CHUO MATH NO.123(2018)

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Apr. 28 , 2018

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Abstract

We remark that there is no smooth function $f(x)$ on $[0,1]$ which is flat at 0 such that the derivative $f^{(n)}$ of any order $n \geq 0$ is positive on $(0,1]$. Moreover, the number of zeros of the n-th derivative $f^{(n)}$ grows to the infinity and the zeros accumulate to 0 when $n \to \infty$.

We consider smooth functions on the interval $[0,1]$ which are flat at the origin, namely of class C^∞ and any derivative $f^{(n)}(x)$ converges to 0 when $x \to 0+0$. Eventually it is equivalent to say that f extends to the whole real line as a smooth function by defining $f(x) = 0$ for $x < 0$. In this short note we make a couple of remarks on the asymptotics of higher derivatives around the origin.

Among non-tirivial flat functions the most well-known might be the one which is defined as follows.

$$f(0) = 0 \quad \text{and} \quad f(x) = e^{-\frac{1}{x}} \ \text{ for } \ x > 0$$

If we imagine its graph, of course it seems smooth enough, and it can be extended as constantly 0 on $(-\infty, 0]$ as a smooth function on the real line $\mathbb{R}$. Its first derivative is positive on $(0, \infty)$, but the second derivative vanishes at $x = \frac{1}{2} = x_2$ and the third vanishes at $x_3 = \frac{1-1/\sqrt{3}}{2} < x_2$, and so on. That is, setting $x_n = \min\{x\,;\, f^{(n)}(x) = 0,\ x > 0\}$ for $n = 2, 3, 4, \ldots$, it is clear that $\{x_n\}_n$ is strictly decreasing, and in fact $\lim_{n\to\infty} x_n = 0$. More over, if we fix any interval $[0, \alpha)$ $(\alpha > 0)$, $f^{(n)}(x)$ tends to behave more and more wildly when $n \to \infty$ on the interval.

Also, if we take $g_0(x) = f(x)(\sin(\frac{1}{x}) + 1)$ and

$$g_n(x) = \int_0^x \int_0^{t_{n-1}} \cdots \int_0^{t_1} g_0(t_0)dt_0 \cdots dt_{n-2}dt_{n-1}\,,$$

then for $n = 1, 2, 3, \cdots$, $g_n(x)$ is positive on $(0, \infty)$ and is flat at $x = 0$, and apparently $g_n^{(k)}(x) > 0$ when $x > 0$ for $0 \leq k \leq n-1$ but there is no interval $(0, \alpha)$ on which $g_n^{(n)}(x)$ is positive.

They seem to exhibit not particular for these examples but rather common or inevitable phenomena of higher derivatives of flat functions.

Theorem 1 There exists no smooth function $f(x)$ on $[0,1]$ which is flat at $x = 0$ and satisfies $f^{(n)}(x) > 0$ on $(0,1]$ for any $n \geq 0$.

This fact is refined as follows.

2010 *Mathematics Subject Classification*. 26A24 (primary) and 26A06 (secondary).

Theorem 2 For a smooth function $f(x)$ on $[0,1]$ which is flat at $x = 0$, put $Z(n) = \{x \in (0,1) \mid f^{(n)}(x) = 0\}$ and $z(n) = \sharp Z(n)$ for $n \geq 0$. Then

$$\lim_{n\to\infty} z(n) = \infty$$

holds, where ∞ might be $\aleph_c$.

Corollary 3 1) In general, $\lim_{n\to\infty} \inf Z(n) = 0$.
2) More strongly, for any $k > 0$ there exist $N > 0$ and $y^{(n)}(l) \in Z(n)$ for $n \geq N$ and $l = 1, \ldots, k$ which are strictly increasing in l and strictly decreasing in n, namely, satisfying

for each fixed n, $y^{(n)}(l) < y^{(n)}(l+1)$ for $1 \leq l \leq k-1$,

for each fixed l, $y^{(n)}(l) > y^{(n+1)}(l)$.

