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and their symmetries I

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Reeb components of leafwise complex foliations and their symmetries I

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Abstract

We review the standard Hopf construction of Reeb components with leafwise complex structure and almost determine the group of leafwise holomorphic smooth automorphisms for Reeb components of certain type in the case of complex leaf dimension one. In particular, it contains an infinite dimensional vector space.

0 Introduction

The aim of this article is to begin a study of Reeb components in leafwise complex foliation of codimension one, especially focused on the symmetry in the real 3-dimensional case. Recall that a $(p+1)$-dimensional Reeb component is a compact manifold $R = D^p \times S^1$ with a (smooth) foliation of codimension one, whose leaves are graphs of smooth functions $f : \mathrm{int}D^p \to \mathbb{R}$ where $\lim_{z\to\partial D^p} f(z) = +\infty$, and a compact leaf which is the boundary $S^{p-1} \times S^1$. Here we identify R with $(D^p \times \mathbb{R})/\mathbb{Z}$.

Foliations of codimension one with complex analytic leaves are drawing attentions in several complex variables because it appears as the Levi foliations of Levi-flat real hypersurfaces in complex manifolds. A simple construction of Hopf manifolds admits a Levi-flat real hypersurface, whose Levi foliation consists of a pair of Reeb components. This construction is generalized in Nemirovskii's examples [Ne]. They have non-trivial linear holonomy along toral leaves. In this paper, from rather topological points of view, we study mainly Reeb components with flat holonomy, which, for example, appears in turbulization or in Dehn surgery.

After fixing basic notions in Section 1, Reeb components and the turbulization are reviewed in the context of leafwise complex foliations in Section 2. Then, we study the symmetries of a Reeb component in a leafwise complex foliation on a 3-manifold with flat holonomy along the boundary toral leaf. In section 4 we study the group of leafwise holomorphic foliated diffeomorphisms and determine it for Reeb components obtained by the Hopf construction with holonomy tangent to the identity to the infinite order at the boundary (Theorem 4.7). In particular the automorphism group contains an infinite dimensional vector space. Also depending on the subtle property of the boundary holonomy, the group can be topologically not complete. These phenomena are common to, for example, leafwise complex foliations on S^3

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or on lens spaces which consist of two Reeb components (see Section 5). The study of moduli space of tame Reeb component due to Meersseman-Verjovsky [MV] exhibits similar phenomena to those of compact complex manifolds to a certain degree. Here the study of automorphisms reveals results which are contrary to it.

Similar results are obtained for Reeb components of complex leaf dimension 2. They are explained in a forthcoming paper [Ho]. Also for a Reeb component whose boundary holonomy is linear or tangent to the identity at the origin only to a finite order the situations are rather different and basically easier. They are discussed in separate articles in preparation.

A key step of the proof relies on a study of functional equations concerning flat functions and expanding diffeomorphisms of the half line which are infinitely tangent to the identity at the origin (Theorem 3.1). In this direction, in order to prepare for [Ho] as well, we describe the results slightly beyond the minimum which we need in this paper. These are treated in Section 3.

Throughout this article, we only consider smooth manifolds and smooth foliations.

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1 Basic definitions

Let M be a $(2n+q)$-dimensional smooth manifold and $\mathcal{F}$ be a smooth foliation of codimension q on M and let $p = 2n$ be the dimension of leaves. We refer general basics for foliation theory to [CC].

Definition 1.1 (Leafwise complex structure, cf. [MV]) $(M, \mathcal{F})$ is said to be equipped with a *leafwise complex structure* if there exists a system of local smooth foliated coordinate charts $(U_\lambda, \varphi_\lambda)$ where $\varphi_\lambda : U_\lambda \to V_\lambda \subset \mathbb{C}^n \times \mathbb{R}^q = \{(z_1, \cdots, z_n, y_1, \cdots, y_q)\}$ is a smooth diffeomorphism onto an open set V_λ such that the coordinate change $(w_1, \cdots, w_n, t_1, \cdots, t_q) = \gamma_{\mu\lambda}(z_1, \cdots, z_n, y_1, \cdots, y_q)$ is smooth, t_j's depend only on y_k's $(j, k = 1, \cdots, q)$, and when y_k's are fixed w_l's are holomorphic in z_m's, where $\gamma_{\mu\lambda} : \varphi_\lambda(V_\lambda \cap U_\mu) \to \varphi_\mu(V_\lambda \cap U_\mu)$. It is equivalent to that the foliation has complex leaves whose complex structures vary smoothly in transverse directions. It is eventually equivalent to that the tangent bundle $\tau\mathcal{F}$ to the foliation is equipped with a smooth integrable almost complex structure J. We call $(M, \mathcal{F}, J)$ a *leafwise complex foliation* and quite often, along the context, $\mathcal{F}$ might be referred to as well.

Accordingly, a diffeomorphism between two foliated manifolds with leafwise complex structures are said to be an *isomorphism* between leafwise complex foliations iff it preserves the foliations and gives rise to biholomorphisms between leaves.

In this paper we are only concerned with foliation of codimension one. In particular, our interest will be focused on Reeb components , especially of real dimension 3, namely in the case of $n = 1$ and $q = 1$. As we see from the examples of Nemirovskii [Ne] even a real analytic Levi-flat hypersurface in a complex manifold can admit Reeb components in its Levi foliation. In such a case, the holonomy along the toral boundary leaf can not be tangent to the identity to the infinite order.

Apart from Levi-flats, for example, if we perform a turbulization we easily find various leafwise complex foliations admitting Reeb components with holonomy infinitely tangent to the identity. See the next section for the detail.

2 Reeb components with complex leaves

In this section we review a particular construction of Reeb component with complex leaves and a process of turbulization which produces a new Reeb component in a leafwise complex foliation.

In order to make pasting construction easier, we introduce the following notions. Let $(R, \mathcal{F}, J)$ (or simply R for short) be a Reeb component with leafwise complex structure of complex leaf dimension n and (H, J_H) be its boundary leaf.

Definition 2.1 The Reeb component R has a *tame* boundary (or 'R is *tame* at boundary' for short, or even shorter 'R is *tame*') with respect to a product coordinate $H \times [0, \varepsilon)$ of a collar neighborhood of H if it gives rise to a smooth foliation with leafwise complex structure when it is pasted with the product foliation $(H \times (-\varepsilon, 0], \{H \times \{x\} | x \in (-\varepsilon, 0]\}, J_H)$ along their boundary. Here each leaf $H \times \{x\}$ has the same complex structure as H when identified with the natural projection. Namely, the Reeb component is extended to the outside as a product foliation.

The notion of tameness was introduced in [MV].

