CHUO MATH NO.111(2015)

Nonautonomous differential equations and Lipschitz evolution operators in Banach spaces

by
Yoshikazu Kobayashi, Naoki Tanaka
and Yukino Tomizawa

DEPARTMENT OF MATHEMATICS
CHUO UNIVERSITY
BUNKYOKU TOKYO JAPAN

MAY.29 , 2015

Nonautonomous differential equations and Lipschitz evolution operators in Banach spaces

Yoshikazu Kobayashi, Naoki Tanaka and Yukino Tomizawa

Abstract. A new class of Lipschitz evolution operators is introduced and a characterization of continuous infinitesimal generators of such evolution operators is given. It is shown that a continuous mapping A from a subset Ω of $[a, b) \times X$ into X, where $[a, b)$ is a real half-open interval and X is a real Banach space, is the infinitesimal generator of a Lipschitz evolution operator if and only if it satisfies a sub-tangential condition, a general type of quasi-dissipative condition with respect to a metric-like functional and a connectedness condition. An application of the results to the initial value problem for the quasilinear wave equation with dissipation is also given.

1. Introduction and Main Theorems

Throughout this paper, $\mathbb{R}$ denotes the set of all real numbers. Let X be a real Banach space with norm $\|\cdot\|$. For a subset Q of $\mathbb{R} \times X$, $Q(t)$ denotes the section of Q at $t \in \mathbb{R}$, that is, $Q(t) = \{x \in X\,;\,(t, x) \in Q\}$.

Let $[a, b)$ be a subinterval of $\mathbb{R}$ and Ω a subset of $[a, b) \times X$ such that $-\infty < a < b \le \infty$ and $\Omega(t) \neq \emptyset$ for $t \in [a, b)$. Let A be a continuous mapping from Ω to X. Given $(\tau, z) \in \Omega$, we consider the following initial value problem:

$$(\mathrm{IVP}; \tau, z) \qquad \begin{cases} u'(t) = A\big(t, u(t)\big) & \text{for} \quad \tau \le t < b, \\ u(\tau) = z. \end{cases}$$

Suppose that the problem $(\mathrm{IVP}; \tau, z)$ has a unique solution $u(\cdot)$ on $[\tau, b)$ for every $(\tau, z) \in \Omega$. Defining $U(t, \tau)z = u(t)$, we have the following properties from the uniqueness of solutions:

(E1) $U(\tau, \tau)z = z$ and $U(t, s)U(s, \tau)z = U(t, \tau)z$ for $z \in \Omega(\tau)$ and $a \le \tau \le s \le t < b$.

The first author is supported by Grant-in-aid for Science Research, No. 25400145.

The second author is supported by Grant-in-aid for Science Research, No. 254001234.

2010 *Mathematics Subject Classification.* Primary 34G20; Secondary 47J35.

Key words and phrases. Lipschitz evolution operator, infinitesimal generator, sub-tangential condition, quasi-dissipative condition, metric-like functional, connectedness condition.

Set $\Delta = \{(t, \tau); a \leq \tau \leq t < b\}$. Usually, we have also the following properties from the continuous dependence of solutions on the initial data $(\tau, z) \in \Omega$:

(E2) Let $(t, \tau) \in \Delta$, $z \in \Omega(\tau)$, $(t_n, \tau_n) \in \Delta$ and $z_n \in \Omega(\tau_n)$ for $n = 1, 2, \ldots$. If $(t_n, \tau_n) \to (t, \tau)$ and $z_n \to z$ as $n \to \infty$, then $U(t_n, \tau_n)z_n \to U(t, \tau)z$ as $n \to \infty$.

By an *evolution operator* on Ω, we mean a family $\{U(t, \tau)\}_{(t,\tau)\in\Delta}$ of operators $U(t, \tau) : \Omega(\tau) \to \Omega(t)$ satisfying (E1) and (E2). Such a family $\{U(t, \tau)\}_{(t,\tau)\in\Delta}$ is called a *Lipschitz evolution operator* on Ω, if the following additional condition is satisfied:

(E3) There exist a number $L \geq 1$ and a continuous function $\omega : [a, b) \to [0, \infty)$ such that

$$\|U(t, \tau)x - U(t, \tau)y\| \leq L \exp\left(\int_\tau^t \omega(\theta)d\theta\right)\|x - y\|$$

for x, $y \in \Omega(\tau)$ and $(t, \tau) \in \Delta$.

The main purpose of this paper is to establish the conditions on the continuous mapping A which are necessary and sufficient to guarantee the existence of the Lipschitz evolution operator associated with A. The obtained results extend that of Kobayashi and Tanaka in [8] concerning the autonomous case where A is independent of t. In particular, a type of generalized quasi-dissipativity condition on A with respect to a metric-like functional is shown to be necessary for the existence of the Lipschitz evolution operator. Sufficient conditions on A for the existence of evolution operators have been studied by many authors and this paper is related with the works of Iwamiya [4], Kato [5], [6], Kenmochi and Takahashi [7], Lakshmikantham, Mitchell and Mitchell [10], Martin [11], [12], [13] , Murakami [15], Pavel and Vrabie [19], Pavel [18] and Cârjă, Necula and Vrabie [22]. Several types of generalized quasi-dissipativity conditions on A are introduced and investigated in [15], [12], [10] , [6], [20] and [2]. Such a kind of generalized quasi-dissipativity conditions was first found by Okamura [17] as a uniqueness criteria for ordinary differential equations. See [1] or [24]. Our results extend the most of them. As in [7], [6] and [4], the domain Ω is allowed to be genuinely noncylindrical and the subtangential condition, which was first found by Nagumo [16], is used to construct approximate solutions to (IVP; τ, z). The advantage of these assumptions is illustrated by an application of the results to the initial value problems for nonlinear wave equations.

Let $J \subset [a, b)$ be a subinterval of the form $[\tau, c]$ or $[\tau, c)$. An X-valued continuous function $u : J \to X$ is called a *solution to* (IVP;τ, z) *on* J, if $u(\tau) = z$, $\big(t, u(t)\big) \in \Omega$ for $t \in J$, u is differentiable on J and $u'(t) = A\big(t, u(t)\big)$ for $t \in J$. A solution to (IVP; τ, z) on $[\tau, b)$ is called a *global solution*.

Let $d(x, D)$ denote the distance from $x \in X$ to $D \subset X$, i.e., $d(x, D) = \inf\{\|x - y\| \,;\, y \in D\}$. We consider the following conditions.

