\alpha, \tilde{\beta})$ curve with $\tilde{\beta} = 5$ and the $\tau = \bar{\tau}_0(\tilde{\alpha}, \beta)$ curve with $\tilde{\alpha} = 1$ are, respectively, illustrated in the (α, τ) plane and the (β, τ) plane. In Figure 2(A) the $\tau = \bar{\tau}_m(\alpha, \tilde{\beta})$ curves for $m = 0, 1, 2, 3$ are depicted for $\alpha < \tilde{\beta}$. Each curve is positive-sloping as

$$\frac{\partial \bar{\tau}_m}{\partial \alpha} = \frac{1 + \alpha \bar{\tau}_m(\alpha, \tilde{\beta})}{\tilde{\beta}^2 - \alpha^2} > 0$$

and shift upward as m increases. It is confirmed that $\bar{\tau}_m(\alpha, \tilde{\beta})$ is asymptotic to the vertical line at $\alpha = \tilde{\beta}$ and

$$\bar{\tau}_m(0, \tilde{\beta}) = \frac{(1+4m)\pi}{2\tilde{\beta}} \text{ and } \lim_{\alpha \to \tilde{\beta}} \bar{\tau}_m(\alpha, \tilde{\beta}) = \infty.$$

In Figure 2(B), condition $\beta \leq \tilde{\alpha}$ holds in the hatched area so that the system is stable due to

Theorem 3. Next, we verify the shape of $\bar{\tau}_m(\tilde{\alpha}, \beta)$. Differentiating equation (12) with respect to β yields

$$\frac{\partial \bar{\tau}_m}{\partial \beta} = -\frac{\tilde{\alpha} + \beta^2 \bar{\tau}_m(\tilde{\alpha}, \beta)}{\beta(\beta^2 - \tilde{\alpha}^2)} < 0 \tag{13}$$

implying that the $\bar{\tau}_m(\tilde{\alpha}, \beta)$ curve is downward-sloping in the (β, τ) plane. Parameter m is increased from 0 to 3 and the corresponding four $\bar{\tau}_m(\tilde{\alpha}, \beta)$ curves are depicted. It is clear from equation (12) that increasing m shifts the $\bar{\tau}_m(\tilde{\alpha}, \beta)$ curve upward. It is also confirmed that each curve is asymptotic to the vertical line at $\beta = \tilde{\alpha}$ as $\beta \to \tilde{\alpha}$ and to the horizontal line at $2m\pi$ as $\beta \to \infty$, since we have

$$\lim_{\beta \to \tilde{\alpha}} \bar{\tau}_m(\tilde{\alpha}, \beta) = \infty \text{ and } \lim_{\beta \to \infty} \bar{\tau}_m(\tilde{\alpha}, \beta) = 2m\pi.$$

We now examine the τ-effect generated by changes in τ. Since a solution of the characteristic equation is a continuous function of delay τ, differentiating equation (8) with respect to τ and rearranging terms yield

$$\frac{\partial \lambda}{\partial \tau} = \frac{\lambda \beta e^{-\lambda\tau}}{1 - \beta\tau e^{-\lambda\tau}}.$$

Substituting $\lambda = i\omega$ and taking the real part give

$$\mathrm{Re}\left[\left.\frac{\partial \lambda}{\partial \tau}\right|_{\lambda = i\omega}\right] = \mathrm{Re}\left[\left.\frac{\lambda \beta e^{-\lambda\tau}}{1 - \beta\tau e^{-\lambda\tau}}\right|_{\lambda = i\omega}\right]$$

$$= \mathrm{Re}\left[\left.-\frac{\lambda(\lambda + \alpha)}{1 + \alpha\tau + \tau\lambda}\right|_{\lambda = i\omega}\right]$$

$$= \mathrm{Re}\left[\frac{\omega^2 - i[\alpha\omega(1 + \alpha\tau) + \tau\omega^3]}{(1 + \alpha\tau)^2 + (\tau\omega)^2}\right]$$

