球面内の完備な部分多様体について
On complete minimal submanifolds in a sphere

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Let $S^{n+p}(c)$ be an $(n+p)$-dimensional Euclidean sphere of constant curvature c and M an n-dimensional minimal submanifold isometrically immersed in $S^{n+p}(c)$. We denote by A_ξ the Weingarten endomorphism associated a normal vector field ξ and T the tensor defined by $T(\xi,\eta) = \mathrm{trace} A_\xi A_\eta$.

Yuan and Matsuyama [13] proved the following: Let M be an n-dimensional compact minimal submanifold isometrically immersed in $S^{n+p}(c)$. Let σ and ψ are the second fundamental form of M in $S^{n+p}(c)$ and the immersion respectively. Then

$$|\sigma|^2 \leq \frac{np(n+2)}{2(n+p+2)}c \quad \text{and} \quad T = k\langle\ ,\ \rangle$$

if and only if one of the following conditions is satisfied:

(A) $|\sigma|^2 \equiv 0$ and M is totally geodesic.

(B) $|\sigma|^2 = \dfrac{np(n+2)}{2(n+p+2)}c$ and M is isotropic and has parallel second fundamental form.

Hence if ψ is full, then ψ is one of the following standard ones: $S^n(c) \to S^n(c)$; $PR^2(\frac{1}{3}c) \to S^4(c)$; $S^2(\frac{1}{3}c) \to S^4(c)$; $CP^2(c) \to S^7(c)$; $QP^2(\frac{3}{4}c) \to S^{13}(c)$; $CP^2(\frac{4}{3}c) \to S^{25}(c)$.

Moreover, they obtain the reseult of the case of M being complete: Let M be an n-dimensional complete minimal submanifold isometrically immersed in $S^{n+p}(c)$. Then

$$|\sigma|^2 \leq \frac{np(n+2)}{2(n+p+2)}c \quad \text{and} \quad T = k\langle\ ,\ \rangle.$$

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(B) $|\sigma|^2 = \dfrac{np(n+2)}{2(n+p+2)}c$ and M is isotropic and has parallel second fundamental form.

Rerated to these results, Li and Li[2] obtained without assumption of $T = k\langle\ ,\ \rangle$, the following: Let $A_1, A_2, ..., A_p$ be symmetric $(n \times n)$-matrices $(p \geq 2)$. Denote $S_{\alpha\beta} = \mathrm{trace}\,{}^tA_\alpha A_\beta, S_\alpha = S_{\alpha\alpha} = N(A_\alpha), S = S_1 + \cdots + S_p$. Then we have

$$\sum_{\alpha,\beta} N(A_\alpha A_\beta - A_\beta A_\alpha) + \sum_{\alpha,\beta} S_{\alpha\beta}^2 \leq \frac{3}{2}S^2,$$

and the equality holds if and only if one of the following conditions holds:

1) $A_1 = A_2 = ... = A_p = 0$,

2) only two of the matrices $A_1, A_2, ..., A_p$ are different from zero. Moreover, assuming $A_1 \neq 0, A_2 \neq 0, A_3 = ... = A_p = 0$, then $S_1 = S_2$, and there exists an orthogonal$(n \times n)$-matrix T such that

$$ {}^tTA_1T = \sqrt{\frac{S_1}{2}} \left(\begin{array}{cc|c} 1 & 0 & \\ 0 & -1 & 0 \\ \hline \multicolumn{2}{c|}{0} & 0 \end{array} \right), $$

$$ {}^tTA_2T = \sqrt{\frac{S_1}{2}} \left(\begin{array}{cc|c} 1 & 0 & \\ 0 & 1 & 0 \\ \hline \multicolumn{2}{c|}{0} & 0 \end{array} \right). $$

Using the result, they proved the following: Let M be an n-dimensional compact minimal submanifold in $S^{n+p}, p \geq 2$. If $|\sigma|^2 \leq \frac{2}{3}n$ everywhere on M, then M is either a totaly geodesic submanifold or a Velonese surface in S^4.

Now let $v \in UM_x$, $x \in M$. If $e_2, ..., e_n$ are orthonormal vectors in UM_x orthogonal to v, then we can consider $\{e_2, ..., e_n\}$ as an orthonormal basis of $T_v(UM_x)$. We remark that $\{v = e_1, e_2, ..., e_n\}$ is an orthonormal basis of T_xM. If we denote the Laplacian of $UM_x \cong S^{n-1}$ by Δ, then $\Delta f = e_2e_2f + \cdots + e_ne_nf$, where f is a differentiable function on UM_x.