Remark 2.2 If we forget the leafwise complex structure and consider the same notion only as foliation of codimension one, it does not depend on the choice of product coordinate on the positive side and the tameness implies exactly that the holonomy is tangent to the identity to the infinite order. This is because the set of expanding diffeomorphisms of the half line $\mathbb{R}_{\geq 0}$ which are infinitely tangent to the identity is an open convex cone and invariant under conjugation by any diffeomorphism. Also remark that the tameness depends only on the smooth projection of the collar neighborhood to the boundary, which the product coordinate defines. If two projections have the same infinite jets on the boundary, the tameness notion coincides for the two.

Definition 2.3 The leafwise complex structure of a Reeb component R is *simple* around boundary (or R has a *simple* complex structures around boundary) if the boundary has a collar neighborhood $U \cong H \times [0, \varepsilon)$ such that the restriction of the projection $U = H \times [0, \varepsilon) \to H$ to each leaf in U is holomorphic.

This notion should also be understood relative to the projection from a collar neighborhood to the boundary.

The notions of tameness and simpleness apply not only to Reeb components but also to more general leafwise complex foliations of codimension one with a compact leaf or a boundary leaf.

Clearly if a Reeb component has simple complex structures around the boundary and the holonomy of the boundary leaf is infinitely tangent to the identity, it is tame with respect to the appropriate projection. The tameness condition prohibits unexpected wild behaviour around boundary. In particular in the case of complex leaf dimension = 1, it induces a strong consequence due to Meersseman and Verjovsky. See the following subsection.

2.1 Reeb component by Hopf construction

Let us present a particular construction of a Reeb component with leafwise complex structure which is tame. If we remove the flat condition on the holonomy, it gives rise to more general Reeb components. These constructions are a kind of folklore.

Construction 2.4 (Hopf construction) Let $\varphi \in \mathit{Diff}^{\infty}(\mathbb{R}_{\geq 0})$ a diffeomorphism of the half line $\mathbb{R}_{\geq 0} = [0, +\infty)$ satisfying $\varphi(x) - x > 0$ for $x > 0$, namely the origin is an expanding unique fixed point. Also take a (local) biholomorphic diffeomorphism $G \in \mathit{Diff}^{hol}(\mathbb{C}^n, O)$ which is expanding. This implies that for some small neighborhood D of the origin O with smooth boundary $G(\mathrm{int} D) \supset \overline{D}$ and $\lim_{k \to -\infty} G^k(D) = \{O\}$. Now take $U = \cup_{k=1}^{\infty} G^k(D) \subset \mathbb{C}^n$. Then on $\tilde{R} = U \times \mathbb{R}_{\geq 0} \setminus \{(O,0)\} \subset \mathbb{C}^n \times \mathbb{R}$, take the restriction $\tilde{\mathcal{F}}$ of the product foliation $\{\mathbb{C}^n \times \{x\}\}$ together with the natural complex structure on leaves and a diffeomorphism $T = G \times \varphi$ on $U \times \mathbb{R}_{\geq 0} \setminus \{(O,0)\})$. Practically we take fairy simple diffeomorphisms such as linear maps as G so that U becomes the whole $\mathbb{C}^n$. Then on the quotient $R = \tilde{R}/T^{\mathbb{Z}}$ a foliation $\mathcal{F}$ with leafwise complex structure is induced.

From the construction, it is simple around the boundary. If the holonomy is infinitely tangent to the identity it is also tame with respect to the coordinate in the construction.

The boundary $U \setminus \{O\}/G^{\mathbb{Z}}$ is a complex manifold which is a so called *Hopf manifold*. In the case $n = 1$ it is an elliptic curve and the construction is equivalent to one with linear map as G.

Theorem 2.5 (Meersseman-Verjovsky, [MV]) Any tame Reeb component with complex leaves of complex dimension 1 is isomorphic to one of those given by the Hopf construction.

We present a couple of extensions (variants) of the above construction.

Construction 2.6 Now, let us take the product not with the half line but with the whole real line $\mathbb{R}$. Let M and $\Phi \in \mathit{Diff}^{\infty}_{+}(\mathbb{R})$ be as follows.

- $M = (U \times \mathbb{R} \setminus \{(O,0)\})/T'^{\mathbb{Z}}$, $\quad T' = G \times \Phi$,
- $x = 0$ is an expanding unique fixed point of Φ.

M consists of two Reeb components and in exactly the same way as above a foliation with leafwise complex structure is induced on M.

Note that in this and above construction, the holonomy of the toral leaf is given by φ and Φ. In these constructions, we can choose them so as to be tangent to the identity at the origin to the infinite order.

Construction 2.7 Next, we take Φ as a simple linear expansion in order to extend it to a biholomorphic expansion $\tilde{\Phi}$ of $\mathbb{C}$. Note that $\mathbb{R}$ is an invariant subspace. Fix the expansion ratio $\mu > 1$ and take the followings;

$$
\begin{aligned}
W &= (U \times \mathbb{C} \setminus \{O\})/T''^{\mathbb{Z}}, & T'' &= G \times \tilde{\Phi}, & \tilde{\Phi}(z) &= \mu z \ \ (z \in \mathbb{C}),\\
M &= (U \times \mathbb{R} \setminus \{(O,0)\})/T'^{\mathbb{Z}}, & T' &= G \times \Phi, & \Phi(x) &= \mu x \ \ (x \in \mathbb{R}).
\end{aligned}
$$

W is an $(n+1)$-dimensional Hopf manifold, M is its Levi-flat real hypersurface with Levi foliation consisting of two Reeb components, and a unique compact leaf is the Hopf manifold $(U \setminus \{O\})/G^{\mathbb{Z}}$ of $\dim_{\mathbb{C}} = n$.

Problem 2.8 The above result by Meersseman and Verjovsky poses the following questions. We assume the complex leaf dimension to be one. If two Reeb components with leafwise complex structures have the same boundary holonomy and their boundary leaves are biholomorphic to each other, are they isomorphic as leafwise complex foliations? Does there exist a Reeb component with complex leaves which is not isomorphic to a tame one but with holonomy infinitely tangent to the identity? Or does there exist one which is not isomorphic to any of those given by the Hopf construction? The second form of question seems not difficult to have negative answers. Anyway, those questions are asking what should be the complete invariants to determine Reeb components without assuming the tameness.

Construction 2.9 We introduce one more construction, which is prepared for turbulization. Take $\tilde{M} = (\mathbb{C}^n \times \mathbb{R}) \setminus \{O\} \times \mathbb{R}_{\leq 0}$ and restrict the product action $\hat{T} = G \times \Psi$ to $\tilde{M}$, where Ψ is an orientation preserving diffeomorphism of $\mathbb{R}$ which fixes 0, expanding on $\mathbb{R}_{\geq 0}$, and *contracting* on the negative side $\mathbb{R}_{\leq 0} = (-\infty, 0]$, *i.e.*, $\Psi(x) > x$ for $x < 0$. On $\tilde{M}$ we take (the restriction of) the horizontal foliation $\tilde{\mathcal{F}}$. Then take the quotient $(M, \mathcal{F}, J_{\mathcal{F}}) = (\tilde{M}, \tilde{\mathcal{F}}, J_{\mathrm{std}})/\hat{T}^{\mathbb{Z}}$.

