8.2 Existence of the solution

In this section we consider the existence of solution for the equation (76) under the assumption $1-\lambda v \neq 0$. The case with $1-\lambda v = 0$ will be treated later.

8.2.1 The 1st column and *m*-th row

The 1st column of the equation (76) contains zero elements and is extracted to be rewritten as

$$\begin{bmatrix} (1-\lambda\nu)z_{11} \\ (1-\lambda\nu)z_{21} \\ \dots \\ (1-\lambda\nu)z_{m-11} \\ (1-\lambda\nu)z_{m1} \end{bmatrix} - \begin{bmatrix} \nu z_{21} \\ \nu z_{31} \\ \dots \\ \nu z_{m1} \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ \dots \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ \dots \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} q_{11} \\ q_{21} \\ \dots \\ q_{m-11} \\ q_{m1} \end{bmatrix},$$

which can be rearranged as

$$\begin{aligned} (1-\lambda\nu)z_{m1} &= 0 + q_{m1}, \\ (1-\lambda\nu)z_{g1} &= \nu z_{g+11} + q_{g1}, (g = \underline{m-1}, \underline{m-2}, \dots, 1). \end{aligned} \tag{77}$$

So that, we obtain the solutions, under the assumption $1-\lambda v \neq 0$,

$$\begin{aligned} z_{m1} &= \frac{q_{m1}}{(1-\lambda\nu)}, \\ z_{m-11} &= \frac{\nu q_{m1}}{(1-\lambda\nu)^2} + \frac{q_{m-11}}{(1-\lambda\nu)}, \\ &\dots \\ z_{21} &= \frac{\nu^{m-2} q_{m1}}{(1-\lambda\nu)^{(m-1)}} + \dots + \frac{\nu^2 q_{41}}{(1-\lambda\nu)^3} + \frac{\nu q_{31}}{(1-\lambda\nu)^2} + \frac{q_{21}}{(1-\lambda\nu)}, \\ z_{11} &= \frac{\nu^{m-1} q_{m1}}{(1-\lambda\nu)^m} + \dots + \frac{\nu^2 q_{31}}{(1-\lambda\nu)^3} + \frac{\nu q_{21}}{(1-\lambda\nu)^2} + \frac{q_{11}}{(1-\lambda\nu)}. \end{aligned} \tag{78}$$

The m-th row of the equation (76) contains zero elements and is extracted to be rewritten as

$$\begin{bmatrix} (1-\lambda\nu)z_{m1} & (1-\lambda\nu)z_{m2} & \dots & (1-\lambda\nu)z_{mn} \end{bmatrix} -$$
$$\begin{bmatrix} 0 & 0 & \dots & 0 \end{bmatrix} - \begin{bmatrix} 0 & \lambda z_{m1} & \lambda z_{m2} & \dots & \lambda z_{mn-1} \end{bmatrix}$$
$$- \begin{bmatrix} 0 & 0 & \dots & 0 \end{bmatrix} = \begin{bmatrix} q_{m1} & q_{m2} & \dots & q_{mn} \end{bmatrix},$$

which can be rearranged as

$$\begin{aligned} (1-\lambda\nu)z_{m1} &= 0 + q_{m1}, \\ (1-\lambda\nu)z_{mh+1} &= \lambda z_{mh} + q_{mh}, (h = 2, ..., m), \end{aligned} \tag{79}$$

Therefore, we obtain the solutions, under the assumption $1 - \lambda v \neq 0$,

$$\begin{aligned} z_{m1} &= \frac{q_{m1}}{(1-\lambda\nu)}, \\ z_{m2} &= \frac{q_{m2}}{(1-\lambda\nu)} + \frac{\lambda q_{m1}}{(1-\lambda\nu)^2}, \\ &\dots \\ z_{mn-1} &= \frac{q_{mn-1}}{(1-\lambda\nu)} + \frac{\lambda q_{mn-2}}{(1-\lambda\nu)^2} + \dots + \frac{\nu^{n-3} q_{m2}}{(1-\lambda\nu)^{(n-2)}} + \frac{\lambda^{n-2} q_{m1}}{(1-\lambda\nu)^{(n-1)}}, \\ z_{mn} &= \frac{q_{mn}}{(1-\lambda\nu)} + \frac{\lambda q_{mn-1}}{(1-\lambda\nu)^2} + \dots + \frac{\lambda^{n-2} q_{m2}}{(1-\lambda\nu)^{n-1}} + \frac{\lambda^{n-1} q_{m1}}{(1-\lambda\nu)^{n}}. \end{aligned} \tag{80}$$