Moreover it satisfies for any l $\lim_{n\to\infty} y^{(n)}(l) = 0$.

The accumulation of $Z(n)$ to 0 ($n \to 0$) must be formulated in many more stronger statements. The above corollary is one of them.

Proof of Corollary 3. There is a zero of $f^{(n+1)}$ between two zeros of $f^{(n)}$. This simple argument, which will be used repeatedly, tells that once $Z(n)$ accumulates to 0 for some n, so does $Z(k)$ for any $k \geq n$. Therefore in this case the proof is done. Otherwise, 0 is always isolated from $Z(n)$ and then we can pick up the least element $y^{(n)}(1) \in Z(n)$. Now it is clear that $y^{(n+1)}(1) < y^{(n)}(1)$ for any n. Then, if $\lim_{n\to\infty} y^{(n)}(1) = c > 0$, $f|_{[0,c]}$ contradicts to Theorem 1. This proves 1).

Now let us prove 2). Theorem 2 implies for any k there is N', $\sharp Z(n) \geq k$ for $n \geq N'$. Like in 1), once 0 is accumulated by $Z(N)$, take any decreasing sequence $y^{(n)}(k) \in Z(n)$ for $n \geq N$, and then it is fairy easy to take $\{y^{(n)}(l)\}$ for other l's so as to satisfy the conditions. Therefore we assume that 0 is isolated from $Z(n)$ for any $n \in \mathbb{N}$.

Next, take $A(n) \subset Z(n)$ to be the set of points which is accumulated from above by points in $Z(n)$. Clearly this set has $\eta(n) = \min A(n)$ whenever $A(n) \neq \emptyset$. If $A(n) = \emptyset$, put $\eta(n) = 1$. If $\eta(n) < 1$, $f^{(n)}$ is flat at $\eta(n)$ and $\eta(n) \in A(n')$ for $n' \geq n$. Therefore the sequence $\{\eta(n)\}_n$ is weakly decreasing.

In the case where $c = \lim_{n\to\infty} \eta(n) > 0$, applying Theorem 2 to $f|_{[0,c]}$, we can find N such that $\sharp(Z(N) \cap (0,c)) \geq k$. Moreover, in this case, for any $n \geq N$ we can take the k least zeros $0 < y^{(n)}(1) < y^{(n)}(2) < \cdots < y^{(n)}(k)$ because there is no accumulation from above. Automatically $\{y^{(n)}(l)\}_n$ is strictly decreasing for each l. If $\lim_{n\to\infty} y^{(n)}(k) = c' > 0$, then again $f|_{[0,c']}$ contradicts to Theorem 2. Therefore this case is done.

In the case where $\lim_{n\to\infty} \eta(n) = 0$, a similar argument in the case where 0 is accumulated by some $Z(n)$ enable us to arrange $\{y^{(n)}(l)\}$ so as to satisfy the conditions. $\square$

Proof of Theorem 1. The theorem is easily deduced from Lemma 4 by contradiction. Assume for some $\alpha > 0$ that $f(x)$ is smooth on $[0, \alpha]$, is flat at $x = 0$, and that its n-th derivative is positive on $(0, \alpha]$ for any $n \in \mathbb{N}$. We adjust the function f into $g(x) = f(\alpha)^{-1} f(\alpha x)$. Then $g(x)$ satisfies the condition of the lemma for any $n \in \mathbb{N}$. Therefore $g(x) \equiv 0$ on $[0, 1)$, and we obtain a contradiction. □

Lemma 4 Let n be an integer and $g(x)$ be a function on $[0, 1]$ of class C^{n+1} with the following properties.

(1) $g^{(k)}(0) = 0$ for $k = 0, ..., n$, and $g(1) = 1$,

(2) $g^{(n+1)}(x) > 0$ for $x > 0$.