The non-negative part is nothing but the Reeb component constructed in 2.4 regarding $\varphi = \Psi|_{\mathbb{R}_{\geq 0}}$. The non-positive side $(N, \mathcal{G}) = (M, \mathcal{F})|_{x \leq 0}$ remains non-compact and is in fact a foliated $\mathbb{R}_{\leq 0}$-bundle with holonomy $\psi = \Psi|_{\mathbb{R}_{\leq 0}}$.

If we remove the boundary compact leaf $\{x = 0\}$ from the non-positive side $(N, \mathcal{G})$, it is isomorphic to $(\mathbb{C}^n \setminus \{O\}) \times S^1$. For a better description of turbulization process, let us be more precise about this identification. This is done by embedding $\hat{T}$ in a 1-parameter family. Take a smooth curve G_t in $GL(2; \mathbb{C})$ and also a smooth curve ψ_t in $\mathit{Diff}^\infty(\mathbb{R}_{\leq 0})$ which is always tangent to the identity to the infinite order at $x = 0$, satisfying the following conditions.

$$\psi_k = \psi^k \;\; (k \in \mathbb{Z}), \quad \psi_{t+1} = \psi \circ \psi_t \;\; (t \in \mathbb{R}), \quad \frac{\partial \psi_t(x)}{\partial t} > 0 \; (\forall x, t),$$

$$G_k = G^k \;\; (k \in \mathbb{Z}), \quad G_{t+1} = G \circ G_t \;\; (t \in \mathbb{R}).$$

Then, fixing (any) $x_0 < 0$, $x = \psi_t(x_0)$ gives a diffeomorphism between $(-\infty, 0)\,(\ni x)$ and $\mathbb{R}(\ni t)$. Then the identification of $(z, x) \in (\mathbb{C}^n \setminus \{O\}) \times \mathbb{R}_{<0}$ with $(w, t) \in (\mathbb{C}^n \setminus \{O\}) \times \mathbb{R}$ by $(z = G_t(w), x = \psi_t(x_0))$ conjugates $\hat{T}|_{x<0}$ into $(w, t) \mapsto (w, t + 1)$.

It is worth remarking that this identification gives rise to a partial compactification of horizontally foliated manifold $((\mathbb{C}^n \setminus \{O\}) \times S^1, \{(\mathbb{C}^n \setminus \{O\}) \times \{t\}\}$ by a Hopf manifold N as a boundary leaf so as to obtain $(N, \mathcal{G})$. Around this boundary the structure is tame. If we take diffeormorphisms ψ_t infinitely tangent to the identity at the origin, we obtain a tame structure. Also on the non-negative side, by taking smilar family φ_t for $\varphi = \varphi_1$, we also obtain a tame structure on the non-negative side.

2.2 Turbulization in $L \times S^1$

Here we review a classic of modification of a foliation of codimension one to produce a (new) Reeb component, namely a turbulization. We start from a standard situation.

Construction 2.10 Let $(M, \mathcal{F})$ be a leafwise complex foliation of codimension one and assume that there is an embedded solid torus $U = \mathrm{int} D^{2n} \times S^1$ on which the the induced foliation is $\{\mathrm{int} D^{2n} \times \{*\}\}$ and the induced complex structure is also the canonical ones on each $\mathrm{int} D^{2n} \times \{*\} \cong \mathrm{int} D^{2n} \subset \mathbb{C}^n$. Let (w, t) denote the natural coordinate of $U = \mathrm{int} D^{2n} \times S^1$ where S^1 is regarded as $\mathbb{R}/\mathbb{Z}$. Then we remove $\{O\} \times S^1$ from U and let U^* denote the result. Using the coordinate (w, t) U^* is identified with an open subset of the negative side of Construction 2.9, together with leafwise complex foliations. Therefore we can compactify this end with the Hopf manifold as in Construction 2.9 and also if we add positive side of Construction 2.9 we obtain a leafwise complex foliated manifold without boundary with a new Reeb component. For this construction we can choose any of G_t, ψ_t, and Φ as in Construction 2.9. The above process including adding the positive side is the leafwise complex version of a *turbulization.* See also Figure 1 below.

2.3 General case

It is easy to find a closed transversal to a foliation of codimension one, namely, an embedded circle which is transverse to the foliation, unless the manifold is open and the foliation is too simple. If we do not regard the leafwise complex structure, it is always possible to perform the turbulization in a tubular neighborhood of the closed transversal. However, even with leafwise complex structures, the situation is almost the same because of the following fact, which also belongs to folklore.

Theorem 2.11 Let $(M^{2n+1}, \mathcal{F}, J)$ be a smooth leafwise complex foliation of codimension one and $K \subset M$ is a closed transvesal, namely there exists a smooth embedding $f : S^1 \to M$ which is transverse to the foliation $\mathcal{F}$ with its image $f(S^1) = K$.

Then, there exists a tubular neighborhood $U \cong K \times \operatorname{int} D^{2n}$ such that the restricted foliation $(U, \mathcal{F}_U, J|_{\mathcal{F}_U})$ is isomorphic to the standard one $(S^1 \times \operatorname{int} D^{2n}, \mathcal{F}_0 = \{t\} \times \operatorname{int} D^{2n}, J_0)$ and through this isomorphism K is identified with $S^1 \times \{O\}$.

In particular, we can perform the standard turbulization 2.10 in U.

This theorem follows from the following lemma.

Lemma 2.12 The group $\mathit{Diff}^{\,hol}(\mathbb{C}^n, O)$ of germs of holomorphic diffeomorphisms of $(\mathbb{C}^n, O)$ which fix the origin is pathwise connected.

The lemma immediately follows from the two facts that $GL(n; \mathbb{C})$ is pathwise connected and that such a germ with identical linear part can be joined by a straight segment to the identity.

2.4 Dehn surgery in $\dim = 3$ vs. higher dimensional turbulization

In order to close the section, this subsection provides with some remarks concerning the possibility of pasting the Reeb component in a different way in a turbulization. In the rest of this section and in fact that of this paper, let us assume the holonomy Ψ and thus accordingly φ and ψ as well to be tangent to the identity to the infinite order at the origin.

Remark 2.13 If we forget the leafwise complex structure and treat foliations only as smooth objects, basically there are two ways to perform the turbulization. The one has been already described above and is indicated in Figure 1. For the other one we can reverse the top and bottom of the Reeb component (Figure 2). This is because the cyclic (universal for $n \geq 2$) covering of the boundary leaf is $\mathbb{R}^{2n} \setminus \{O\} \cong S^{2n} \setminus \{N, S\}$ and two ends are exchangeable by a diffeomorphism. However, as a complex manifold, $\mathbb{C}^n \setminus \{O\}$ has one convex end and one concave. For the case of leaf dimension n is greater than 1, these two ends are not exchangeable. In particular, for

$n \geq 2$, the turbulization for leafwise complex foliations can not change the homotopy class of the tangent bundle.