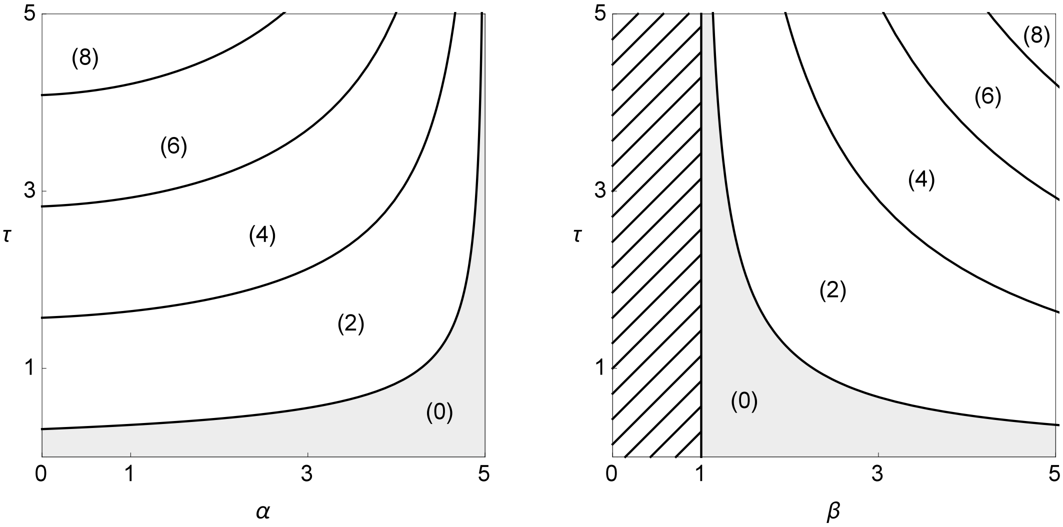
Hence

$$\left.\frac{\partial(\mathrm{Re}\lambda)}{\partial \tau}\right|_{\lambda = i\omega} = \frac{\omega^2}{(1 + \alpha\tau)^2 + (\tau\omega)^2} > 0. \tag{14}$$

This inequality implies that all solutions crossing the imaginary axis at $i\omega$ cross from left to right as τ increases, that is, the real part becomes positive from negative.

Focusing on Figure 2(B), we see how the stability is lost when the value of τ increases. Given $\alpha > 0$, selecting β such as $\beta > \alpha$ and increasing the value of τ along the vertical line at this β, it sooner or later crosses the $\bar{\tau}_m$ curve with $m = 0$, the lowest downward-sloping curve in Figure 2(B). Let the τ-value of the intersection be $\bar{\tau}_0$. We have already confirmed that the equilibrium is stable for $\tau = 0$. Thus for $0 < \tau < \bar{\tau}_0$, all solutions of equation (8) have

Figure 2 Stability switching curves in the 2 D plane



(A) The (α, β) plane (B) The (β, τ) plane

strictly negative real parts and thus the equilibrium is still stable under the positive delay. Since the selection of β is arbitrary as far as $\beta > \alpha$ holds, the equilibrium is stable in the region below the $\bar{\tau}_{m=0}$ curve. This region is colored in gray in Figure 2(B). On the $\bar{\tau}_{m=0}$ curve, the real part of one solution becomes zero and $\partial \mathrm{Re}\lambda / \partial \tau > 0$. The real part of this solution becomes positive and the equilibrium loses stability for $\tau > \bar{\tau}_0$. Further it can be shown by the Hopf bifurcation theorem that a periodic cycle emerges for $\tau > \bar{\tau}_0$. Stability is switched to instability at the first intersection. In other words, in the gray region, the number of the positive roots is zero and the number increases to two in the region just above as indicated by the figures in the parentheses in Figure 2(B). The $\bar{\tau}_{m=0}$ curve is the boundary between these two regions and we call it the *stability switching curve*. With further increasing τ, the vertical line crosses the $\bar{\tau}_m$ curve with $m = 1$, the second lowest curve. At the τ-value of this intersection, the real part of another solution becomes zero. No stability switch occurs at the second intersection (i.e., instability is preserved) but the number of the positive roots becomes four in the region just above this $\bar{\tau}_{m=1}$ curve. The same phenomenon can be observed at each intersection of the $\bar{\tau}_m$ curve with $m > 1$ and the number of the positive roots increases accordingly when we increase τ further. When τ increases and the value of β is fixed, the same phenomenon can be observed in Figure 2 (A) in which the system is stable in the gray region. We may go on from these observations to the following well-known results on the τ-effect : a larger delay has a destabilizing effect.