Thus we can get the solution (78) and (80) for the variables z indicated in the matrix

$$Z = \begin{bmatrix} z_{11} & \square & \square & \dots & \square & \square \\ z_{21} & \square & \square & \dots & \square & \square \\ . & . & . & \dots & . & . \\ z_{m-11} & \square & \square & \dots & \square & \square \\ z_{m1} & z_{m2} & z_{m3} & \dots & z_{mn-1} & z_{mn} \end{bmatrix}, \tag{81}$$

where $\square$ means the variable not yet obtained by the 1st column and the m-th row of the equation system (76).

8.2.2 The remaining part

The remaining variables indicated as $\square$ in (81) are determined by the equation system (76) other than its 1st column and m-th row. They are rewritten as

$$(1-\lambda\nu)z_{gh} - \nu z_{g+1h} - \lambda z_{gh-1} - z_{g+1h-1} = q_{gh},$$

$$(g = \underline{m-1}, \underline{m-2}, ..., 1, h = 2, ..., n), \tag{82}$$

Under the assumption $1 - \lambda v \neq 0$, we can rewrite (82),

$$z_{gh} = (q_{gh} + \nu z_{g+1h} + \lambda z_{gh-1} + z_{g+1h-1}) / (1-\lambda\nu),$$

$$(g = \underline{m-1}, \underline{m-2}, ..., 1, h = 2, ..., n). \tag{83}$$

The system can be solved sequentially as illustrated in the following table.

row \ col.	$h-1$	...	h
g	z_{gh-1}	$\longrightarrow$	z_{gh}
$\vdots$	$\vdots$	$\nearrow$	$\uparrow$
$g+1$	z_{g+1h-1}	...	z_{g+1h}

(84)

That is, the 1st column of Z, $\{z_{11}, z_{21}, \dots, z_{m-11}, z_{m1}\}$, and the m-th row, $\{z_{m1}, z_{m2}, \dots, z_{mn-1}, z_{mn}\}$, are first determined as (78) and (80), then the remaining variables are determined by the equations (83). This process is illustrated by the arrows in (84).

The other process might be possible, though we don't have to consider it. It is only noted that the north east variable z_{gh} can not be determined by the other process than the one illustrated in (84).

First, the variable appearing in (84) are determined by (77) and (79), then given these as initial values the remaining variables are determined by (83) or (82). Thus, under the assumption $1-\lambda v \neq 0$, we obtain the solution of Z as a whole. In other words, we have a proposition:

Lemma20 *(Without the subscripts i and j,) there is* $Z = [Z_{gh}]$ *satisfying (74) under the assumption* $1-\lambda v \neq 0$.

Recovering the subscripts i and j, under the assumption $1-\lambda_i v_j \neq 0$, *(* $i = 1, \dots, r$, $j = 1, \dots, s$*), there is* Z_{ij} *satisfying (73) or (72). Furthermore, under the assumption* $1-\lambda_i v_j \neq 0$, *there is* $Z = [Z_{ij}]$ *satisfying (71), and* $X = TZS^{-1}$ *with T and S defined in (69) and (70) is a solution of (67), see Lemma 19.*

8.3 Uniqueness of the solution

We here discuss, under $1-\lambda v \neq 0$ without the subscripts i and j, the uniqueness of the solution for the equation (74) $Z-(\lambda I + H(m))\, Z(vI + H(n)) = Q$ or (75). To start with, let us consider a special case with $Q = [q_{gh}] = 0$ for (75). In this case, we have

$$z_{m1} = z_{m-11} = \ldots = z_{21} = z_{11} = 0,$$

in view of (78), and there is no other solution. In addition, from (80), we have

$$z_{m1} = z_{m2} = \ldots = z_{mn-1} = z_{mn} = 0,$$

and there is no other solution. Furthermore, we have, from (83),

$$z_{gh} = 0, (g = \underline{m-1}, \ldots, 1, h = 2, \ldots, n),$$

and there is no other solution. In summary, we obtain

Lemma 21 *Under the assumption* $1 - \lambda v \neq 0$*, if* $Q = 0$*, then the equation (74) or (75) has a unique solution* $Z = 0$.