Then $g(x) < x^n$ holds on $(0, 1)$.

Proof of Lemma 4. It is enough to show that $g(x)/x^n$ is increasing on $[0, 1]$. As $\dfrac{d}{dx}\left(\dfrac{g(x)}{x^n}\right) = \dfrac{xg'(x) - ng(x)}{x^{n+1}}$, it is also sufficient to show that the numerator $xg'(x) - ng(x)$ is positive on $(0, 1)$.

Then because $(xg'(x) - ng(x))^{(n)} = xg^{(n+1)}(x)$ is positive on $(0, 1]$ from our condition, we see successively that each k-th derivative $(xg'(x) - ng(x))^{(k)} = xg^{(k+1)} - (n-k)g^{(k)}(x)$ vanishes at $x = 0$ and therefore is positive on $(0, 1]$ for $k = n-1, n-2, \ldots, 0$. This completes the proof. □

A variant of this lemma is used to prove Theorem 2.

Proof of Theorem 2. The key idea is not to look at $z(n)$ bt at the number $s(n)$ of the quasi-positive and quasi-negative intervals of $f^{(n)}$. For a smooth (continuous) function g on $[0, 1]$ a connected component of the closure of $g^{-1}(0, \infty)$ [*resp.* $g^{-1}(-\infty, 0)$] is called a *quasi-positive* [*resp. quasi-negative*] interval. Such intervals are exactly maximal ones on which the primitive $\int g(x)dx$ of g is strictly monotone. Let us define $s(g) \in \mathbb{N} \cup \{\infty\}$ to be the number of all the quasi-positive and quai-negative intervals of g. Then we put $s(n) = s(f^{(n)})$.

If we have $s(n) = \infty$ for some n , $s(k) = \infty$ for $k \geq n$ as follows. If $Z(n)$ has interior points for some n, so does $Z(k)$ for $k \geq n$, If for some n we have $z(n) = \infty$ and $\operatorname{int} Z(n) = \emptyset$, the complement $[0, 1] \setminus Z(n)$ consists of infinitely many intervals and possibly of one half open interval. Each open interval contains an element in $Z(n+1)$.

Therefore, eliminating such cases, , we can assume $z(n) < \infty$ for any $n \in \mathbb{N}$. Consequently $s(n) < \infty$ ($\forall n \in \mathbb{N}$) holds as well. We want to prove $\lim_{n\to\infty} s(n) = \infty$ under this assumption.

Let $x^{(n)}(l)$ ($l = 1, 2, \ldots, s(n) - 1$, $n = 0, 1, 2, \ldots$) denotes the bigger end point of the l-th of quasi-positive/negative intervals for $f^{(n)}$, namely $[0, x^{(n)}(1)], [x^{(n)}(1), x^{(n)}(2)], \ldots, [x^{(n)}(l-1), x^{(n)}(l)], \ldots, [x^{(n)}(s(n)-1), 1]$ are the maximal intervals. Except for the final one, any quasi-positive [*resp.* quasi-negative] interval contains a maximal [*resp.* minimal] point in its interior. From this observation it is easy to see the following, among which

1) is a conclusion of Theorem 1, because it implies $z(n) \geq 1$ for some n and then we have $s(n+1) \geq 2$.

Assertion 5 1) $s(n) \geq 2$ for some n.
2) $\{x^{(k)}(1)\}_k$ is strictly decreasing ($k = n,\, n+1,\, n+2, \dots$) for n in 1).
3) Also for any m and $0 < l < s(m)$, the sequence $\{x^{(k)}(l)\}_k$ is strictly decreasing ($k = m,\, m+1,\, m+2, \dots$).
4) $\{s(n)\}_n$ is weakly increasing, namely, $s(n) \leq s(n+1)$ for any n.