Note that for these arguments we have to pay attentions only to complex structures of the boundary leaves, because we are dealing with flat structures.

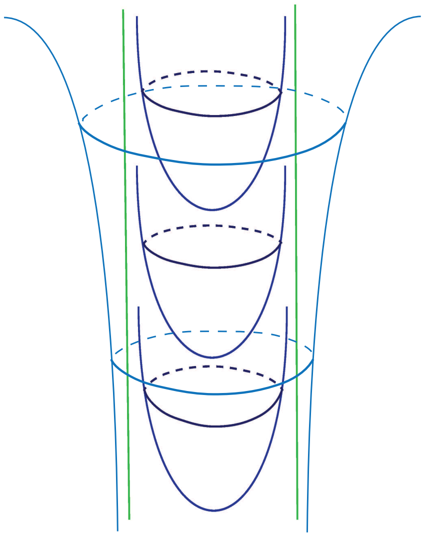

Figure 1

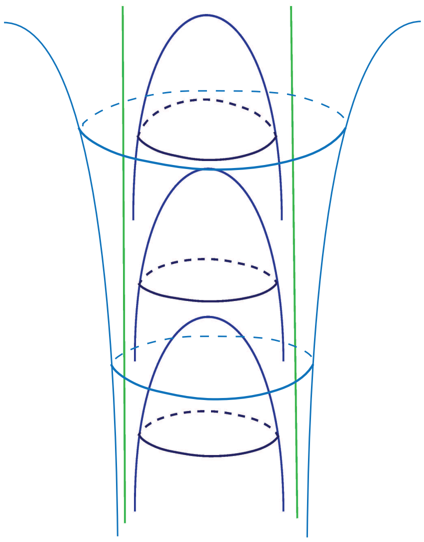

Figure 2

The green lines indicate the bounndary leaves of Reeb components. The axes of the rotational symmetries of the Reeb components, which are not drawn but the readers could imagine, correspond to $\{O\} \times \mathbb{R}_+$.

Remark 2.14 In the case of complex leaf dimension $= 1$, the 'upside-down' construction always works. Namely, in Construction 2.4, $z \leftrightarrow z^{-1}$ always induces an biholomorphism on the boundary elliptic curve.

If the boundary elliptic curve admits a complex multiplication, namely finite but discrete symmetries of order 2, 3 or 4, removing the Reeb component and pasting it back with one of those symmetries is a special kind of Dehn surgeries.

More generally, in the turbulization, remove the Reeb component and first leave it. We prepare another Reeb component with a different complex structure. If their boundaries match up through some diffeomophism, we can fill up the boundary with that Reeb component. In this way, a Dehn twist corresponding to any element of the mapping class group $\mathcal{M}_1 (\cong SL(2;\mathbb{Z}))$ of a 2-dimensional torus T^2 is realized for a closed transversal in a leafwise complex codimension one foliation of $n = 1$.

3 Functional equations on flat functions

In this section some preliminaries for the determination of the automorphisms of a Reeb component concerning certain functional equations for flat functions wich involve the holonomy diffeomorphisms. We include slight extensions of what we minimally need to have in this paper.

Let $\varphi \in \mathit{Diff}^{\infty}(\mathbb{R}_{\geq 0})$ be a diffeomorphism of the half line $\mathbb{R}_{\geq 0} = [0, \infty)$ which is tangent to the identity to the infinite order at $x = 0$ and satisfies $\varphi(x) - x > 0$ for $x > 0$. Also we fix a complex number λ with $|\lambda| > 1$. Let us consider the following (system of) functional equations on β, β_1 and β_2 $\in C^{\infty}(\mathbb{R}_{\geq 0}; \mathbb{C})$ concerning φ and λ. In the case where λ is a real number, we can consider the same equations for $\beta_2 \in C^{\infty}(\mathbb{R}_{\geq 0}; \mathbb{R})$.

Equation (I) : $\quad \beta(\varphi(x)) = \lambda \beta(x)$.

Equation (II) : $\quad \beta_1(\varphi(x)) = \lambda \beta_1(x) + \beta_2(x), \quad \beta_2(\varphi(x)) = \lambda \beta_2(x)$.

First consider these equations on $(0, \infty)$. Then, Equation (I) has a lot of solutions and if we fix any solution $\beta^*(x) \in C^{\infty}((0, \infty); \mathbb{C})$ which never vanishes, *i.e.*, $\beta^*(x) \neq 0$ for $x > 0$, then each solution corresponds to a smooth function on $S^1 = (0, \infty)/\varphi^{\mathbb{Z}}$ by taking $\beta \mapsto \beta/\beta^*$. This gives a bijective correspondence between the space $\mathcal{Z} = \mathcal{Z}_{\varphi,\lambda}$ of solutions to (I) on $(0, \infty)$ and $C^{\infty}(S^1; \mathbb{C})$ as vector space.

Also take the space $\mathcal{S} = \mathcal{S}_{\varphi,\lambda}$ of solutions to Equation (II) on $(0, \infty)$. If we assign β_2 to a solution $(\beta_1, \beta_2) \in \mathcal{S}$, we obtain the projection $P_2 : \mathcal{S} \to \mathcal{Z}$. Here the kernel of P_2 is nothing but $\mathcal{Z}$. We also see that the projection P_2 is surjective because for any $\beta_2 \in \mathcal{Z}$

$$\beta_1(x) = \frac{1}{\lambda \log \lambda} \beta_2(x) \log \beta^*(x)$$

gives a solution $(\beta_1, \beta_2) \in \mathcal{S}$, where for $\log \beta^*(x)$ any smooth branch can be taken. Therefore, as a vector space, $\mathcal{S}$ has a structure such that

$$0 \to \mathcal{Z} \to \mathcal{S} \to \mathcal{Z} \to 0$$

is a short exact sequence.

Theorem 3.1 1) Any solution $\beta \in \mathcal{Z}$ extends to $\mathbb{R}_{\geq 0}$ so as to be a smooth function which is flat at $x = 0$, *i.e.*, k-th jet satisfies $j^k\beta(0) = 0$ for any $k = 0, 1, 2, \cdots$.
2) The same applies to any solution $(\beta_1, \beta_2) \in \mathcal{S}$.

In the rest of this section we prove the above theorem. In order to clarify the strategy it might be suggested to the readers to check $\lim_{x \to +0} \beta(x) = 0$ and $\lim_{x \to +0} \beta'(x) = 0$ *i.e.*, for $k = 0, 1$, which are almost trivial, and then the the second jet $k = 2$. Looking at the case $k = 3$ might make the roll of the following lemma clearer.