Based on this lemma we get another proposition

Lemma 22 *Under the assumption* $1 - \lambda v \neq 0$*, the equation (75)* $Z - (\lambda I + H(m)) Z (vI + H(n)) = Q$ *or (76) has a unique solution.*

Proof. Suppose that Z_1 and Z_2 are two solutions for the equation (75) or its equivalent (72) $Z - JZK = Q$, without the subscripts i and j for Z_{ij}, J_i and K_j. That is, suppose that

$$Z_1 - JZ_1K = Q,$$

$$Z_2 - JZ_2K = Q,$$

hold. Then we have

$$[Z_1 - Z_2] - J[Z_1 - Z_2]K = 0,$$

which means $Z_1 = Z_2$, from Lemma 21 under the assumption $1 - \lambda v \neq 0$. The two solutions are identical. ■

Lemma 23 *Under the assumption* $1 - \lambda_j v_j \neq 0$*, (* $i = 1, \ldots, r$*,* $j = 1, \ldots, s$*), the equation (72)* $Z_{ij} - J_i Z_{ij} K_j = Q_{ij}$*, (* $i = 1, \ldots, r$*,* $j = 1, \ldots, s$*) has a unique solution. And the equation (71)* $Z - JZK = Q$ *has a unique solution. Furthermore, the equation (67)* $X - UXV = W$ *has a unique solution.*

Proof. This is a consequence of Lemmas 20 and 22. ■

8.4 Matrix series solution

Returning to the equation (67) $X - UXV = W$, we consider its solution given by a matrix series. Calculating this series solution does not have to have a knowledge about Jordan transformation of U and V. Only the values of U, V and W are necessary. The discussion here aims at analyzing the property of the solution for (67). We also consider a special case $X - UXU' = W$.

8.4.1 Solution

We try a series

$$X = W + UWV + UUWVV + UUUWVVV + \dots, \tag{85}$$

for a solution for the equation (67), and consider its property.

Lemma 24 *If X defined by (85) converges, then X is a solution for (67).*

Proof. Suppose that (85) converges, and substitute this X into the left hand side of (67). Then we have

$$\begin{aligned} X - UXV &= (W + UWV + UUWVV + UUUWVVV + \dots) \\ &\quad - U(W + UWV + UUWVV + UUUWVVV + \dots)V \\ &= W + UWV - UWV + UUWVV - UUWVV + \dots \\ &= W, \end{aligned}$$

which shows that X of (85) is the solution for (67). ■

Note that for a special case with $V = U'$, the equation (67) is

$$X - UXU' = W, \tag{86}$$

and the solution (85) for this case is

$$X = W + UWU' + UUWU'U' + UUUWU'U'U' + \dots. \tag{87}$$

Then we have

Lemma 25 *If W is positive definite and X defined by (87) converges, then X of (87) is a solution for (86), and is positive definite.*

Proof. Lemma 24 shows the first part. To show the second part, consider a quadratic form of X provided that (87) is convergent. For $X = (m \times m)$, we have a quadratic form

$$\xi' X \xi = \xi' W \xi + \xi' U W U' \xi + \xi' U U W U' U' \xi + \xi' U U U W U' U' U' \xi + \dots,$$

with a non-zero vector $\xi' = [\xi_1, \dots, \xi_m]$. Letting $\eta'_k = \xi' U^k$ we have

$$\xi' X \xi = \xi' W \xi + \eta'_1 W \eta_1 + \eta'_2 W \eta_2 + \eta'_3 W \eta_3 + \dots,$$

where the first term is positive by the assumption and the other terms are at least non-negative. Therefore, we have $\xi' X \xi > 0$. ■

Condition for the convergence. In Lemmas 24 and 25, the convergent series (85) is assumed. We now consider the condition for the convergence. For that matter, it is reminded that the equation in question is (67) $X - UXV = W$, and that the characteristic roots of U and V are $\lambda_1, \dots, \lambda_r$ and $\upsilon_1, \dots, \upsilon_s$ which include distinct roots only, and that U and V both can be transformed into Jordan canonical forms.