Theorem 4 *If* β *and* α *are selected such as* $\beta > \alpha$, *then the positive steady state of the dynamic equation* (6) *is* *asymptotically stable for* $\tau < \bar{\tau}_0(\alpha, \beta)$

while it is

unstable for $\tau > \bar{\tau}_0(\alpha, \beta)$

where it loses stability at $\tau = \bar{\tau}_0(\alpha, \beta)$ *and a limit cycle emerges for* $\tau > \bar{\tau}_0(\alpha, \beta)$ *with*

$$\bar{\tau}_0(\alpha, \beta) = \frac{1}{\sqrt{(\beta+\alpha)(\beta-\alpha)}}\cos^{-1}\left(-\frac{\alpha}{\beta}\right) > 0.$$

Notice that $\beta \leq \alpha$ means $s_2 \leq d_2$, which then implies that the rate of increase of the supply is not greater than the rate of decrease of the demand. As is known, this is the stability condition of the cobweb model in discrete-time scales. It is already seen that the continuous-time cobweb model is always stable for any positive values of α and β. For $\beta > \alpha$, the discrete-time cobweb model is unstable while the delay cobweb model is stable for $\tau < \bar{\tau}_0$ and unstable otherwise. Hence the delay cobweb model is more stable than the discrete-time model and less stable than the continuous-time model.

3.2 Root crossing curve

We reconsider the behavioral parameter effects from a different view point in the (α, β) plane. With $\tau = 1$,[4] the loci of $\bar{\tau}_m(\alpha, \beta) = 1$ divide the (α, β) space into subregions by the number of unstable characteristic roots. We first allow α and β to be negative to examine the global plane division and then restrict them to nonnegative values. First of all, the equilibrium is stable in the region defined as

$$\overline{S}_1 = \{(\alpha, \beta) \mid |\beta| \leq \alpha \text{ and } 0 \leq \alpha\}$$

in which all eigenvalues have negative real parts or the number of the roots with nonnegative real parts is zero. We then turn attention to a boundary case in which $\omega = 0$ and $\tau > 0$. The first equation of system (10) leads to

$$\beta = -\alpha \text{ for } \omega = 0. \tag{15}$$

We then move to a non-boundary case in which $\omega > 0$ and $\tau > 0$,[5]

$$\text{If } \omega\tau \neq j\pi \text{ for } j \in N, \text{ then } \tilde{\alpha}(\omega) = -\frac{\omega \cos \tau\omega}{\sin \tau\omega} \text{ and } \tilde{\beta}(\omega) = \frac{\omega}{\sin \tau\omega} \tag{16}$$

where $\tilde{\alpha}(\omega)$ and $\tilde{\beta}(\omega)$ solve system (10) simultaneously. So the locus of $\tilde{\alpha}(\omega)$ and $\tilde{\beta}(\omega)$ is given as $L(\omega) = \{\tilde{\alpha}(\omega), \tilde{\beta}(\omega)\}$ for $\omega > 0$.

4) $\tau = 1$ is selected only for convenience. The results to be obtained below can hold for any other value of τ.

5) If $\omega\tau = j\pi$ holds for $j \in N$, then the second equation of (10) gives $\omega = 0$ that violates $\omega > 0$. This case is eliminated from futher considerations.