Now let us procede by contradiction. We assume that $s(n)$ does not grow to ∞, *i.e.*, for some N, $s(n) \equiv s(N)(= S)$ for any $n \geq N$. For fixed $l \in \{1, \cdots, S\}$, the quasi-positivity/negativity of the l-th interval is independent of $n \geq N$. Under the assumption we also see the following.

Assertion 6 For $n \geq N$,
1) $f^{(n)}$ is strictly monotone on the final interval $[x^{(n)}(S-1), 1]$.
2) In particular $f^{(n)}(1) \neq 0$. More precisely, if the final interval is quasi-positive [*resp.* quasi-negative] we have $f^{(n)}(1) > 0$ [*resp.* $f^{(n)}(1) < 0$].
3) The final intervals are increasing, namely, we have
$x^{(N)}(S-1) > x^{(N+1)}(S-1) > \cdots x^{(n)}(S-1) > x^{(n+1)}(S-1) > \cdots$.

By multiplying a non-zero constant to f, we assume that for any $n \geq N$ $f^{(n)}$ is weakly increasing on the final interval and that $f^{(N)}(1) = 1$.

Lemma 7 Under these assumptions, the following estimate holds.

$$f^{(N)}(x) \leq x^p \quad \text{on} \quad [x^{(N)}(S-1), 1] \quad \text{for any} \quad p \in \mathbb{N}.$$

This lemma apparently implies $f^{(N)}|_{[x^{(N)}(S-1),1]} \equiv 0$ and contradicts to our assumption. This completes the proof of Theorem 2. □

Proof of Lemma 7. We adjust the proof of Lemma 4 in order to apply to $f^{(N)}$. Put $a_n = x^{(n)}(S-1)$ to simplify the notation.

It is enough to show that for any $p \geq 0$, $f^{(N)}(x) \cdot x^{-p}$ is strictly increasing on $[a_N, 1]$ because $f^{(N)}(x) \cdot x^{-p}|_{x=1} = 1$. So it suffices to show $\left(f^{(N)}(x) \cdot x^{-p}\right)' > 0$, namely, $xf^{(N+1)}(x) - pf^{(N)}(x) > 0$ on $(a_N, 1)$.

For this purpose we prove inductively for $k = p,\, p-1,\, p-2,\, \cdots, 0$

$$\left(xf^{(N+1)}(x) - pf^{(N)}(x)\right)^{(k)} > 0 \quad \text{on} \quad (a_{N+k}, 1)\,.$$

For $k = p$, on $(a_{N+p+1}, 1)$ and in particular on $(a_{N+p}, 1)$, we have clearly

$$\left(xf^{(N+1)}(x) - pf^{(N)}(x)\right)^{(p)} = xf^{(N+p+1)}(x) > 0\,.$$

Then on each step, as $f^{(N+k+1)}(a_{N+k}) > 0$ and $f^{(N+k)}(a_{N+k}) = 0$,

$$\left(xf^{(N+1)}(x) - pf^{(N)}(x)\right)^{(k)} = xf^{(N+1+k)}(x) - (p-k)f^{(N+k)}(x)$$

is positive at $x = a_{N+k}$. Because the inductive hypothesis implies its derivative is positive on $(a_{N+k}, 1)$ (and even on $(a_{N+k+1}, 1)$), the induction is completed. □

Problem 8 1) For some smooth functions on $[0,1]$ which are flat at 0, $\cup_{n=1}^{\infty} Z(n)$ seems to be dense in $[0,1]$. However, we do not see which kind of further properties as flat functions are essential for this phenomena, because it discusses points away from 0. Verify this phenomena for certain f's and explain the reason.
2) Does there exist a smooth function on $[0,1]$ which is flat at 0 such that $\lim_{n\to\infty} \max Z(n) = 0$? Or how about flat at 0 such that the derived set of $\cup_{n=1}^{\infty} Z(n)$ coincides with $\{0\}$? It seems plausible that such functions do not exist, while we do not know how to prove it.

References

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