(Ω1) A is continuous on Ω.
(Ω2) If $(t_n, x_n) \in \Omega$, $t_n \uparrow t \in [a, b)$ in $\mathbb{R}$ and $x_n \to x$ in X as $n \to \infty$, then $(t, x) \in \Omega$.
(Ω3) $\liminf_{h\downarrow 0} h^{-1} d\big(x + hA(t, x), \Omega(t + h)\big) = 0$ for $(t, x) \in \Omega$.
(Ω4) There exists a functional $V : [a, b) \times X \times X \to [0, \infty)$ satisfying the following properties $(V1)$–$(V4)$ and a continuous function $\omega : [a, b) \to [0, \infty)$ such that

$$D_+V(t, x, y)\big(A(t, x), A(t, y)\big) \le \omega(t)V(t, x, y)$$

for $x, y \in \Omega(t)$ and $t \in [a, b)$. Here, for $(t, x, y) \in [a, b) \times X \times X$ and $(\xi, \eta) \in X \times X$,

$$D_+V(t, x, y)(\xi, \eta) = \liminf_{h\downarrow 0} \frac{1}{h}\big(V(t + h, x + h\xi, y + h\eta) - V(t, x, y)\big),$$

where the values ∞ and $-\infty$ are not excluded.
$(V1)$ There exists a number $L > 0$ such that $|V(t, x, y) - V(t, \hat{x}, \hat{y})| \le L(\|x - \hat{x}\| + \|y - \hat{y}\|)$ for $(x, y), (\hat{x}, \hat{y}) \in X \times X$ and $t \in [a, b)$.
$(V2)$ $V(t, x, x) = 0$ for $t \in [a, b)$ and $x \in \Omega(t)$.
$(V3)$ If $\{t_n\}$ is a sequence in $[a, b)$ and $\{(x_n, y_n)\}$ is a sequence in $X \times X$ such that $(x_n, y_n) \in \Omega(t_n) \times \Omega(t_n)$ for $n \ge 1$, $t_n \to t \in [a, b)$ and $(x_n, y_n) \to (x, y) \in \Omega(t) \times \Omega(t)$ as $n \to \infty$, then $V(t, x, y) \le \liminf_{n\to\infty} V(t_n, x_n, y_n)$.
$(V4)$ If $\{t_n\}$ is a sequence in $[a, b)$ and $\{(x_n, y_n)\}$ is a sequence in $X \times X$ such that $(x_n, y_n) \in \Omega(t_n) \times \Omega(t_n)$ for $n \ge 1$, $t_n \to t \in [a, b)$ and $V(t_n, x_n, y_n) \to 0$ as $n \to \infty$, then $\|x_n - y_n\| \to 0$ as $n \to \infty$.
(Ω5) For any $(\tau, z) \in \Omega$, there exists a connected component C of Ω such that $(\tau, z) \in C$ and $C(t) \ne \emptyset$ for $t \in (\tau, b)$.

Remark 1. *Condition $(V1)$ with $(V2)$ implies the following:*

$|V(t, x, y)| \le L\|x - y\|$ *for* $(x, y) \in \Omega(t) \times \Omega(t)$ *and* $t \in [a, b)$.

The following are our main theorems.

Theorem 1. *Let A be a mapping from Ω into X such that conditions (Ω1)–(Ω4) are satisfied. Let C be a connected component of Ω and set $d = \sup\{t \in [a, b) \,;\, C(t) \ne \emptyset\}$. Then the following assertions hold true:*

(i) *For $(\tau, z) \in C$, (IVP; τ, z) has a unique solution $u(t; \tau, z)$ on $[\tau, d)$ and the interval $[\tau, d)$ is the maximal interval of existence of solution.*

(ii) *For* $z, \hat{z} \in C(\tau)$ *and* $t \in [\tau, d)$,

$$V(t, u(t;\tau,z), u(t;\tau,\hat{z})) \le \exp\left(\int_\tau^t \omega(\theta)d\theta\right) V(\tau, z, \hat{z}).$$

THEOREM 2. *Let A be a mapping from Ω into X such that* $(\Omega 1)$ *and* $(\Omega 2)$ *are satisfied. Then there exists a Lipschitz evolution operator* $\{U(t,\tau)\}_{(t,\tau)\in\Delta}$ *on* Ω *such that* $u(t) := U(t,\tau)z$ *is a global solution to* (IVP; τ, z) *for any* $(\tau, z) \in \Omega$ *if and only if conditions* $(\Omega 3)$–$(\Omega 5)$ *are satisfied, where condition* $(V4)$ *is replaced by the following condition:*

$(V4)'$ *For any* $t \in [a,b)$ *and* $x, y \in \Omega(t)$, $\|x-y\| \le V(t,x,y)$.

Theorem 1 consists of the uniqueness and local existence of solutions to initial value problems (IVP; τ, z) and the global existence theorem as well as the continuous dependence of solutions on initial data. They are discussed in Sections 2 and 3 respectively. The proof of Theorem 2 is given in Section 4. An application of our results to the initial value problem for quasi-linear wave equations is given in Section 5.

2. Uniqueness and Local Existence of Solutions

In this section, we construct the solutions to the initial value problem (IVP; τ, z). We assume that conditions $(\Omega 1)$–$(\Omega 4)$. The following proposition ensures the uniqueness of solutions.

PROPOSITION 1. *Let* $[\tau, c) \subset [a,b)$ *and* $z_i \in \Omega(\tau)$ *for* $i = 1, 2$. *Let* u_i *be solutions to* (IVP; τ, z_i) *on* $[\tau, c)$, *for* $i = 1, 2$, *respectively. Then*

$$V\big(t, u_1(t), u_2(t)\big) \le \exp\Big(\int_\tau^t \omega(s)ds\Big) V(\tau, z_1, z_2)$$

for $t \in [\tau, c)$. *In particular, if* $z_1 = z_2$, *then* $u_1(t) = u_2(t)$ *for* $t \in [\tau, c)$.

PROOF. Set $w(t) = V\big(t, u_1(t), u_2(t)\big)$ for $t \in [\tau, c)$. From $(V3)$ we see that w is lower semi-continuous on $[\tau, c)$. Let $t \in [\tau, c)$ and $h \in (0, c-t)$. From $(V1)$ it follows that

$$\begin{aligned}
&(w(t+h) - w(t))/h - \Big(V(t+h, u_1(t) + hA(t, u_1(t)), u_2(t) + hA(t, u_2(t)))\\
&\quad - V(t, u_1(t), u_2(t))\Big)/h \le \big|V(t+h, u_1(t+h), u_2(t+h))\\
&\quad - V(t+h, u_1(t) + hA(t, u_1(t)), u_2(t) + hA(t, u_2(t)))\big|/h\\
&\le L\big(\|u_1(t+h) - u_1(t) - hA\big(t, u_1(t)\big)\|/h\\
&\quad + \|u_2(t+h) - u_2(t) - hA\big(t, u_2(t)\big)\|/h\big).
\end{aligned}$$

Taking the inferior limit as $h \downarrow 0$ yields

$$\liminf_{h\downarrow 0}(w(t+h)-w(t))/h \le D_+V(t,u_1(t),u_2(t))(A(t,u_1(t)),A(t,u_2(t))).$$