Then we have

Lemma 26 *If* $1 > |\lambda_i \upsilon_j|$, $(i = 1, \dots, r, j = 1, \dots, s)$, *then the series (85) is convergent.*

Proof. The series (85) is transformed as

$$\begin{aligned} TXS^{-1} &= TWS^{-1} + TUWVS^{-1} + TUUWVVS^{-1} \\ &\quad + TUUUWVVVS^{-1} + \dots \\ &= TWS^{-1} + TUT^{-1}TWS^{-1}SVS^{-1} \\ &\quad + TUT^{-1}TUT^{-1}TWS^{-1}SVS^{-1}SVS^{-1} \\ &\quad + TUT^{-1}TUT^{-1}TUT^{-1}TWS^{-1}SVS^{-1}SVS^{-1}SVS^{-1} \\ &\quad + \dots, \end{aligned}$$

by T and S defined in (69) and (70). This can be rewritten as

$$Z = Q + JQK + JJQKK + JJJQKKK + \dots .$$

In partitioned form we have

$$\begin{bmatrix} Z_{11} & \dots & Z_{1s} \\ . & \dots & . \\ Z_{r1} & . & Z_{rs} \end{bmatrix} = \begin{bmatrix} Q_{11} & \dots & Q_{1s} \\ . & \dots & . \\ Q_{r1} & \dots & Q_{rs} \end{bmatrix}$$
$$+ \begin{bmatrix} J_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & J_r \end{bmatrix} \begin{bmatrix} Q_{11} & \dots & Q_{1s} \\ . & \dots & . \\ Q_{r1} & \dots & Q_{rs} \end{bmatrix} \begin{bmatrix} K_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & K_s \end{bmatrix}$$
$$+ \begin{bmatrix} J_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & J_r \end{bmatrix}^2 \begin{bmatrix} Q_{11} & \dots & Q_{1s} \\ . & \dots & . \\ Q_{r1} & \dots & Q_{rs} \end{bmatrix} \begin{bmatrix} K_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & K_s \end{bmatrix}^2$$
$$+ \quad J^3QK^3 + \dots . \tag{88}$$

Now the (i, j) – block of (88) is

$$Z_{ij} = Q_{ij} + J_iQ_{ij}K_j + J_iJ_iQ_{ij}K_jK_j + J_iJ_iJ_iQ_{ij}K_jK_jK_j + \dots .$$

Rewriting $\boldsymbol{Z_{ij}}$, $\boldsymbol{Q_{ij}}$, $\boldsymbol{J_i}$, K_i, λ_i, v_j, m_i and n_j as $\boldsymbol{Z}$, $\boldsymbol{Q}$, $\boldsymbol{J}$, K, λ, v, m and n by removmg the subscripts i and j, (88) can be rewritten as

$$\begin{aligned} Z &= \sum_{k=0}^{\infty} J^kQK^k \\ &= \sum_{k=0}^{\infty} (\lambda I + H(m))^k\, Q\, (\nu I + H(n))^k , \end{aligned} \tag{89}$$

The convergence problem is now referred to the series (89). (As noted above, we only deal with the case where $m_i > 1$ and $n_i > 1$.)

If we apply the binomial expansion to both $(\lambda I + H(m))^k$ and $(\upsilon I + H(n))^k$, then we have

$$\begin{aligned} Z &= \sum_{k=0}^{\infty}\left[\sum_{a=0}^{k}\frac{k!}{a!(k-a)!}H(m)^a\lambda^{k-a}\right]Q\left[\sum_{b=0}^{k}\frac{k!}{b!(k-b)!}H(n)^b\nu^{k-b}\right] \\ &= \sum_{k=0}^{\infty}\left[\sum_{a=0}^{k}\sum_{b=0}^{k}\left(\frac{k!}{a!(k-a)!}\lambda^{k-a}\frac{k!}{b!(k-b)!}\nu^{k-b}\right)H(m)^aQH(n)^b\right]. \quad (90) \end{aligned}$$

Here the matrices $H(m)$ and $H(n)$ are nil-potent in the sense that

$$H(m)^aQH(n)^b = \begin{cases} 0, & \text{if } a \geqq m \text{ or } b \geqq n, \\ Q^{(a,b)}, & \text{if } a < m \text{ and } b < n, \end{cases} \quad (91)$$

where

$$Q^{(a,b)} = \begin{bmatrix} 0 & \dots & 0 & q_{a+11} & q_{a+12} & \cdots & q_{a+1n-b} \\ 0 & \dots & 0 & q_{a+21} & q_{a+22} & \cdots & q_{a+2n-b} \\ . & \dots & . & . & . & \dots & . \\ 0 & \dots & 0 & q_{m1} & q_{m2} & \cdots & q_{mn-b} \\ 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ . & \dots & . & . & . & \dots & . \\ 0 & \dots & 0 & 0 & 0 & \dots & 0 \end{bmatrix}.$$