Lemma 3.2 The n-th derivative $\{\beta(\varphi(x))\}^{(n)}$ is written in the following form for $n \in \mathbb{N}$.

$$\{\beta(\varphi(x))\}^{(n)} = (\varphi'(x))^n \cdot \beta^{(n)}(\varphi(x)) + \sum_{k=1}^{n-1} \Phi_{n,k} \cdot \beta^{(k)}(\varphi(x)) \, .$$

Here, $\Phi_{n,k}$ is an integral polynomial in $\varphi'(x)$, $\varphi''(x), \cdots$, $\varphi^{(n)}(x)$, without constant term and no term is of monomial only in $\varphi'(x)$.

This lemma is easily seen by the induction, but in fact it is a corollary to the well-known formula of Faà di Bruno (*e.g.*, see [Ri], [Ro], or textbooks on calculus). It is independent of our assumption on φ and is valid for any composite functions. On the other hand the flatness of φ at the origin implies $(\varphi'(x))^n \to 1$ and $\Phi_{n,k} \to 0$ when $x \to 0+0$.

Now let us prove 1) of Theorem 3.1. Let β be a solution to (I) on $(0, \infty)$. From the equation it is easy to see that $\beta(x) \to 0$ when $x \to 0+0$.

Now fix any integer N. $\beta'(x) \to 0$ is also easy to see, but for higher derivatives, in a natural estimate the lower derivatives are involved. Thus the basic strategy is not to estimate the higher derivatives by induction on the order, but to estimate them all together up to the fixed order N.

From Equation (I) and the above lemma we have the following computation.

$$\begin{aligned} \sum_{n=1}^{N} |\beta^{(n)}(x)| &= \frac{1}{|\lambda|} \sum_{n=1}^{N} \left| \{\beta(\varphi(x))\}^{(n)} \right| \\ &\leq \frac{1}{|\lambda|} \sum_{n=1}^{N} \left\{ (\varphi'(x))^n \cdot |\beta^{(n)}(\varphi(x))| + \sum_{k=1}^{n-1} |\Phi_{n,k}| \cdot |\beta^{(k)}(\varphi(x))| \right\} \\ &\leq \frac{1}{|\lambda|} \sum_{k=1}^{N} \left((\varphi'(x))^k + \sum_{n=k+1}^{N} |\Phi_{n,k}| \right) \cdot |\beta^{(k)}(\varphi(x))| \end{aligned}$$

As is remarked above, we know $(\varphi'(x))^k \to 1$ and $\sum_{n=k+1}^{N} |\Phi_{n,k}| \to 0$ when $x \to 0$. Therefore there exists $b_N > 0$ such that for $x \in (0, b_N]$ we have

$$(\varphi'(x))^k + \sum_{n=k+1}^{N} |\Phi_{n,k}| \leq \sqrt{|\lambda|} \quad \text{for} \quad k = 1, 2, \cdots, N.$$

This implies for any $x \in (0, b_N]$

$$\sum_{n=1}^{N} |\beta^{(n)}(x)| \leq \frac{1}{\sqrt{|\lambda|}} \sum_{n=1}^{N} |\beta^{(n)}(\varphi(x))|.$$

Put $M = \max\{\sum_{n=1}^{N} |\beta^{(n)}(x)| \, ; \, x \in [b_N, \varphi(b_N)]\}$ and define $m(x) \in \mathbb{N}$ for $x \in (0, b_N)$ so that $\varphi^{m(x)} \in [b_N, \varphi(b_N))$. Then, the above inequality implies

$$\sum_{n=1}^{N} |\beta^{(n)}(x)| \leq M \cdot \sqrt{|\lambda|}^{\,-m(x)}$$

for $x \in (0, b_N)$. Because '$x \to 0+0$' is equivalent to '$m(x) \to \infty$', we obtained the convergence

$$\beta^{(n)}(x) \to 0 \quad (x \to 0+0) \qquad \text{for} \quad n = 1, \cdots, N.$$

This completes the proof of 1).

Let us outline the proof of 2). We extend the basic strategy of the proof of 1) in the following sense. When we estimate the derivatives of β_1, naturally those of β_2 are involved. Therefore we will estimate the derivatives of β_1 and β_2 all together up to a fixed order N, even though the flatness of β_2 is already proved in 1).

First we fix $\varepsilon > 0$ so small that $\varepsilon \le |\lambda|^{\frac{5}{4}} - |\lambda|$ is satisfied. Now take any solution $(\beta_1, \beta_2) \in \mathcal{S}$ and put $\tilde{\beta}_2 = \varepsilon^{-1}\beta_2$. Then instead of Equation (II), β_1 and $\tilde{\beta}_2$ satisfy

$$\text{Equation } (\tilde{\text{II}}): \quad \beta_1(\varphi(x)) = \lambda\beta_1(x) + \varepsilon\tilde{\beta}_2(x), \quad \tilde{\beta}_2(\varphi(x)) = \lambda\tilde{\beta}_2(x).$$

Then, from $(\tilde{\text{II}})$ we have

$$\beta_1(\varphi(x)) + \frac{e^{i\theta}\lambda - \varepsilon}{\lambda}\tilde{\beta}_2(\varphi(x)) \quad = \quad \lambda\beta_1(x) + e^{i\theta}\lambda\tilde{\beta}_2(x)$$

and consider the n-th derivatives of both sides. For any $\theta \in \mathbb{R}$ and $n = 1, \cdots, N$, we have

$$|\beta_1^{(n)}(x) + e^{i\theta}\tilde{\beta}_2^{(n)}(x)| \le \frac{1}{|\lambda|}\left(|\{\beta_1(\varphi(x))\}^{(n)}| + \frac{|\lambda| + \varepsilon}{|\lambda|}|\{\tilde{\beta}_2(\varphi(x))\}^{(n)}|\right)$$

Because the right hand side is independent of θ, using the inequality

$$\left|\frac{e^{i\theta}\lambda - \varepsilon}{\lambda}\right| \le \frac{|\lambda| + \varepsilon}{|\lambda|} \le |\lambda|^{\frac{1}{4}} \quad \text{for any} \;\; \theta \in \mathbb{R}$$

we obtain

$$|\beta_1^{(n)}(x)| + |\tilde{\beta}_2^{(n)}(x)| \le \frac{1}{|\lambda|^{\frac{3}{4}}}\left(|\{\beta_1(\varphi(x))\}^{(n)}| + |\{\tilde{\beta}_2(\varphi(x))\}^{(n)}|\right).$$

Applying Lemma 3.2 to $\beta_1(\varphi(x))$ and to $\tilde{\beta}_2(\varphi(x))$ for $n = 1, \cdots, N$, from the same argument as in 1) we obtain

$$\sum_{n=1}^{N}\left(|\beta_1^{(n)}(x)| + |\tilde{\beta}_2^{(n)}(x)|\right) \le \frac{1}{|\lambda|^{\frac{1}{4}}}\sum_{n=1}^{N}\left(|\beta_1^{(n)}(\varphi(x))| + |\tilde{\beta}_2^{(n)}(\varphi(x))|\right)$$

for $x \in (0, b_N]$, where b_N is exactly the same as in the proof of 1). □

Now it is almost straight forward to generalize these facts to the following case. Let $\boldsymbol{\beta}(x)$ be a $\mathbb{C}^n$-valued function on the open halfline $(0, +\infty)$ and consider the equation $\boldsymbol{\beta}(\varphi(x)) = A\boldsymbol{\beta}(x)$ where $A = (a_{ij})$ is an $n \times n$ matrix any of its eigenvalues has the absolute value greater than 1. Of course it is enough to consider the case where A is a nontrivial Jordan block, *i.e.*, $a_{ij} = \lambda$ for $i = j$, $a_{ij} = 1$ for $i + 1 = j$, and $a_{ij} = 0$ otherwise.