From (Ω4) we have $D_+w(t) \le \omega(t)w(t)$, where $D_+w(t)$ denotes the lower right derivative of $w(t)$. Therefore, we see that the function

$$t \to \exp\bigl(-\int_\tau^t \omega(s)\,ds\bigr)w(t)$$

is lower semicontinuous on $[\tau, c)$ and $D_+\Bigl(\exp\bigl(-\int_\tau^t \omega(s)\,ds\bigr)w(t)\Bigr) \le 0$ for $t \in [\tau, c)$. By [3, Lemma 6.3], we have $w(t) \le \exp\Bigl(\int_\tau^t \omega(s)ds\Bigr)w(\tau)$ for $t \in [\tau, c)$. Refer to [9] or [21] for the same kind of differential inequalities. □

For each $(t,x) \in \mathbb{R} \times X$ and $r > 0$, we define $S_r(t,x) = \{(s,y) \in \mathbb{R} \times X\,;\, |s-t| < r, \|y-x\| < r\}$. We need the following lemmas which are proved in [7] without using condition (Ω4).

Lemma 1 ([7, Lemma 1]). *Let $(t,x) \in \Omega$ and $\eta > 0$. Let $r > 0$ be a number such that $\|A(s,y) - A(t,x)\| \le \eta$ for $(s,y) \in \Omega \cap S_r(t,x)$. Let $M > 0$ be a number such that $\|A(s,y)\| \le M$ for $(s,y) \in \Omega \cap S_r(t,x)$. Set $h_0 = \min\{r, r/M, b-t\}$. Then*

$$d\bigl(x + hA(t,x), \Omega(t+h)\bigr) \le h\eta \quad \text{for } h \in (0, h_0).$$

Lemma 2 ([7, Lemma 2]). *Let $(t,x) \in \Omega$ and $\varepsilon \in (0,1)$. Let $r > 0$ and $M > 0$ be numbers such that $t + r < b$ and such that $\|A(s,y) - A(t,x)\| \le \varepsilon/3$ and $\|A(s,y)\| \le M$ for $(s,y) \in \Omega \cap S_r(t,x)$. Let $h \in (0, r/(M+1)]$. Let $\{s_k\}_{k=0}^n$ be a partition of $[t, t+h]$: $t = s_0 < s_1 < \cdots < s_n = t+h$. Then there exists a sequence $\{y_k\}_{k=0}^n$ of elements in X such that*

(i) *$y_0 = x$ and $(s_k, y_k) \in \Omega$ for $0 \le k \le n$;*
(ii) *$\|y_k - x\| \le (M+\varepsilon)(s_k - t)$ for $0 \le k \le n$;*
(iii) *$\|y_{k-1} + (s_k - s_{k-1})A(s_{k-1}, y_{k-1}) - y_k\| \le \varepsilon(s_k - s_{k-1})$ for $1 \le k \le n$.*

We also need the following lemma.

Lemma 3. *Let $(t,x) \in \Omega$ and $\varepsilon \in (0,1)$. Let $r > 0$ and $M > 0$ be numbers such that $t + r < b$ and $\|A(s,y)\| \le M$ for $(s,y) \in \Omega \cap S_r(t,x)$. Let $\sigma \in (0, r/(M+1)]$. Then the following assertions hold true:*

(i) *If a sequence $\{(s_i, y_i)\}_{i=0}^n$ in Ω satisfies*

$$t = s_0 < s_1 < \cdots < s_n \le t + \sigma, \tag{2.1}$$

$$\|y_{i-1} + (s_i - s_{i-1})A(s_{i-1}, y_{i-1}) - y_i\| \le \varepsilon(s_i - s_{i-1}) \quad \text{for } 1 \le i \le n, \text{ where } y_0 = x, \tag{2.2}$$

then

$$\|y_i - y_j\| \le (M+\varepsilon)(s_i - s_j) \quad \text{for } 0 \le j \le i \le n,$$
$$\|A(s_i, y_i)\| \le M \quad \text{for } 0 \le i \le n.$$

Moreover, if $\eta > 0$ *and* $\|A(s,y) - A(t,x)\| \le \eta$ *for* $(s,y) \in \Omega \cap S_r(t,x)$*, then*

$$\|x + (s_n - t)A(t,x) - y_n\| \le (\varepsilon + \eta)(s_n - t). \tag{2.3}$$

(ii) *Let* $\eta > 0$ *and* $\|A(s,y) - A(t,x)\| \le \eta$ *for* $(s,y) \in \Omega \cap S_r(t,x)$*. If a sequence* $\{(s_i, y_i)\}_{i=0}^{\infty}$ *in* Ω *satisfies*

$$t = s_0 < s_1 < \cdots < s_i < \cdots < t + \sigma \quad \text{and} \quad \lim_{i\to\infty} s_i = t + \sigma, \tag{2.4}$$

$$\|y_{i-1} + (s_i - s_{i-1})A(s_{i-1}, y_{i-1}) - y_i\| \le \varepsilon(s_i - s_{i-1}) \quad \text{for } i \ge 1, \text{ where } y_0 = x, \tag{2.5}$$

then $\hat{y} = \lim_{i\to\infty} y_i$ *exists in* X*,* $\hat{y} \in \Omega(t+\sigma)$ *and*

$$\|x + \sigma A(t,x) - \hat{y}\| \le (\varepsilon + \eta)\sigma. \tag{2.6}$$

PROOF. To prove (i), let $\{(s_i, y_i)\}_{i=0}^{n}$ be a sequence in Ω satisfying (2.1) and (2.2). We first show inductively that $(s_i, y_i) \in S_r(t,x)$ for $0 \le i \le n$. It is obvious that $(s_0, y_0) \in S_r(t,x)$. Let k be a nonnegative integer such that $k < n$ and assume that $(s_i, y_i) \in S_r(t,x)$ for $0 \le i \le k$. From (2.2) we obtain

$$\|y_{i-1} - y_i\| \le (s_i - s_{i-1})\|A(s_{i-1}, y_{i-1})\| + \varepsilon(s_i - s_{i-1})$$

for $1 \le i \le n$. Since $\|A(s_i, x_i)\| \le M$ for $0 \le i \le k$ by assumption, we have

$$\|y_i - y_{i-1}\| \le (M+\varepsilon)(s_i - s_{i-1})$$

for $1 \le i \le k+1$. Summing up this inequality from $i = 1$ to $i = k+1$, we find that

$$\|y_{k+1} - x\| \le (M+\varepsilon)(s_{k+1} - t) < (M+1)\sigma \le r.$$

It is obvious that $s_{k+1} - t \le \sigma < \sigma(M+1) \le r$. These mean that $(s_{k+1}, y_{k+1}) \in S_r(t,x)$. Thus, we inductively prove that $(s_i, y_i) \in S_r(t,x)$ for $0 \le i \le n$.