We have assumed that $m \leqq n$ for the indexes m and n and that a runs over the set of values { 0, 1, ... , m, ... , k} while b runs over the set { 0, 1, ... , m, ... , n, ... , k}. The equation (91) implies that when a takes a value m or larger after taking values 0, 1, ... , $m-2$ and $m-1$, $H(m)^aQH(n)^b$ becomes zero (matrix), and that when b takes a value n or larger after taking values 0, 1, ... , $n-2$ and $n-1$, $H(m)^aQH(n)^b$ becomes zero (matrix), So that, the double summation term including $H(m)^aQH(n)^b$ in (90) is truncated or

$$\sum_{a=0}^{k}\sum_{b=0}^{k}=\sum_{a=0}^{m-1}\sum_{b=0}^{n-1}.$$

Then therefore we have, for (90),

$$Z=\sum_{k=0}^{\infty}\left[\sum_{a=0}^{m-1}\sum_{b=0}^{n-1}\left(\frac{k!}{a!(k-a)!}\lambda^{k-a}\frac{k!}{b!(k-b)!}\nu^{k-b}\right)Q^{(a,b)}\right].$$

Partitioning the sum over k, we get

$$\begin{aligned} Z &= \sum_{k=0}^{m-1}\left[\sum_{a=0}^{m-1}\sum_{b=0}^{n-1}\left(\frac{k!}{a!(k-a)!}\lambda^{k-a}\frac{k!}{b!(k-b)!}\nu^{k-b}\right)Q^{(a,b)}\right] \\ &\quad +\sum_{k=m}^{\infty}\left[\sum_{a=0}^{m-1}\sum_{b=0}^{n-1}\left(\frac{k!}{a!(k-a)!}\lambda^{k-a}\frac{k!}{b!(k-b)!}\nu^{k-b}\right)Q^{(a,b)}\right]. \end{aligned}$$

Denoting the second term on the right hand side by G, we have

$$\begin{aligned} G &= \sum_{a=0}^{m-1}\sum_{b=0}^{n-1}Q^{(a,b)}\sum_{k=m}^{\infty}\left(\frac{k!}{a!(k-a)!}\lambda^{k-a}\frac{k!}{b!(k-b)!}\nu^{k-b}\right) \\ &= \sum_{a=0}^{m-1}\sum_{b=0}^{n-1}Q^{(a,b)}\sum_{k=0}^{\infty}\left(\frac{(k+m)!}{a!(k+m-a)!}\lambda^{k+m-a}\frac{(k+m)!}{b!(k+m-b)!}\nu^{k+m-b}\right). \end{aligned} \quad (92)$$

Letting the infinite series in the equation (92) be ϕ, we get

$$\phi=\sum_{k=0}^{\infty}\frac{(k+m)!}{a!(k+m-a)!}\frac{(k+m)!}{b!(k+m-b)!}\lambda^{k+m-a}\nu^{k+m-b},$$

and we consider the convergence property of ϕ.[5)]

If we denote the k-th term of ϕ by θ_k, we find the ratio

$$\left|\frac{\theta_{k+1}}{\theta_k}\right|=\left|\frac{k+m+1}{k+m+1-a}\right|\left|\frac{k+m+1}{k+m+1-b}\right||\lambda\nu|.$$

5) See, for instance, W. Rudin(1976) : *Principles of Mathematical Analyses*, Third ed. , McGraw Hill, p.66.

Then, if $1 > |\lambda v|$, then ϕ and therefore G, Z and X are convergent. Whereas, for an arbitrary k, we have

$$\frac{k+m+1}{k+m+1-a} \geqq 1, (a = 0, ..., m-1),$$
$$\frac{k+m+1}{k+m+1-b} \geqq 1, (b = 0, ..., n-1),$$

and therefore if $1 \leqq |\lambda v|$, then

$$\left|\frac{\theta_{k+1}}{\theta_k}\right| \geqq 1, (a = 0, ..., m-1, b = 0, ..., n-1),$$

so that ϕ and therefore G, Z and X are divergent. ■

Now we are in a position to summarize the results obtained so far in this section concerning the equations (67) $X - UXV = W$ and (86) $X - UXU' = W$:

Theorem 27 *For the equations (67) and (86), assume that U and V are transformed into Jordan canonical forms with the distinct characteristic roots* $\lambda_1, \dots, \lambda_r$ *of U and those* $v_1, \dots, v_s$ *of V, we have the following propositions (a), (b), (c) and (d):*

(a) Under the assumption $1 - \lambda_i v_j \neq 0$, $(i = 1, \dots, r, j = 1, \dots, s)$, *there is a solution for (67). (See Lemmas 20 and 23).*

(b) Under the assumption, $1 - \lambda_i v_j \neq 0$, $(i=1, \dots, r, j=1, \dots, s)$, *the solution of (67) is unique. (See Lemma 23).*

(c) Under the assumption $1 - |\lambda_i v_j| > 0$, $(i = 1, \dots, r, j = 1, \dots, s)$, *the series (85) converges, and is a unique solution for (67). (See Lemmas 23,24 and 26).*

(d) Under the assumptions $1 - |\lambda_i \lambda_j| > 0$, $(i, j = 1, \dots, r)$ *and W is positive definite, the series (87) converges, and is a unique solution for (86), and is positiue definite. (See Lemmas 23, 24, 25 and 26).*

Theorem 27 is an extended form of Lemma 12, which is the basis of the foregoing analyses. While Lemma 12 presupposes the diagonalizability of U and V, Theorem 27 does not. Therefore, the main propositions of the previous sections hold with the assumption of the possibility of Jordan canonical transformation of U and V.

8.5 Divergent case

There still remains one case to be discussed to complete the analysis. We here deal with a case where $1-\lambda v=0$, for the equation (74) $Z-(\lambda I+H(m))\, Z(vI+H(n))=Q$. Recall that it is shown that the series solution (85) does not converge if $1-\lambda v=0$. Thought it is thought that other types of solution than the one given as a matrix series might be tenable, this subsection shows that the equation (74) does not have a unique solution when $1-\lambda v=0$.

When $1-\lambda v=0$, the equation (74) or its equivalent (76) is simplified and written as

$$-\begin{bmatrix} \nu z_{21} & \nu z_{22} & \dots & \nu z_{2n} \\ \nu z_{31} & \nu z_{32} & \dots & \nu z_{3n} \\ . & . & \dots & . \\ \nu z_{m1} & \nu z_{m2} & \dots & \nu z_{mn} \\ 0 & 0 & \dots & 0 \end{bmatrix} - \begin{bmatrix} 0 & \lambda z_{11} & \lambda z_{12} & \dots & \lambda z_{1n-1} \\ 0 & \lambda z_{21} & \lambda z_{22} & \dots & \lambda z_{2n-1} \\ . & . & . & \dots & . \\ 0 & \lambda z_{m-11} & \lambda z_{m-12} & \dots & \lambda z_{m-1n-1} \\ 0 & \lambda z_{m1} & \lambda z_{m2} & \dots & \lambda z_{mn-1} \end{bmatrix}$$

$$-\begin{bmatrix} 0 & z_{21} & \dots & z_{2n-1} \\ 0 & z_{31} & \dots & z_{3n-1} \\ . & . & \dots & . \\ 0 & z_{m1} & \dots & z_{mn-1} \\ 0 & 0 & \dots & 0 \end{bmatrix} = \begin{bmatrix} q_{11} & q_{12} & \dots & q_{1n} \\ q_{21} & q_{22} & \dots & q_{2n} \\ . & . & \dots & . \\ q_{m-11} & q_{m-12} & \dots & q_{m-1n} \\ q_{m1} & q_{m2} & \dots & q_{mn} \end{bmatrix}. \qquad (93)$$

Note that if $1-\lambda v=0$, then $\lambda=0$ and/or $v=0$ are impossible.

8.5.1 The 1st column and *m*-th row

Some parts of the equation (93) can be simplified further, since it contains the zero elements. To see this, let us rewrite the 1st column of (93),

$$-\begin{bmatrix} \nu z_{21} \\ \nu z_{31} \\ . \\ \nu z_{m1} \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ . \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ . \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} q_{11} \\ q_{21} \\ . \\ q_{m-11} \\ q_{m1} \end{bmatrix}, \tag{94}$$

so that it is seen that q_{m1} must be zero and that the solution for (94) is

$$\begin{bmatrix} z_{21} \\ z_{31} \\ . \\ z_{m1} \end{bmatrix} = -\frac{1}{\nu} \begin{bmatrix} q_{11} \\ q_{21} \\ . \\ q_{m-11} \end{bmatrix}. \tag{95}$$