For α and β on the line $\beta = -\alpha$, the characteristic root is real, crossing the imaginary axis at 0. On the other hand, a locus $L(\omega)$ of $\tilde{\alpha}(\omega)$ and $\tilde{\beta}(\omega)$ is associated with a complex conjugate pair of characteristic roots in the form $\lambda = i\omega$. These curves divide the parameter plane (α, β) into regions where the numbers of unstable characteristic roots are identical. In Figure 3, $L(\omega)$ for $\omega \in (0, \pi)$ constructs a positive sloping locus with

$$\lim_{\omega_+ \to 0} \tilde{\alpha}(\omega) = \lim_{\omega_+ \to 0} \tilde{\beta}(\omega) = \frac{1}{\tau}$$

and

$$\lim_{\omega_- \to \pi} \tilde{\alpha}(\omega) = \lim_{\omega_- \to \pi} \tilde{\beta}(\omega) = \infty$$

where $\omega_+ \to 0$ and $\omega_- \to \pi$ means that ω approaches zero from above and π from below. These boundary values imply that the locus starts at point $(1/\tau, 1/\tau)$ and becomes asymptotic to the diagonal line from above as ω goes to π. Hence, the region defined as

$$\overline{S}_2 = \{ (\alpha, \beta) \mid |\alpha| < \beta < \tilde{\beta}(\omega_\alpha) \text{ and } -\frac{1}{\tau} < \alpha \}$$

is the stability region where ω_α solves $\alpha = \tilde{\alpha}(\omega)$. In Figure 3, $\overline{S}_1$ is hatched and $\overline{S}_2$ is colored in gray. In the same way, $L(\omega)$ for $\omega \in (pi, 2\pi)$ is illustrated as a mound-shaped curve with

$$\lim_{\omega_+ \to \pi} \tilde{\alpha}(\omega) = \lim_{\omega_+ \to \pi} \tilde{\beta}(\omega) = -\infty$$

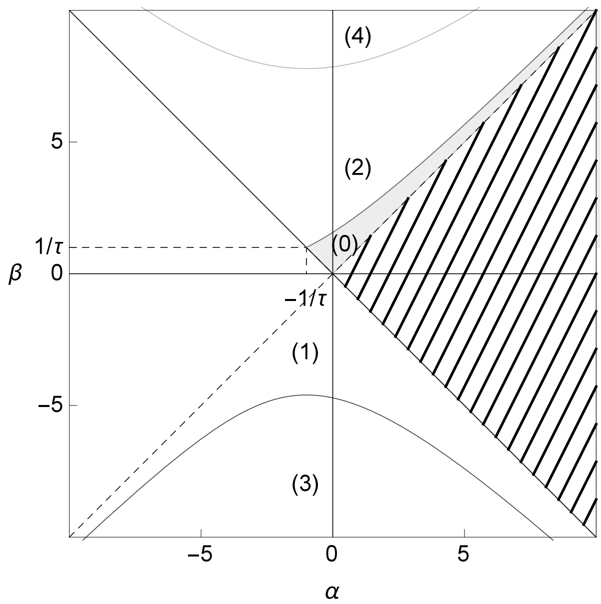
and

$$\lim_{\omega_- \to 2\pi} \tilde{\alpha}(\omega) = \infty, \lim_{\omega_- \to 2\pi} \tilde{\beta}(\omega) = -\infty.$$

This curve is asymptotic from below to the $\beta = \alpha$ line as $\omega_+ \to \pi$ and to the $\beta = -\alpha$ line as $\omega_- \to 2\pi$. A U-shaped curve is described by $L(\omega)$ for $\omega \in (2\pi, 3\pi)$ and is asymptotic from above to the $\beta = -\alpha$ line as $\omega_+ \to 2\pi$ and to the $\beta = \alpha$ line as $\omega_- \to 3\pi$. $L(\omega)$ for ω in the any other interval is defined as well.