Since $(s_k, y_k) \in S_r(t,x)$ for $0 \le k \le n$, we have $\|A(s_k, y_k)\| \le M$ for $0 \le k \le n$ and $\|y_k - y_{k-1}\| \le (M+\varepsilon)(s_k - s_{k-1})$ for $1 \le k \le n$. Therefore, we find that

$$\|y_i - y_j\| \le (M+\varepsilon)(s_i - s_j)$$

for $0 \le j \le i \le n$. To prove (2.3), let $\eta > 0$ and assume that $\|A(s,y) - A(t,x)\| \le \eta$ for $(s,y) \in \Omega \cap S_r(t,x)$. Since $\{(s_i, y_i)\,;\, 0 \le$

$i \le n\} \subset \Omega \cap S_r(t,x)$, we have $\|A(s_i,y_i) - A(t,x)\| \le \eta$ for $0 \le i \le n$. From (2.2) we see that

$$\begin{aligned}
&\|y_{i-1} + (s_i - s_{i-1})A(t,x) - y_i\| \\
&\quad \le \|y_{i-1} + (s_i - s_{i-1})A(s_{i-1},y_{i-1}) - y_i\| \\
&\qquad + \|(s_i - s_{i-1})\big(A(t,x) - A(s_{i-1},y_{i-1})\big)\| \\
&\quad \le \varepsilon(s_i - s_{i-1}) + \eta(s_i - s_{i-1}) = (\varepsilon + \eta)(s_i - s_{i-1})
\end{aligned}$$

for $1 \le i \le n$. Hence

$$\begin{aligned}
\|x + (s_n - t)A(t,x) - y_n\| &\le \sum_{i=1}^{n} \|y_{i-1} + (s_i - s_{i-1})A(t,x) - y_i\| \\
&\le (\varepsilon + \eta)(s_n - t).
\end{aligned}$$

To prove (ii), let $\{(s_i,y_i)\}_{i=0}^{\infty}$ be a sequence in Ω satisfying (2.4) and (2.5). From (i) we obtain $\|y_i - y_j\| \le (M+\varepsilon)(s_i - s_j)$ for $0 \le j \le i$. This implies that $\hat{y} = \lim_{i\to\infty} y_i$ exists in X and is in $\Omega(t+\sigma)$ by (Ω2). By (i) again, we note that the inequality (2.3) holds for $n \ge 0$. Passing to the limit in (2.3) as $n \to \infty$, we obtain

$$\begin{aligned}
\|x + \sigma A(t,x) - \hat{y}\| &= \lim_{n\to\infty} \|x + (s_n - t)A(t,x) - y_n\| \\
&\le \lim_{n\to\infty} (\varepsilon + \eta)(s_n - t) = (\varepsilon + \eta)\sigma,
\end{aligned}$$

namely, the desired inequality (2.6) is proved. □

The local existence of approximation solutions to (IVP; τ, z) is given by the following proposition, which is essentially shown in [7] and [4]. We give the proof for completeness.

Proposition 2. *Let $(t,x) \in \Omega$ and $\varepsilon \in (0,1)$. Let $r > 0$ and $M > 0$ be numbers such that $t + r < b$ and $\|A(s,y)\| \le M$ for $(s,y) \in \Omega \cap S_r(t,x)$. Let $\sigma \in (0, r/(M+1)]$. Then there exists a sequence $\{(s_i,y_i)\}_{i=0}^{\infty}$ in Ω such that*

(i) *$t = s_0 < s_1 < \cdots < s_i < \cdots < t + \sigma$ and $\lim_{i\to\infty} s_i = t + \sigma$;*
(ii) *$s_i - s_{i-1} \le \varepsilon$ for $i \ge 1$;*
(iii) *$\|y_{i-1} + (s_i - s_{i-1})A(s_{i-1},y_{i-1}) - y_i\| \le \varepsilon(s_i - s_{i-1})/2$ for $i \ge 1$, where $y_0 = x$;*
(iv) *if $(s,y) \in \Omega \cap S_{(M+1)(s_i - s_{i-1})}(s_{i-1},y_{i-1})$, then*

$$\|A(s,y) - A(s_{i-1},y_{i-1})\| \le \varepsilon/4 \text{ for } i \ge 1.$$

Proof. Set $(s_0,y_0) = (t,x)$. Let k be a positive integer and assume that there exists a sequence $\{(s_i,y_i)\}_{i=0}^{k-1}$ in Ω which satisfies the first half of (i) and (ii)–(iv) for $1 \le i \le k-1$. We consider a nonnegative number $\hat{h}_k$ defined by the supremum of $h \in [0,\varepsilon]$ such that $h < t + \sigma - s_{k-1}$ and

$$\|A(s,y) - A(s_{k-1},y_{k-1})\| \le \varepsilon/4 \ \text{ for } (s,y) \in \Omega \cap S_{h(M+1)}(s_{k-1},y_{k-1}).$$

By the continuity of A, we have $\hat{h}_k > 0$. Thus there exists a number $h_k \in (0, \varepsilon]$ such that $\hat{h}_k/2 < h_k < t + \sigma - s_{k-1}$ and

$$\|A(s, y) - A(s_{k-1}, y_{k-1})\| \le \varepsilon/4 \quad \text{for } (s, y) \in \Omega \cap S_{r_k}(s_{k-1}, y_{k-1}), \tag{2.7}$$

where $r_k = h_k(M + 1)$. Set $s_k = s_{k-1} + h_k$. Then $s_{k-1} < s_k < t + \sigma$ and conditions (ii) and (iv) with $i = k$ are satisfied. By Lemma 3, $\|A(s_i, y_i)\| \le M$ for $0 \le i \le k - 1$. The inequality (2.7) implies that $\|A(s, y)\| \le M + \varepsilon/4$ for $(s, y) \in \Omega \cap S_{r_k}(s_{k-1}, y_{k-1})$. Hence, Lemma 1, with $(t, x), r, M$ and η replaced by $(s_{k-1}, y_{k-1}), r_k, M + \varepsilon/4$ and $\varepsilon/4$ respectively, implies that

$$d\big(y_{k-1} + h_k A(s_{k-1}, y_{k-1}), \Omega(s_k)\big) \le \varepsilon h_k/4.$$

Thus there exists an element $y_k \in \Omega(s_k)$ satisfying (iii) with $i = k$.