In a similar way, we have

$$-\begin{bmatrix} 0 & 0 & \dots & 0 \end{bmatrix} - \begin{bmatrix} 0 & \lambda z_{m1} & \lambda z_{m2} & \dots & \lambda z_{mn-1} \end{bmatrix}$$
$$-\begin{bmatrix} 0 & 0 & \dots & 0 \end{bmatrix} = \begin{bmatrix} q_{m1} & q_{m2} & \dots & q_{mn} \end{bmatrix}, \tag{96}$$

for the m-th row of (93). It is also seen that q_{m1} must be zero and that the solution for (96) is

$$\begin{bmatrix} z_{m1} & z_{m2} & \dots & z_{mn-1} \end{bmatrix} = -\frac{1}{\lambda} \begin{bmatrix} q_{m2} & q_{m3} & \dots & q_{mn} \end{bmatrix}. \tag{97}$$

Up to now we get the solution uniquely for the variables shown as □ in the matrix

$$Z = \begin{bmatrix} \square & \square & \square & \dots & \square & \square \\ z_{21} & \square & \square & \dots & \square & \square \\ . & . & . & \dots & . & . \\ z_{m-11} & \square & \square & \dots & \square & \square \\ z_{m1} & z_{m2} & z_{m3} & \dots & z_{mn-1} & \square \end{bmatrix}, \qquad (98)$$

which shows that z_{11} and z_{mn} are not determined yet.

8.5.2 The remaining part

The remaining equations of (93) other than (94) or (96) are

$$-\nu z_{g+1h} - \lambda z_{gh-1} - z_{g+1h-1} = q_{gh},$$

$$(g = \underline{m-1}, \underline{m-2}, ..., 1, h = 2, ..., n). \qquad (99)$$

The variables $\square$ so indicated in (98) may be possibly determined by this system. However, the equation system (99) has a logical structure, which can not determine the $\square$ variables in (98).

The equation system (99) indicates three ways of determining one variable given the others. For instance, the first way is to detemine z_{gh-1} given z_{g+1h} and z_{g+1h-1} or the solution would be $z_{gh-1} = -(v/\lambda)z_{g+1h} - (1/\lambda) q_{gh}$. This way is illustrated by (i) of figure (100). Given z_{g+1h} and $(1/\lambda) z_{g+1h-1}$, the north-west variable z_{gh-1} is to be determined. The second and third ways are illustrated by (ii) and (iii) of the figure (100) in a similar fashion. Though, these three ways can not determine the north-east variables z_{gh} given the initial values which are obtained in (95) and (97). The variable z_{gh} can take any arbitrary value when $1 - \lambda v = 0$. Therefore, the equation system (74) does not have a unique solution.

(i)

row \ col.	$h-1$	$\cdots$	h
g	z_{gh-1}	$\cdots$	z_{gh}
$\vdots$	$\uparrow$	$\nwarrow$	$\vdots$
$g+1$	z_{g+1h-1}	$\cdots$	z_{g+1h}

(ii)

row \ col.	$h-1$	$\cdots$	h
g	z_{gh-1}	$\cdots$	z_{gh}
$\vdots$	$\vdots$	$\searrow$	$\vdots$
$g+1$	z_{g+1h-1}	$\longrightarrow$	z_{g+1h}

(100)

(iii)

row \ col.	$h-1$	$\cdots$	h
g	z_{gh-1}	$\cdots$	z_{gh}
$\vdots$	$\downarrow$	$\cdots$	$\vdots$
$g+1$	z_{g+1h-1}	$\longleftarrow$	z_{g+1h}

9 Concluding Remarks

The remaining problems are listed below. These points have to be further developed

i) The conditions for Theorem 25 are given in terms of the characteristic roots $\lambda_1, \ldots, \lambda_r$ of $U\,(=B\Delta)$ and those $v_1, \ldots, v_s$ of $V\,(=\Delta B)$. It is not yet certain whether the condition can be given in terms of B and Δ separately, or in another form.

ii) It is clear that the necessary conditions for the case of multivariate distributions is not treated good enough.

iii) The analysis leads us from the notion of integral equation to those of matrix equation, matrix diagonalization and Jordan canonical form.Then the analysis stops there. It is not certain now that any further development is needed beyond the notion of Jordan canonical form.