Study on the stability of the solution of the linear one delay equation in the parameter region was started by Hayes (1950) that gives the stability criterion using the given coefficients of the linear delay equation. His result includes a solution of a transcendental equation, which makes it uneasy to apply except very special cases. Burger (1956) eliminates this obstacle and presents another stability conditions. Having the characteristic equation (8) with the simplified assumption $\tau = 1$, we can show that Burger's conditions are fulfilled in the stability region of Figure 3. We state his theorem in our notation.

Figure 3 Division of the (α, β) space



Theorem 5 (*Theorem* 1 *of Burger (1956)*) *For all roots* λ *of* $\lambda+\alpha+\beta e^{-\lambda}=0$ *to possess negative real parts, it is sufficient and necessary*

(1) *in the case of* $\beta \leq 1$ *that* $\beta > -\alpha$,

(2) *in the case of* $\beta > 1$ (*i*) *that* $\beta < \alpha$ *or* (*ii*) *that* $-\beta < \alpha \leq \beta$ *and* $\cos^{-1}(-\alpha/\beta) > \sqrt{\beta^2-\alpha^2}$ *where the value of the function* $\cos^{-1}$ *is restricted by* $0 < \cos^{-1}(-\alpha/\beta) < \pi$

It is apparent that *(1)* and *(2)*-(*i*) determine the region $\overline{S}_1$. From (12), it is seen that $\sqrt{\beta^2-\alpha^2} = \cos^{-1}(-\alpha/\beta)$ determines the locus of α and β satisfying $\overline{\tau}_m(\alpha, \beta) = 1$. From (16), on the $L(\omega)$ locus,

$$\tilde{\beta}(\omega) = -\frac{\tilde{\alpha}(\omega)}{\cos \tau\omega}$$

which can be rewritten as

$$\tau\omega = \cos^{-1}\left(-\frac{\tilde{\alpha}(\omega)}{\tilde{\beta}(\omega)}\right)$$

or

$$\sqrt{\beta^2-\alpha^2} = \cos^{-1}\left(-\frac{\alpha}{\beta}\right)$$

where $\tau = 1$ and $\alpha = \tilde{\alpha}(\omega)$, $\beta = \tilde{\beta}(\omega)$. The red locus of $L(\omega)$ in Figure 3 is depicted under condition $\tau = 1$. Since increasing τ value shifts the $L(\omega)$ locus downward, Burger's condition*(2)-(ii)* is satisfied in the region $\overline{S}_2$. Therefore the stability region can be constructed under Burger's stability conditions (as well as Hayes' conditions).

Boese (1993) points out that Hayes' results do not determine the stability region explicitly and provides the explicit conditions to construct the stability region. Again in our notation, his theorem is stated as follows :

Theorem 6 (*Theorem of Boese (1993)*) *The stability set is determined by parameter triples* (α, β, τ) *such as*

$$\tau < \tau_0(\alpha, \beta)$$

where

$$\tau_0(\alpha, \beta) = \begin{cases} 0 & if -\beta \geq \alpha, \\ \infty & if -\alpha < \beta \leq \alpha \ (>0), \\ \dfrac{1}{\overline{\omega}}\left[\dfrac{\pi}{2} - tan^{-1}\left(\dfrac{\alpha}{\overline{\omega}}\right)\right] & if\ |\alpha| < \beta \end{cases}$$

and $\overline{\omega} = \sqrt{\beta^2 - \alpha^2}$.

Notice that the last expression on the right hand side of $\tau_0(\alpha, \beta)$ is equivalent to the expression of $\overline{\tau}_{k=0}$ if we remember the relation,

$$\tan^{-1}(x) = \tan^{-1}\left(\frac{1}{x}\right) + \frac{\pi}{2}.$$

$\tau_0(\alpha, \beta) = \infty$ means that the delay is harmless and describes the striped region of Figure 3. The gray region of Figure 3 is described by the third conditions. The union of these regions is the stability region described by Hayes (1950).