We shall show that $\lim_{i\to\infty} s_i = t + \sigma$. Assume to the contrary that $\hat{s} = \lim_{i\to\infty} s_i < t + \sigma$. By Lemma 3 (i) we obtain $\|y_i - y_j\| \le (M + \varepsilon/2)(s_i - s_j)$ for $0 \le j \le i$. Hence, $\lim_{i\to\infty} y_i$ exists in X, and we denote its limit by $\hat{y}$. Since $(\hat{s}, \hat{y}) = \lim_{i\to\infty}(s_i, y_i)$ in $\mathbb{R} \times X$ and $(s_i, y_i) \in \Omega$ for $i \ge 1$, we have $(\hat{s}, \hat{y}) \in \Omega$ by (Ω2). The continuity of A enables us to choose $\eta \in (0, \varepsilon]$ such that

$$\eta \le t + \sigma - \hat{s} \quad \text{and} \quad \|A(s, y) - A(\hat{s}, \hat{y})\| \le \varepsilon/8 \quad \text{for } (s, y) \in \Omega \cap S_{\hat{r}}(\hat{s}, \hat{y}),$$

where $\hat{r} = 2(M + 1)\eta$. Choose an integer $i_0 \ge 1$ so that $\hat{s} - s_{i-1} \le \eta$ and $\|\hat{y} - y_{i-1}\| \le (M + 1)\eta$ for $i \ge i_0$. Then, for $i \ge i_0$ and $(s, y) \in S_{(M+1)\eta}(s_{i-1}, y_{i-1})$, we have

$$|s - \hat{s}| \le |s - s_{i-1}| + |s_{i-1} - \hat{s}| < (M + 1)\eta + \eta \le 2(M + 1)\eta,$$
$$\|y - \hat{y}\| \le \|y - y_{i-1}\| + \|y_{i-1} - \hat{y}\| < 2(M + 1)\eta.$$

Hence $S_{(M+1)\eta}(s_{i-1}, y_{i-1}) \subset S_{\hat{r}}(\hat{s}, \hat{y})$ for $i \ge i_0$. By the choice of η, we see that if $i \ge i_0$, then

$$\begin{aligned}\|A(s, y) - A(s_{i-1}, y_{i-1})\| &\le \|A(s, y) - A(\hat{s}, \hat{y})\| + \|A(\hat{s}, \hat{y}) - A(s_{i-1}, y_{i-1})\| \\ &\le \varepsilon/8 + \varepsilon/8 = \varepsilon/4\end{aligned}$$

for $(s, y) \in \Omega \cap S_{(M+1)\eta}(s_{i-1}, y_{i-1})$. Since $\eta < t + \sigma - s_{i-1}$ for $i \ge 1$, the definition of $\hat{h}_i$ implies that $\eta \le \hat{h}_i < 2h_i = 2(s_i - s_{i-1})$ for $i \ge i_0$ and the right-hand side tends to zero as $i \to \infty$. This contradicts the fact that η is positive. □

In what follows, we write $\overline{\omega}([\hat{a}, \hat{b}]) = \sup_{s\in[\hat{a},\hat{b}]} \omega(s)$ for $[\hat{a}, \hat{b}] \subset [a, b)$. To prove the convergence of the approximate solutions, we need the following Propositions, which are the refinements of the results in [11], [10], [6] and [8].

PROPOSITION 3. *Let $t \in [a,b)$, $(x,\hat{x}) \in \Omega(t) \times \Omega(t)$ and $\eta, \hat{\eta} \in (0,1)$. Let $r > 0$ and $M > 0$ be numbers such that $t + r < b$,*

$$\|A(s,z)\| \le M \quad \text{and} \quad \|A(s,z) - A(t,x)\| \le \eta/4 \quad \text{for } (s,z) \in \Omega \cap S_r(t,x),$$
$$\|A(s,\hat{z})\| \le M \quad \text{and} \quad \|A(s,\hat{z}) - A(t,\hat{x})\| \le \hat{\eta}/4 \quad \text{for } (s,\hat{z}) \in \Omega \cap S_r(t,\hat{x}).$$

Let $\sigma \in (0, r/(M+1)]$. Then there exists a pair $(y,\hat{y}) \in \Omega(t+\sigma) \times \Omega(t+\sigma)$ such that

$$\|x + \sigma A(t,x) - y\| \le \eta\sigma, \tag{2.8}$$
$$\|\hat{x} + \sigma A(t,\hat{x}) - \hat{y}\| \le \hat{\eta}\sigma, \tag{2.9}$$
$$V(t+\sigma, y, \hat{y}) \le \exp\big(\sigma\overline{\omega}([t,t+\sigma])\big)\big(V(t,x,\hat{x}) + L(\eta+\hat{\eta})\sigma\big). \tag{2.10}$$

PROOF. We shall show that there exist two sequences $\{(s_j,z_j)\}_{j=0}^{\infty}$ and $\{(s_j,\hat{z}_j)\}_{j=0}^{\infty}$ in Ω such that

$$t = s_0 < s_1 < \cdots < s_j < \cdots < t+\sigma \quad \text{and} \quad \lim_{j\to\infty} s_j = t+\sigma, \tag{2.11}$$
$$\|z_{j-1} + (s_j - s_{j-1})A(s_{j-1},z_{j-1}) - z_j\| \le 3\eta(s_j - s_{j-1})/4 \quad \text{for } j \ge 1, \text{ where } z_0 = x, \tag{2.12}$$
$$\|\hat{z}_{j-1} + (s_j - s_{j-1})A(s_{j-1},\hat{z}_{j-1}) - \hat{z}_j\| \le 3\hat{\eta}(s_j - s_{j-1})/4 \quad \text{for } j \ge 1, \text{ where } \hat{z}_0 = \hat{x}, \tag{2.13}$$
$$(V(s_j,z_j,\hat{z}_j) - V(s_{j-1},z_{j-1},\hat{z}_{j-1}))/(s_j - s_{j-1}) \le \omega(s_{j-1})V(s_{j-1},z_{j-1},\hat{z}_{j-1}) + L(\eta+\hat{\eta}) \quad \text{for } j \ge 1. \tag{2.14}$$

Set $(s_0,z_0,\hat{z}_0) = (t,x,\hat{x})$ and assume that sequences $\{(s_j,z_j)\}_{j=0}^{i-1}$ and $\{(s_j,\hat{z}_j)\}_{j=0}^{i-1}$ in Ω with $i \ge 1$ satisfy the first half of (2.11) and (2.12)–(2.14) for $1 \le j \le i-1$. Then we need to show that there exist $s_i \in \mathbb{R}$, $z_i \in \Omega(s_i)$ and $\hat{z}_i \in \Omega(s_i)$ such that $s_{i-1} < s_i < t+\sigma$ and (2.12)–(2.14) with $j = i$ are satisfied. Let $\hat{h}_i$ denote the supremum of all $h \ge 0$ such that $h < t + \sigma - s_{i-1}$ and

$$V(s_{i-1}+h, z_{i-1} + hA(s_{i-1},z_{i-1}), \hat{z}_{i-1} + hA(s_{i-1},\hat{z}_{i-1})) - V(s_{i-1},z_{i-1},\hat{z}_{i-1}) \le h\big(\omega(s_{i-1})V(s_{i-1},z_{i-1},\hat{z}_{i-1}) + (\eta+\hat{\eta})L/4\big).$$