Define functions $f_1(v), f_2(v), \cdots, f_{16}(v)$ on UM_x, $x \in M$, by

$$f_1(v) = \sum_{i=1}^{n} \langle A_{\sigma(v,e_i)}v, A_{\sigma(v,v)}e_i \rangle$$

$$f_2(v) = \sum_{i,j=1}^{n} \langle A_{\sigma(e_j,e_i)}e_j, A_{\sigma(v,v)}e_i \rangle,$$

$$f_3(v) = \sum_{i=1}^{n} \langle A_{\sigma(v,v)}v, A_{\sigma(v,e_i)}e_i \rangle,$$

$$f_4(v) = \sum_{i,j=1}^{n} \langle A_{\sigma(e_j,e_i)}e_j, A_{\sigma(v,e_i)}v \rangle,$$

$$f_5(v) = \sum_{i,j=1}^{n} \langle A_{\sigma(e_i,v)}e_i, A_{\sigma(v,e_j)}e_j \rangle,$$

$$f_6(v) = \sum_{i=1}^{n} \langle A_{\sigma(v,v)}e_i, A_{\sigma(v,v)}e_i \rangle,$$

$$f_7(v) = |\sigma(v,v)|^2,$$

$$f_8(v) = \sum_{i,j=1}^{n} \langle A_{\sigma(v,e_i)}e_j, A_{\sigma(e_j,v)}e_i \rangle,$$

$$f_9(v) = \sum_{i,j=1}^{n} \langle A_{\sigma(e_j,v)}e_i, A_{\sigma(e_j,v)}e_i \rangle,$$

$$f_{10}(v) = \sum_{i=1}^{n} \langle A_{\sigma(v,e_i)}e_i, v \rangle,$$

$$f_{11}(v) = |A_{\sigma(v,v)}v|^2.$$

$$f_{12}(v) = \sum_{i=1}^{n} \langle A_{\sigma(v,e_i)}v, A_{\sigma(v,e_i)}v \rangle$$

$$f_{13}(v) = |\sigma(v,v)|^4$$

$$f_{14}(v) = \sum_{i=1}^{n} \langle A_{\sigma(v,e_i)}e_i, v \rangle |\sigma(v,v)|^2$$

$$f_{15}(v) = (\sum_{i=1}^{n} \langle A_{\sigma(v,e_i)}e_i, v \rangle)^2$$

$$f_{16}(v) = |\sigma|^2 |\sigma(v,v)|^2,$$

The following generalized maximum principle due to Omori [11] and Yau [18] will be used in order to prove our theorem.

Generalized Maximum Principle. (Omori [11] and Yau [18])*Let M^n be a complete Riemannian manifold whose Ricci curvature is bounded from below and $f \in C^2(M)$ a function bounded from above on M^n. Then, for any $\epsilon > 0$, there exists a point $p \in M^n$ such that*

$$f(p) \geq \sup f - \epsilon, \qquad ||\text{grad } f|| < \epsilon, \qquad \Delta f(p) < \epsilon.$$

We have the following (See [7] and [8])

Lemma. *Let M be an n-dimensional minimal submanifold isometrically immersed in $S^{n+p}(c)$. Then for $v \in UM_x$ we have*

$$\begin{aligned}\frac{1}{2}\sum_{i=1}^{n}(\nabla^2 f_7)(e_i, e_i, v) =& \sum_{i=1}^{n}|(\nabla\sigma)(e_i, v, v)|^2 + nc|\sigma(v, v)|^2 \\ &+ 2\sum_{i=1}^{n}\langle A_{\sigma(v,v)}e_i, A_{\sigma(e_i,v)}v\rangle - 2\sum_{i=1}^{n}\langle A_{\sigma(v,e_i)}e_i, A_{\sigma(v,v)}v\rangle \\ &- \sum_{i=1}^{n}\langle A_{\sigma(v,v)}e_i, A_{\sigma(v,v)}e_i\rangle \\ =& \sum_{i=1}^{n}|(\nabla\sigma)(e_i, v, v)|^2 + nf_7(v) + 2f_1(v) - 2f_3(v) - f_6(v)\end{aligned}$$

Using this Lemma and the result [2], we obtained: **Theorem 1.** *Let M be an n-dimensional complete minimal submanifold in $S^{n+p}, p \geq 2$. If $|\sigma|^2 \leq \frac{2}{3}n$ everywhere on M, then M is isotropic and either a totally geodesic submanifold or a Veronese surface in S^4*

On the other hand, in Yuan and Matsuyama [13], we assume codimension $= 2$ and

$$\text{trace} A_\alpha^2 \leq \frac{n(n+2)}{2(n+4)}c \quad \text{for} \quad {}^\forall\alpha$$

every where on M, we obtained:

Theorem 2. *Let M be an n-dimensional complete minimal submanifold in S^{n+2}. If $\text{trace} A_\alpha^2 \leq \dfrac{n(n+2)}{2(n+4)}c$ for ${}^\forall\alpha$, then M is isotropic and either a totally geodesic submanifold or isotropic and has parallel second fundamental form.*

Especially, $n = 2 \Rightarrow S^2(\frac{1}{3}c) \to S^4(c)$ *and* $n = 5 \Rightarrow S^5 \to S^7(c)$.

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