Let $\mathcal{S}_n$ denote the set of solutions. For $1 \le m < n$, $\mathcal{S}_m$ is naturally identified with a quotient $\{{}^t(\beta_{n-m+1} \cdots \beta_n) \,|\, {}^t(\beta_1 \cdots \beta_n) \in \mathcal{S}_n\}$ of $\mathcal{S}_n$. Each projection $\mathcal{S}_m \to \mathcal{S}_{m-1}$ is surjective because the multiplication $\times \frac{1}{\lambda \log \lambda} \beta^*$ is a linear right inverse and its kernel coincides with $\mathcal{Z}$. $\mathcal{S}_1$ is nothing but $\mathcal{Z}$ and $\mathcal{S}_2$ the above $\mathcal{S}$ as well. Then any $\boldsymbol{\beta} \in \mathcal{S}_n$ extends to $\mathbb{R}_{\ge 0}$ by defining $\boldsymbol{\beta}(0) = 0$ and is then flat at $x = 0$.

The proof of this generalization is also easy and is left to the readers. It appears if we deal with higher dimensional Reeb components, while it is described mainly because of the curiosity, its easiness, and clarity.

4 Symmetries of 3-dimensional Reeb component

In this section we compute the group of automorphisms of a Reeb component of dimension 3 which is given by a Hopf construction. In order to fix notations, we present our objects again. Let $\tilde{R}$ be $\mathbb{C} \times \mathbb{R}_{\ge 0} \setminus \{(0,0)\}$, take $\lambda \in \mathbb{C}$ with $|\lambda| > 1$ and a diffeomorphism $\varphi \in \mathit{Diff}^{\infty}(\mathbb{R}_{\ge 0})$ which is tangent to the identity to the infinite order at 0 and satisfies $\varphi(x) - x > 0$ for $x > 0$. Let G denote the linear expansion of $\mathbb{C}$ defined as the multiplication by λ and $T : \tilde{R} \to \tilde{R}$ be $T = G \times \varphi$. Then we obtain a Reeb component $(R, \mathcal{F}, J) = (\tilde{R}, \tilde{\mathcal{F}}, J_{\mathrm{std}})/T^{\mathbb{Z}}$ as the quotient, as well as the boundary elliptic curve $H = \mathbb{C} \setminus \{0\}/G^{\mathbb{Z}}$. Here, on the upstairs the leaves of the foliation $\tilde{\mathcal{F}} = \{\mathbb{C} \times \{x\} | x > 0\} \sqcup \{\mathbb{C}^* \times \{0\}\}$ are equipped with the natural complex structure J_{std} which is inherited by those of $\mathcal{F}$.

4.1 Lift to $\tilde{R}$ and restriction to H

Let us consider the group $Aut(R, \mathcal{F}, J)$, which is also denoted by $AutR$ for short, of all foliation preserving diffeomorphisms of R whose restriction to each leaf is holomorphic. Also we consider the group of holomorphic diffeomorphisms $AutH$ of the boundary elliptic curve H as well as its identity component Aut_0H which is isomorphic to T^2 and can be identified with H itself.

Propoition 4.1 The image of the restriction map $r_H : AutR \to AutH$ is exactly Aut_0H.

Proof. If we regard $AutH/Aut_0H$ as a subgroup of $SL(2;\mathbb{Z})$, in most cases it is just $\{\pm E\}$ where E denotes the identity matrix. In a few cases where the elliptic curve H admits complex multiplications, they are of order 3,4, or of 6 and a kind of 'rotations' on the universal covering, *i.e.*, elliptic matrices in $SL(2;\mathbb{Z})$. In any of those cases, no element in $AutH \setminus Aut_0H$ preserves the direction of holonomy and thus none extends to R as a foliation preserving diffeomorphism.

On the other hand, any element in $Aut_0 H$ is obtained as the quotient of the scalar multiplication $m_a : \mathbb{C}^* \to \mathbb{C}^*$ by some nonzero complex number a. The automorphism $m_a \times \mathrm{id}_{\mathbb{R}_{\geq 0}}$ of $\tilde{R}$ clearly descends to R and defines an element in $AutR$. $\square$

By this proposition, the study of the structure of $AutR$ breaks into two parts, that of the kernel $Aut(R, H)$ and the study of the restriction map r_H.

Now it is easier to look at the lifts of automorphisms on $\tilde{R}$. Any element $f \in AutR$ has a lift $\tilde{f} \in Aut(\tilde{R}, \tilde{\mathcal{F}}, J_{\mathrm{std}})$ ($= Aut\tilde{R}$) which takes the form

$$\tilde{f}(z, x) = (\xi(z, x), \eta(x))$$

in $\mathbb{C} \times \mathbb{R}_{\geq 0}$-coordinate. A lift $\tilde{f}$ should commutes with the covering transformation T, because, $T \circ \tilde{f} = \tilde{f} \circ T^k$ for some $k \in \mathbb{Z}$ but it is easy to see that $k = 1$ when it is restricted to the boundary. Therefore an element in $Aut\tilde{R}$ is a lift of some element in $AutR$ if and only if it commutes with T. Let $Aut(\tilde{R}; T)$ denote the centralizer of T in $Aut\tilde{R}$, namely, the group of all such lifts. It contains an abelian subgroup $\{m_a \times \mathrm{id}_{\mathbb{R}_{\geq 0}} | a \in \mathbb{C}^*\} \cong \mathbb{C}^*$. This subgroup injectively descends to a subgroup of $AutR$ which restricts exactly to $Aut_0 H \cong \mathbb{C}^*/\lambda^{\mathbb{Z}}$. It is important to remark that whether $Aut_0 H$ admits a homomorphic section is not a trivial question. Postponing this question until the end of this section, we go on an easier way.

Let us introduce one more subgroup $Aut(\tilde{R}, \tilde{H}; T)$ of $Aut(\tilde{R}; T)$ which consists of all elements which act trivially on the boundary $\tilde{H}$. Any element $f \in Aut(R, H)$ has a unique lift to an element $\tilde{f} \in Aut(\tilde{R}, \tilde{H}; T)$ Namely,

Corollary 4.2 $Aut(R, H)$ is isomorphic to $Aut(\tilde{R}, \tilde{H}; T)$.