Since $\hat{h}_i > 0$ by (Ω4), there exists a number $h_i > 0$ such that $\hat{h}_i/2 < h_i < t + \sigma - s_{i-1}$ and

$$V(s_{i-1}+h, z_{i-1} + hA(s_{i-1},z_{i-1}), \hat{z}_{i-1} + hA(s_{i-1},\hat{z}_{i-1})) - V(s_{i-1},z_{i-1},\hat{z}_{i-1}) \le h\big(\omega(s_{i-1})V(s_{i-1},z_{i-1},\hat{z}_{i-1}) + (\eta+\hat{\eta})L/4\big). \tag{2.15}$$

Set $s_i = s_{i-1} + h_i$. It is obvious that $s_{i-1} < s_i < t + \sigma$. To prove that $S_{(M+1)h_i}(s_{i-1},z_{i-1}) \subset S_r(t,x)$, we note by Lemma 3 (i) with

$\varepsilon = 3\eta/4$ that

$$\|z_{i-1} - x\| \le (M + 3\eta/4)(s_{i-1} - t) < (M+1)(s_{i-1} - t).$$

If $(s, z) \in S_{(M+1)h_i}(s_{i-1}, z_{i-1})$, then

$$\begin{aligned}|s - t| &\le |s - s_{i-1}| + |s_{i-1} - t| < (M+1)(h_i + s_{i-1} - t)\\ &= (M+1)(s_i - t) \le (M+1)\sigma \le r\end{aligned}$$

and

$$\|z - x\| \le \|z - z_{i-1}\| + \|z_{i-1} - x\| < (M+1)(h_i + s_{i-1} - t) \le r.$$

This means that $S_{(M+1)h_i}(s_{i-1}, z_{i-1}) \subset S_r(t, x)$. By assumption, we have

$$\|A(s,z)\| \le M \quad \text{and} \quad \|A(s,z) - A(t,x)\| \le \eta/4 \tag{2.16}$$

for $(s, z) \in \Omega \cap S_{(M+1)h_i}(s_{i-1}, z_{i-1})$. From the second inequality of (2.16), we see that if $(s, z) \in \Omega \cap S_{(M+1)h_i}(s_{i-1}, z_{i-1})$, then

$$\begin{aligned}&\|A(s,z) - A(s_{i-1}, z_{i-1})\| \le \|A(s,z) - A(t,x)\| + \|A(s_{i-1}, z_{i-1}) - A(t,x)\|\\ &\le \eta/4 + \eta/4 = \eta/2.\end{aligned}$$

Hence, by Lemma 1 with $r = (M+1)h_i$, $(t, x) = (s_{i-1}, z_{i-1})$ and $h = h_i$, we find that

$$d(z_{i-1} + h_i A(s_{i-1}, z_{i-1}), \Omega(s_i)) \le h_i\eta/2 = \eta(s_i - s_{i-1})/2.$$

This implies that there exists $z_i \in \Omega(s_i)$ such that (2.12) holds true for $j = i$. Similarly, we can show that there exists $\hat{z}_i \in \Omega(s_i)$ satisfying (2.13) with $j = i$.

By $(V1)$ we obtain (2.14) with $j = i$ by the inequality (2.15) combined with (2.12) and (2.13) with $j = i$. Indeed, we have

$$\begin{aligned}&\big(V(s_i, z_i, \hat{z}_i) - V(s_{i-1}, z_{i-1}, \hat{z}_{i-1})\big)/h_i\\ &= \big(V(s_i, z_i, \hat{z}_i) - V(s_i, z_{i-1} + h_i A(s_{i-1}, z_{i-1}), \hat{z}_{i-1} + h_i A(s_{i-1}, \hat{z}_{i-1}))\big)/h_i\\ &\quad + \big(V(s_i, z_{i-1} + h_i A(s_{i-1}, z_{i-1}), \hat{z}_{i-1} + h_i A(s_{i-1}, \hat{z}_{i-1}))\\ &\quad - V(s_{i-1}, z_{i-1}, \hat{z}_{i-1})\big)/h_i\\ &\le L(\|z_i - (z_{i-1} + h_i A(s_{i-1}, z_{i-1}))\| + \|\hat{z}_i - (\hat{z}_{i-1} + h_i A(s_{i-1}, \hat{z}_{i-1}))\|)/h_i\\ &\quad + \omega(s_{i-1}) V(s_{i-1}, z_{i-1}, \hat{z}_{i-1}) + (\eta + \hat{\eta})L/4\\ &\le 3(\eta + \hat{\eta})L/4 + \omega(s_{i-1}) V(s_{i-1}, z_{i-1}, \hat{z}_{i-1}) + (\eta + \hat{\eta})L/4\\ &\le \omega(s_{i-1}) V(s_{i-1}, z_{i-1}, \hat{z}_{i-1}) + L(\eta + \hat{\eta}).\end{aligned}$$

It remains to prove the second half of (2.11). Assume to the contrary that $s_\infty = \lim_{j\to\infty} s_j < t + \sigma$. Lemma 3 (i) asserts that $\{z_j\}$ and $\{\hat{z}_j\}$ are Cauchy sequences in X, since

$$\limsup_{i,j\to\infty} \|z_i - z_j\| \le \limsup_{i,j\to\infty} (M + 3\eta/4)(s_i - s_j) = 0,$$

$$\limsup_{i,j\to\infty} \|\hat{z}_i - \hat{z}_j\| \le \limsup_{i,j\to\infty} (M + 3\hat{\eta}/4)(s_i - s_j) = 0.$$

This implies that $z_\infty = \lim_{j\to\infty} z_j$ and $\hat{z}_\infty = \lim_{j\to\infty} \hat{z}_j$ exist in X and are in $\Omega(s_\infty)$ by $(\Omega 2)$. By $(\Omega 4)$, we choose a number $h > 0$ so that $h < t + \sigma - s_\infty$ and

$$\{V(s_\infty + h, z_\infty + hA(s_\infty, z_\infty), \hat{z}_\infty + hA(s_\infty, \hat{z}_\infty)) - V(s_\infty, z_\infty, \hat{z}_\infty)\}/h \\ \leq \omega(s_\infty)V(s_\infty, z_\infty, \hat{z}_\infty) + (\eta + \hat{\eta})L/8. \tag{2.17}$$