We now shift to changing the number of the positive roots. Taking the partial derivative of the two equations in (9) with respect to β and considering that $\gamma = 0$ along the $L(\omega)$ locus gives

$$\begin{pmatrix} 1 - \beta\tau \cos \tau\omega & -\beta\tau \sin \tau\omega \\ \beta\tau \sin \tau\omega & 1 - \beta\tau \cos \tau\omega \end{pmatrix} \begin{pmatrix} \gamma_\beta \\ \omega_\beta \end{pmatrix} = \begin{pmatrix} -\cos \tau\omega \\ \sin \tau\omega \end{pmatrix}$$

where γ_β and ω_β are the partial derivatives of γ and ω with respect to β. Since we are interested in the sign of the partial derivative of γ, we solve it for γ_β,

$$\gamma_\beta = \frac{\beta\tau - \cos \tau\omega}{(1 - \beta\tau \cos \tau\omega)^2 + (\beta\tau \sin \tau\omega)^2}. \tag{17}$$

If $\tau\omega = 2j\pi$, then $\sin \tau\omega = 0$ and $\cos \tau\omega = 1$, both of which simplify equation (17),

$$\gamma_\beta = -\frac{1}{1-\beta\tau}.$$

It is apparent that the sign of γ_β is determined by the sign of $1-\beta\tau$. If $\tau\omega=(2j+1)\pi$, then $\sin\tau\omega=0$ and $\cos\tau\omega=-1$, so from (17)

$$\gamma_\beta = \frac{1}{1+\beta\tau}.$$

The sign of γ_β is determined by the sign of $1+\beta\tau$.

Theorem 7
(*i*) *when* $\tau\omega=2j\pi$, *then* $\gamma_\beta>0$ *if* $\beta>1/\tau$ *and* $\gamma_\beta<0$ *if* $\beta<1/\tau$.
(*ii*) *when* $\tau\omega=(2j+1)\pi$, *then* $\gamma_\beta>0$ *if* $\beta>-1/\tau$ *and* $\gamma_\beta<0$ *if* $\beta<-1/\tau$.

If $\omega\tau\neq j\pi$ for $j\in N$, then

$$sgn\,[\gamma_\beta]=sgn\,[\beta\tau-\cos\tau\omega]$$

To see dependency of the sign of the right hand side on a value of ω, we divide the basic interval $(0,\,2\pi)$ of ω into two parts $(0,\,\pi)$ and $(\pi,\,2\pi)$. If $\omega\in(0,\,\pi)$, then $\tau\omega>\sin\tau\omega>0$, so from (16)

$$\beta\tau=\frac{\tau\omega}{\sin\tau\omega}>1.$$

This inequality with $|\cos\tau\omega|\leq 1$ implies $\beta\tau-\cos\tau\omega>0$. Therefore, we have

$$\gamma_\beta>0 \text{ for } \omega\in(0,\,\pi).$$

This means that the complex conjugate pair of characteristic roots crosses the imaginary axis from left to right : two stable roots becomes unstable as β increases. It follows that the same holds for $\omega\in(2j\pi,\,(2j+1)\pi)$ for all $j\in N$.

On the other hand, if $\omega\in(\pi,\,2\pi)$, then $-1\leq\sin\tau\omega<0$ and $\tau\omega>|\sin\tau\omega|$ for $\omega>0$ imply that

$$\beta\tau=\frac{\tau\omega}{\sin\tau\omega}<-1$$

In the same way, this inequality with $|\cos\tau\omega|\leq 1$ leads to $\beta\tau-\cos\tau\omega<0$. Therefore, we have

$$\gamma_\beta<0 \text{ for } \omega\in(\pi,\,2\pi).$$

This means that the complex conjugate pair of characteristic roots crosses the imaginary axis from right to left : two unstable roots become stable as β increases. It can be shown that the same result holds for $\omega \in ((2j+1)\pi, 2(j+1)\pi)$ for all $j \in N$. As indicated in Figure 3, the number of the unstable roots are increased to two from zero if a pair of (α, β) crosses the curve from below. The $L(\omega)$ locus for $\omega \in (0, \pi)$ is called the *root crossing curve* in the (α, β) plane.