Again, let us present an element $g \in Aut\tilde{R}$ in the form $g(z, x) = (\xi(z, x), \eta(x))$.

Lemma 4.3 The element g belongs to $Aut(\tilde{R}; T)$ if and only if it satisfies the following conditions.

(1) $\xi(z, x) = az + b(x)$, $b(0) = 0$ for some $b \in C^\infty([0, +\infty), \mathbb{C})$ and $a \in \mathbb{C}^*$.

(2) $b(\varphi(x)) = \lambda b(x)$.

(3) $\varphi \circ \eta = \eta \circ \varphi$, namely, $\eta \in Z_\varphi =$ the centralizer of φ in $\mathit{Diff}^\infty(\mathbb{R}_{\geq 0})$.

Further more, g belongs to $Aut(\tilde{R}, \tilde{H}; T)$ if and only if the above conditions are satisfied with $a = 1$.

Proof. Let us first show the *only if* direction, then the *if* direction will become almost trivial.

Assume $g \in Aut(\tilde{R}; T)$. $\xi(z, x)$ is smooth and holomorphic in z. If x is fixed, $\xi(\,\cdot\,, x) : \mathbb{C} \to \mathbb{C}$ is a holomorphic automorphism even in the case where $x = 0$ because the origin is a removable singularity, it is a linear map with nontrivial linear term. Therefore it is written in the following form; $\xi(z,x) = a(x)z + b(x)$ where $a(x), b(x) \in C^\infty(\mathbb{R}_{\geq 0}, \mathbb{C})$ with $a(x) \neq 0$ and $b(0) = 0$. These also apply to elements in $Aut\tilde{R}$.

Now look at the commutation relation with T. $g \circ T = T \circ g$ implies

$$(a(\varphi(x))\lambda z + b(\varphi(x)), \eta(\varphi(x)) = (\lambda a(x)z + \lambda b(x), \varphi(\eta(x)))\,.$$

Thus we obtain (2) and (3). This also tells us that $a(\varphi(x)) = a(x)$, so that for any $x \geq 0$ we have $a(x) = \lim_{n\to\infty} a(\varphi^{-n}(x)) = a(0)$ and (1) is concluded.

For $g \in Aut(\tilde{R}, \tilde{H}; T)$ we just need to confirm that $a = 1$. □

Corollary 4.4 $AutR$ is naturally isomorphic to $Aut(\tilde{R}; T)/T^{\mathbb{Z}}$.

4.2 Centralizer of φ in $Diff^\infty(\mathbb{R}_{\geq 0})$

In general, the centralizer Z_φ of φ in $Diff^\infty(\mathbb{R}_{\geq 0})$ is known to be fairy wild (see [Ey]). Under our assumption on φ it is known that there exists a unique C^1-vector field $X_\varphi = \rho(x)\dfrac{d}{dx}$ on the half line $\mathbb{R}_{\geq 0}$, which is of class C^∞ on $(0, +\infty)$, in such a way that the exponential map $\exp X_\varphi$ (namely the time one map of the generated flow) coincides with φ (see [Sz] and [Na]). This vector field is often called the *Szekerez vector field* of φ. If the Szekerez vector field X_φ is of class C^∞ on $\mathbb{R}_{\geq 0}$, namely $\rho(x)$ is flat at $x = 0$, then the centralizer Z_φ coincides with the 1-parameter family $\{\exp(tX_\varphi) = \varphi^t\,;\, t \in \mathbb{R}\}$ generated by X_φ.

In general case, using the order of real numbers, the centralizer Z_φ turns out to be a totally ordered abelian group which contains $\{\varphi^{\mathbb{Z}}\} \cong \mathbb{Z}$. Therefore it is uniquely identified with a certain subgroup of the additive group $\mathbb{R}$ under the identification $\{\varphi^{\mathbb{Z}}\} \cong \mathbb{Z}$. Depending on φ, Z_φ can be $\mathbb{Z}$, $\mathbb{Q}$, or *e.g.*, $\mathbb{Z} \oplus \mathbb{Z}\alpha$ where α is a Liouville number [Ey], or far more complicated. The topology on Z_φ through this identification with natural topology of $\mathbb{R}$ coincides with the one induced from the C^0-topology on $Diff^\infty(\mathbb{R}_{\geq 0})$.

We should also remark that any element of Z_φ is tangent to the identity at the origin to the infinite order.

At present it is not known whether $Z_\varphi \cong \mathbb{R}$ implies the smoothness of X_φ at $x = 0$.

4.3 Structure of $AutR$

Upon all the previous preparations we are able to describe the structure of $AutR$ as follows.

Propoition 4.5 Let R be a Reeb component of real dimension 3 which is given by the Hopf construction as indicated in the beginning of this section.

1) The group $AutR$ of automorphisms of the Reeb component R admits a following sequence of extensions by abelian groups,

$$0 \to Aut(R, H) \to AutR \to Aut_0H \to 0$$

$$0 \to \mathcal{Z}_{\varphi,\lambda} \to Aut(R, H) \to Z_\varphi \to 0$$

where $Aut_0H \cong \mathbb{C}^*/\lambda^{\mathbb{Z}}$ is represented by the constant linear part a described in Lemma 4.3, Z_φ is the centralizer of φ which is explained in the previous subsection, and $\mathcal{Z}_{\varphi,\lambda}$ is an infinite dimensional vector space described in Section 3 which is the set of functions $b(x)$ in Lemma 4.3.

The following can be regarded as a corollary to the arguments already done up to the previous subsection, in particular to those done on the space $\mathcal{Z}_{\varphi,\lambda}$ in the previous section.

Corollary 4.6 If we paste $H \times (-\varepsilon, 0]$ to R along the boundary H, any element of $AutR$ extends to the other side, being the identity on $H \times (-\varepsilon, 0]$, as a diffeomophisms of class C^∞.

The first step of the extensions is obtained by looking at the action on the boundary, and once we assume that the action on the boundary is trivial, the second extension is obtained by looking at the action on the vertical line $\{0\} \times \mathbb{R}_{\geq 0}$. We can interpret it as an action on the leaf space. However, the first extension does not yield non-abelian group. Using the identification $AutR \cong Aut(\tilde{R}; T)/T^{\mathbb{Z}}$ in Corollary 4.4, we obtain a better description not only from the above point of view but also from that of the question whether the restriction map $r_H : AutR \to Aut_0H$ admits a homomorphic section. Note that Z_φ admits a section to $Aut(R, H) \subset AutR$.