Let $r_j = s_\infty + h - s_{j-1}$ for $j \geq 1$. Then we have $r_j < t + \sigma - s_{j-1}$ for $j \geq 1$ and $r_j \to h$ as $j \to \infty$. Since $\hat{h}_j < 2h_j = 2(s_j - s_{j-1}) \to 0$ as $j \to \infty$, there exists an integer $j_0 \geq 1$ such that $\hat{h}_j < r_j$ for $j \geq j_0$. By the definition of $\hat{h}_j$, we have

$$\{V(s_{j-1} + r_j, z_{j-1} + r_jA(s_{j-1}, z_{j-1}), \hat{z}_{j-1} + r_jA(s_{j-1}, \hat{z}_{j-1})) \\ - V(s_{j-1}, z_{j-1}, \hat{z}_{j-1})\}/r_j > \omega(s_{j-1})V(s_{j-1}, z_{j-1}, \hat{z}_{j-1}) + (\eta + \hat{\eta})L/4$$

for $j \geq j_0$. Since $s_{j-1} \to s_\infty$, $z_{j-1} \to z_\infty$, $\hat{z}_{j-1} \to \hat{z}_\infty$ and $r_j \to h$ as $j \to \infty$ and $s_{j-1} + r_j = s_\infty + h$ for $j \geq 1$, from $(V1)$ and $(V3)$ we obtain

$$\{V(s_\infty + h, z_\infty + hA(s_\infty, z_\infty), \hat{z}_\infty + hA(s_\infty, \hat{z}_\infty)) - V(s_\infty, z_\infty, \hat{z}_\infty)\}/h \\ \geq \omega(s_\infty)V(s_\infty, z_\infty, \hat{z}_\infty) + (\eta + \hat{\eta})L/4,$$

which contradicts to (2.17).

We now turn to the proof of the existence of pair $(y, \hat{y}) \in \Omega(t) \times \Omega(t)$ satisfying (2.8)–(2.10). We apply Lemma 3 (ii) to show that $y = \lim_{j\to\infty} z_j$ and $\hat{y} = \lim_{j\to\infty} \hat{z}_j$ exist in X and are in $\Omega(t+\sigma)$ and that they satisfy (2.8) and (2.9), that is,

$$\|x + \sigma A(t, x) - y\| \leq (3\eta/4 + \eta/4)\sigma \leq \eta\sigma,$$
$$\|\hat{x} + \sigma A(t, \hat{x}) - \hat{y}\| \leq (3\hat{\eta}/4 + \hat{\eta}/4)\sigma \leq \hat{\eta}\sigma.$$

We note here that $1 + t \leq \mathrm{e}^t$ for $t \geq 0$. We deduce from (2.14) that

$$V(s_j, z_j, \hat{z}_j) \leq \exp\bigl(h_j\overline{\omega}([t, t + \sigma])\bigr)\bigl(V(s_{j-1}, z_{j-1}, \hat{z}_{j-1}) + h_jL(\eta + \hat{\eta})\bigr)$$

for $j \geq 1$. Hence, we inductively show that

$$V(s_j, z_j, \hat{z}_j) \leq \exp\bigl((s_j - t)\overline{\omega}([t, t + \sigma])\bigr)\bigl(V(t, x, \hat{x}) + L(\eta + \hat{\eta})(s_j - t)\bigr)$$

for $j \geq 0$. Thus we obtain (2.10) by letting $j \to \infty$. □

Proposition 4. *Let $(\tau, z) \in \Omega$ and $\lambda, \mu \in (0, 1/2)$. Let $R > 0$ and $M > 0$ be numbers such that $\tau + R < b$ and $\|A(s, y)\| \leq M$ for $(s, y) \in \Omega \cap S_R(\tau, z)$. Let $\sigma \in (0, R/(M+1)]$. For each $\varepsilon \in \{\lambda, \mu\}$, let $\{(t_i^\varepsilon, x_i^\varepsilon)\}_{i=0}^\infty$ be a sequence in Ω satisfying the following conditions:*

(i) $\tau = t_0^\varepsilon < t_1^\varepsilon < \cdots < t_i^\varepsilon < \cdots < \tau + \sigma$ *and* $\lim_{i\to\infty} t_i^\varepsilon = \tau + \sigma$;

(ii) $t_i^\varepsilon - t_{i-1}^\varepsilon \leq \varepsilon$ *for* $i \geq 1$;

(iii) $\|x_{i-1}^\varepsilon + (t_i^\varepsilon - t_{i-1}^\varepsilon)A(t_{i-1}^\varepsilon, x_{i-1}^\varepsilon) - x_i^\varepsilon\| \leq \varepsilon(t_i^\varepsilon - t_{i-1}^\varepsilon)/2$ *for* $i \geq 1$, *where* $x_0^\varepsilon = z$;

(iv) *if* $(s,y) \in \Omega \cap S_{(M+1)(t_i^\varepsilon - t_{i-1}^\varepsilon)}(t_{i-1}^\varepsilon, x_{i-1}^\varepsilon)$, *then*

$$\|A(s,y) - A(t_{i-1}^\varepsilon, x_{i-1}^\varepsilon)\| \le \varepsilon/4 \quad \text{for } i \ge 1.$$

Let $\{s_k\}_{k=0}^\infty$ *be a sequence such that* $s_k < s_{k+1}$ *for* $k \ge 0$ *and*

$$\{s_k\,;\,k = 0,1,2,\ldots\} = \{t_i^\lambda\,;\,i = 0,1,2,\ldots\} \cup \{t_j^\mu\,;\,j = 0,1,2,\ldots\}.$$

Then there exists a sequence $\{(z_k^\lambda, z_k^\mu)\}_{k=0}^\infty$ *in* $X \times X$ *such that* $(z_k^\lambda, z_k^\mu) \in \Omega(s_k) \times \Omega(s_k)$ *for each* $k \ge 0$ *and the following three properties are satisfied:*

(a) *if* $s_k = t_i^\lambda$, *then* $z_k^\lambda = x_i^\lambda$; *if* $s_k = t_j^\mu$, *then* $z_k^\mu = x_j^\mu$;

(b) *for each* $\varepsilon = \lambda, \mu$, *we have*

$$\sum_{j=q}^{k} \|z_{j-1}^\varepsilon + (s_j - s_{j-1})A(s_{j-1}, z_{j-1}^\varepsilon) - z_j^\varepsilon\|$$
$$\le 2\varepsilon(s_k - s_{q-1}) + 3\varepsilon \sum_{t_i^\varepsilon \in \{s_q,\ldots,s_k\}} (t_i^\varepsilon - t_{i-1}^\varepsilon)$$

for $1 \le q \le k$ *and* $k \ge 1$;

(c) *for* $k \ge 0$,

$$V(s_k, z_k^\lambda, z_k^\mu) \le \exp\big((s_k - \tau)\overline{\omega}([\tau, s_k])\big)\big\{2L(\lambda+\mu)(s_k - \tau) + \eta_k(\lambda,\mu)\big\},$$

where

$$\eta_k(\lambda,\mu) = 3L\Big(\lambda \sum_{t_i^\lambda \in \{s_1,\ldots,s_k\}} (t_i^\lambda - t_{i-1}^\lambda) + \mu \sum_{t_j^\mu \in \{s_1,\ldots,s_k\}} (t_j^\mu - t_{j-1}^\mu)\Big).$$