Theorem 8 *$\gamma_\beta > 0$ on the $L(\omega)$ locus defined for $\omega \in (2j\pi, (2j+1)\pi)$ and $\gamma_\beta < 0$ on the $L(\omega)$ locus defined for $\omega \in ((2j+1)\pi, 2(j+1)\pi)$ for $j = 0, 1, 2, \ldots$*

3.3 Comparison

In this section we confine our analysis to the region with $\alpha > 0$ and $\beta > 0$ again and examine the relation between the stability switching curve and the root-crossing curve. Figure 4(A) is an enlargement of Figure 2(B) where $\alpha = 1$ is set. The equilibrium is stable in the hatched rectangle and in the gray region in which $\beta > \alpha$ holds. Consider dynamics at point (β, π) on the $\bar{\tau}_{m=0}$ curve where $\tau = 1$ and $\beta = \beta_1 \simeq 2.262$.[6] Inequality (14) indicates that the equilibrium becomes unstable if τ is increased from 1. This is described by the upwards arrow at point $(\beta_1, 1)$. To detect the β-effect, we differentiate equation (8) with respect to β and then follow the similar procedure driving $\partial \operatorname{Re}\lambda / \partial \tau$ to obtain

$$\left.\frac{\partial \operatorname{Re}\lambda}{\partial \beta}\right|_{\lambda = i\omega} = \frac{\alpha/\beta + \beta\tau}{(1+\alpha\tau)^2 + (\tau\omega)^2} > 0. \tag{18}$$

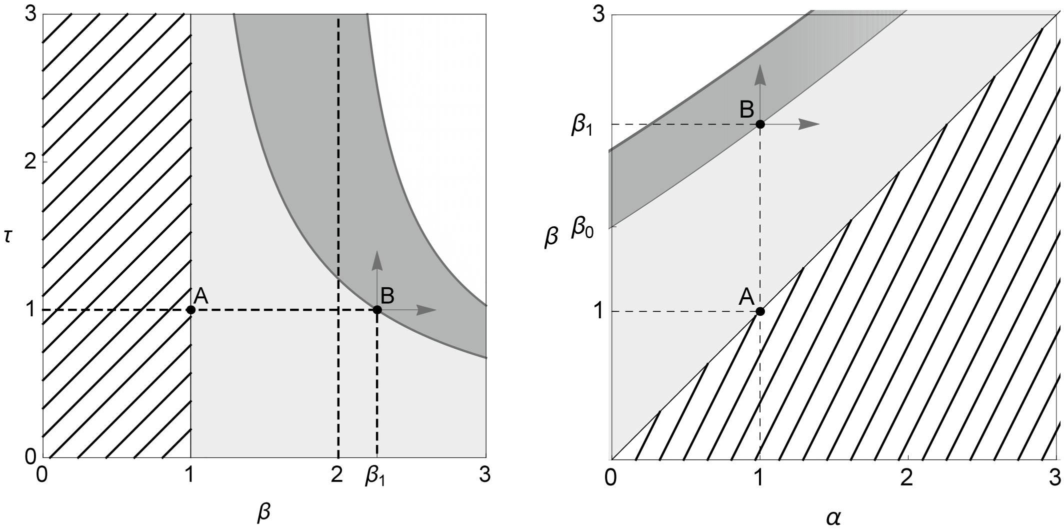
This inequality implies that increasing β along the $\tau = 1$ line destabilizes the equilibrium for $\beta > \beta_1$. This is also described by the rightward arrow at point $(\beta_1, 1)$. Concerning the α-effect, as already seen, increasing α value shifts not only the $\bar{\tau}_{m=0}$ curve upward but also the boundary line between the stability regions, S_1 and S_2, rightward as shown in Figure 4(A). The dark gray region above the gray region is a newly created stability region by increasing α to 2 from 1. Increasing τ and β have a destabilizing effect in the sense that such movements bring the point on the stability switching curve to the instability region. Increasing α, on the other hand, has a stabilizing effect as it enlarges the stability region and thus makes the boundary point $(\beta_1, 1)$ an interior point of the enlarged stability region.