An element $f \in Aut(\tilde{R}; T)/T^{\mathbb{Z}}$ admits a presentation $f(z, x) = (az + b(x), \eta(x))$ up to $T^{\mathbb{Z}}$ where $T(z, x) = (\lambda z, \varphi(x))$. Therefore ignoring $b(x)$ from this presentation and assigning $f \mapsto (a, \eta) \pmod{(\lambda, \varphi)^{\mathbb{Z}}}$, we obtain a surjective homomorphism $Aut(\tilde{R}; T)/T^{\mathbb{Z}} \twoheadrightarrow (\mathbb{C}^* \times Z_\varphi)/(\lambda, \varphi)^{\mathbb{Z}}$ to an abelian group. Also, by setting $b(x) = 0$, we see this abelian group can be realized as a subgroup of $Aut(\tilde{R}; T)/T^{\mathbb{Z}}$. This enables us to describe the structure of $AutR$ as follows.

Theorem 4.7 The automorphism group $AutR \cong Aut(\tilde{R}; T)/T^{\mathbb{Z}}$ is isomorphic to the semi-direct product

$$\mathcal{Z}_{\varphi,\lambda} \rtimes \left\{(\mathbb{C}^* \times Z_\varphi)/(\lambda, \varphi)^{\mathbb{Z}}\right\}$$

where $a \in \mathbb{C}^*$ acts on $b(x) \in \mathcal{Z}_{\varphi,\lambda}$ by multiplication $b(x) \mapsto a^{-1}b(x)$, *i.e.*, the conjugation in the affine transformations of each leaf, and $\eta \in Z_\varphi$ acts by $b(x) \mapsto b(\eta(x))$.

proof. Let us only verify the action of a. The conjugation by the multiplication by a is $[z \mapsto z + b(x)] \mapsto [z \mapsto a^{-1}(az + b(x)) = z + a^{-1}b(x)]$. □

To close this section, consider the liftability of $Aut_0 H$ to $Aut R$. This is nothing but the liftability of the surjective homomorphism

$$(\mathbb{C}^* \times Z_\varphi)/(\lambda, \varphi)^{\mathbb{Z}} \twoheadrightarrow \mathbb{C}^*/\lambda^{\mathbb{Z}}\,.$$

Here we assume the continuity of splitting, otherwise the question should include thinking about non-continuous homomorphism $\mathbb{R} \to \mathbb{R}$ with $1 \mapsto 1$. If the centralizer Z_φ is the total of $\mathbb{R}$, it implies Z_φ is a C^0-family of 1-parameter subgroup $\{\eta_t\,;\, t \in \mathbb{R}\}$ in $\mathit{Diff}^\infty(\mathbb{R}_{\geq 0})$ with $\varphi = \eta_1$. Then we obtain easily a lift defined as

$$a(\mathrm{mod}\,\lambda^{\mathbb{Z}}) \mapsto (a, \eta_{t(a)})\,(\mathrm{mod}\,(\lambda,\varphi)^{\mathbb{Z}})\,, \quad t(a) = \frac{\log|a|}{\log|\lambda|}\,.$$

The converse is almost the same. If we have a continuous lift to $Aut(\tilde{R};T)/T^{\mathbb{Z}}$, choose a value of $\log\lambda$ and look at the lift of a circle subgroup $e^{t\log\lambda}$ $(0 \leq t \leq 1)$ to a continuous path in $Aut(\tilde{R};T)$ starting from the identity. Then its projection to Z_φ gives rise to a 1-parameter family in Z_φ starting from the identity which ends at φ. If this curve is smooth, it implies that the Szekeres vector field X_φ of φ is smooth. Thus we obtain the following.

Theorem 4.8 The restriction map $r_H : Aut R \to Aut_0 H$ admits a continuous [*resp.* smooth] homomorphic section if and only if the centralizer Z_φ is isomorphic to $\mathbb{R}$ as an ordered abelian group [*resp.* the Szekeres vector field X_φ is smooth].

Moreover in this case, $Aut R$ admits a structure of semi-direct product of two abelian groups as follows;

$$Aut R \cong \mathcal{Z}_{\varphi,\lambda} \rtimes (Aut_0 H \times Z_\varphi).$$

5 Reeb foliations

The automorphism group of a leafwise complex foliation on a closed 3-manifold which consists of two Reeb components is now easy to compute.

Let $R_{\varphi,\lambda}$ be the Reeb component which we dealt with in the previous section. For another diffeomorphism $\psi \in \mathit{Diff}^\infty(\mathbb{R}_{\geq 0})$ which is also expanding and tangent to the identity at the origin to the infinite order and another constant $\mu \in \mathbb{C}$ with $|\mu| > 1$, take the Reeb component $R_{\psi,\mu}$ and let $\overline{R}_{\psi,\mu}$ denote the mirror of $R_{\psi,\mu}$, namely the one which we obtain by reversing the the transverse orientation. It is done by replacing x with $-x$ in the Hopf construction.

For example if $\lambda = \mu$ we can paste $R_{\varphi,\lambda}$ and $\overline{R}_{\psi,\mu}$ along the common boundary $H = \mathbb{C}^*/\lambda^{\mathbb{Z}}$ by the identity of H to obtain a leafwise complex foliation on $S^2 \times S^1$. In general according to the pasting element $\in SL(2;\mathbb{Z})$ we can choose appropriately λ and μ and paste them. The foliation on S^3 obtained in such a way is called the Reeb foliation.

Corollary 4.6 yields the following results.

Theorem 5.1 Let $(M, \mathcal{F}, J)$ be a leafwise complex foliation which is obtained by pasting $R_{\varphi,\lambda}$ and $\overline{R}_{\psi,\mu}$. Then its group of automorphism is naturally isomorphic to the fibre product of $AutR_{\varphi,\lambda}$ and $Aut\overline{R}_{\psi,\mu}$ with respect to $Aut_0 H$.

If the centralizer Z_ψ is isomorphic to $\mathbb{R}$ as an ordered abelian group, then $AutR_{\varphi,\lambda}$ is continuously realized as a subgroup in the resulting group of automorphisms.

Theorem 5.2 If $(M^3, \mathcal{F}_1, J_1)$ is obtained from $(M^3, \mathcal{F}_0, J_0)$ by turbulization along a closed transversal and the resulting Reeb component is isomorphic to $AutR_{\varphi,\lambda}$, the group $Aut(M^3, \mathcal{F}_1, J_1)$ naturally contains a subgroup which is isomorphic to $Aut(R_{\varphi,\lambda}, H)$.

Remark 5.3 In both of above theorems, the automorphism group contains an infinite dimensional vector space $\mathcal{Z}$ or one more copy and can be incomplete when the holonomy is not nice. Thus even in the case of closed manifolds, the automorphism group of leafwise complex foliation can be fairy large. This presents a clear contrast between the study of leafwise complex foliations and that of compact complex manifolds or to that of moduli space of leafwise complex foliations with compactness properties such as tameness. It might be also possible to interpret that the largeness of the automorphism group mirrors the finite dimensionality of the moduli space in the tame case.

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