Figure 4(B) is an enlargement of the first quadrant of Figure 3 where $\tau = 1$ is taken. In the hatched right triangle, $\beta \leq \alpha$ holds and thus the equilibrium is stable. The remaining region with $\beta > \alpha$ is divided by the locus of $L(\omega)$ for $\omega \in (0, \pi)$ into the gray stability region and the white and dark gray regions. We consider dynamics at point (α, β) on the $L(\omega)$ curve where $\alpha = 1$ and $\beta = \beta_1$, which is the same as the β_1 value in Figure 4(A).[7] The

6) Solving equation $1 = \bar{\tau}_m(\alpha, \beta)$ with $\alpha = 1$ and $m = 0$ for β gives this value.

7) We arrive at the same value in a different way. First we solve $1 = \tilde{\alpha}(\omega)$ to obtain $\omega \simeq 2.029$ and then

Figure 4 Plane divisions



(A) the (β, τ) plane (B) the (α, β) plane

inequality $\gamma_\beta > 0$ indicates that increasing β leads to a positive real part by moving a point on the $L(\omega)$ locus to the instability region and thus destabilizing the equilibrium for $\beta > \beta_1$. This is described by the upwards arrow at point $(1, \beta_1)$. To detect the α-effects, we differentiate equations (9) with respect to α to obtain

$$\gamma_\alpha = \frac{\beta\tau \cos \beta\tau - 1}{(1 - \beta\tau \cos \tau\omega)^2 + (\beta\tau \sin \tau\omega)^2}$$

that is, with $\omega > 0$, it is reduced to

$$\gamma_\alpha = -\frac{1 + \alpha\tau}{(1 + \alpha\tau)^2 + (\tau\omega)^2} < 0.$$

This inequality implies that the real part crosses the imaginary axis from right to left as α increases. The $L(\omega)$ locus with $\omega \in (0, \pi)$ in the (α, β) plane is identical with the $\bar{\tau}_m$ locus with $m = 0$ and 1 in the (β, τ) plane. Decreasing the value of τ to 1/2 from 1 shifts the $L(\omega)$ locus upward resulting in the enlargement of the stability region that is colored in dark gray in Figure 4(B). Since Figures 4(A) and 4(B) describe the same dynamics from a different view point, we have the following identities concerning the parameter effects on the real parts by using the relations described in (10) :

$$\mathrm{Re}\left[\left. \frac{\partial \lambda}{\partial \alpha} \right|_{\lambda = i\omega} \right] = \gamma_\alpha ,$$

substitute it into $\tilde{\beta}(\omega)$ to obtain this value.

$$\text{Re}\left[\left.\frac{\partial\lambda}{\partial\beta}\right|_{\lambda=i\omega}\right]=\gamma_\beta$$

$$\text{Re}\left[\left.\frac{\partial\lambda}{\partial\tau}\right|_{\lambda=i\omega}\right]=\gamma_\tau.$$

Hence the results obtained are summarized :

Theorem 9 *Increasing the value of α has a stabilizing effect while increasing β and increasing τ have destabilizing effects.*

4 Concluding Remarks

This study constructs a simple cobweb model with one time delay and conducts stability analysis with respect to the length of time delay τ and the behavioral parameters of the model, namely the slope of the demand curve, α and the slope of the supply curve β. It is shown that stability depends on α, β and τ in the following way :

(1) When the demand curve is steeper than the supply curve (i.e., $\alpha \geq \beta$), time delay becomes harmless.
(2) When the inequality is reversed (i.e., $\alpha < \beta$), there is a threshold value of the time delay and the model is asymptotically stable if the time delay is smaller than the threshold value and unstable if larger.
(3) Given the length of delay, a locus of α and β defined in such a way that the threshold value is equal to the given length divides the (α, β) region into the stable and unstable subregions.
(4) When the model is unstable, a periodic cycle emerges via Hopf bifurcation with respect to τ, α and β.

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