PROOF. Set $z_0^\varepsilon = z$ for each $\varepsilon = \lambda, \mu$. Assume that sequences $\{(s_k, z_k^\lambda)\}_{k=0}^{l-1}$ and $\{(s_k, z_k^\mu)\}_{k=0}^{l-1}$ in Ω with $l \ge 1$ satisfy properties (a)–(c) for $0 \le k \le l-1$. Let i and j be positive integers such that $t_{i-1}^\lambda < s_l \le t_i^\lambda$ and $t_{j-1}^\mu < s_l \le t_j^\mu$, respectively. By Lemma 3 (i) with $\varepsilon = \lambda/2$ we obtain $\|x_{i-1}^\lambda - z\| \le (M + \lambda/2)(t_{i-1}^\lambda - \tau)$. If $(s,y) \in S_{(M+1)(t_i^\lambda - t_{i-1}^\lambda)}(t_{i-1}^\lambda, x_{i-1}^\lambda)$, then we get

$$|s - \tau| \le |s - t_{i-1}^\lambda| + |t_{i-1}^\lambda - \tau| < (M+1)(t_i^\lambda - t_{i-1}^\lambda) + (t_{i-1}^\lambda - \tau)$$
$$\le (M+1)\sigma \le R$$

and

$$\|y - z\| \le \|y - x_{i-1}^\lambda\| + \|x_{i-1}^\lambda - z\|$$
$$< (M+1)(t_i^\lambda - t_{i-1}^\lambda) + (M + \lambda/2)(t_{i-1}^\lambda - \tau) < (M+1)\sigma \le R.$$

Hence $S_{(M+1)(t_i^\lambda - t_{i-1}^\lambda)}(t_{i-1}^\lambda, x_{i-1}^\lambda) \subset S_R(\tau, z)$. This implies that

$$\|A(s,y)\| \le M \quad \text{for } (s,y) \in \Omega \cap S_{(M+1)(t_i^\lambda - t_{i-1}^\lambda)}(t_{i-1}^\lambda, x_{i-1}^\lambda). \tag{2.18}$$

We shall show that for each $\varepsilon = \lambda, \mu$,

$$\|A(s,y)\| \le M \quad \text{and} \quad \|A(s,y) - A(s_{l-1}, z_{l-1}^\varepsilon)\| \le \varepsilon/2 \tag{2.19}$$

for $(s,y) \in \Omega \cap S_{(M+1)(s_l - s_{l-1})}(s_{l-1}, z^{\varepsilon}_{l-1})$. By the definition of $\{s_k\}$ we observe that

$$t^{\lambda}_{i-1} \le s_{l-1} < s_l \le t^{\lambda}_i, \quad t^{\mu}_{j-1} \le s_{l-1} < s_l \le t^{\mu}_j,$$

$$t^{\lambda}_{i-1} = s_p \text{ for some } 0 \le p \le l-1, \text{ and } t^{\mu}_{j-1} = s_q \text{ for some } 0 \le q \le l-1.$$

By the hypothesis (a) of induction, we have $z^{\lambda}_p = x^{\lambda}_{i-1}$ and $z^{\mu}_q = x^{\mu}_{j-1}$. If $0 \le p < l-1$, then the set $\{s_{p+1}, \dots, s_{l-1}\}$ contains no points t^{λ}_i. By the hypothesis (b) of induction, we have

$$\|z^{\lambda}_{k-1} + (s_k - s_{k-1})A(s_{k-1}, z^{\lambda}_{k-1}) - z^{\lambda}_k\| \le 2\lambda(s_k - s_{k-1}) \qquad (2.20)$$

for $k = p+1, \dots, l-1$. By (2.18) and (2.20), we use Lemma 3 (i) with $(t,x) = (t^{\lambda}_{i-1}, x^{\lambda}_{i-1}) = (s_p, z^{\lambda}_p), \varepsilon = 2\lambda$ and $r = (M+1)(t^{\lambda}_i - t^{\lambda}_{i-1})$ to obtain $\|z^{\lambda}_{l-1} - z^{\lambda}_p\| \le (M+2\lambda)(s_{l-1} - s_p)$. This is valid for $p = l-1$. If $(s,y) \in S_{(M+1)(s_l - s_{l-1})}(s_{l-1}, z^{\lambda}_{l-1})$, then we get

$$\begin{aligned} |s - t^{\lambda}_{i-1}| &\le |s - s_{l-1}| + |s_{l-1} - t^{\lambda}_{i-1}| \\ &< (M+1)(s_l - s_{l-1}) + (s_{l-1} - t^{\lambda}_{i-1}) \le (M+1)(t^{\lambda}_i - t^{\lambda}_{i-1}), \\ \|y - x^{\lambda}_{i-1}\| &\le \|y - z^{\lambda}_{l-1}\| + \|z^{\lambda}_{l-1} - x^{\lambda}_{i-1}\| \\ &< (M+1)(s_l - s_{l-1}) + (M+2\lambda)(s_{l-1} - s_p) \le (M+1)(t^{\lambda}_i - t^{\lambda}_{i-1}). \end{aligned}$$

This means that

$$S_{(M+1)(s_l - s_{l-1})}(s_{l-1}, z^{\lambda}_{l-1}) \subset S_{(M+1)(t^{\lambda}_i - t^{\lambda}_{i-1})}(t^{\lambda}_{i-1}, x^{\lambda}_{i-1}). \qquad (2.21)$$

Thus, the claim (2.19) with $\varepsilon = \lambda$ follows from (2.18) and condition (iv). Indeed,

$$\begin{aligned} &\|A(s,y) - A(s_{l-1}, z^{\lambda}_{l-1})\| \\ &\le \|A(s,y) - A(t^{\lambda}_{i-1}, x^{\lambda}_{i-1})\| + \|A(t^{\lambda}_{i-1}, x^{\lambda}_{i-1}) - A(s_{l-1}, z^{\lambda}_{l-1})\| \\ &\le \lambda/4 + \lambda/4 = \lambda/2 \end{aligned}$$

for $(s,y) \in \Omega \cap S_{(M+1)(s_l - s_{l-1})}(s_{l-1}, z^{\lambda}_{l-1})$. We apply the above argument again, with p and i replaced by q and j, to show that (2.19) holds true for $\varepsilon = \mu$.

By virtue of (2.19), we deduce from Proposition 3 with $t = s_{l-1}$, $(x, \hat{x}) = (z^{\lambda}_{l-1}, z^{\mu}_{l-1})$, $\eta = 2\lambda$, $\hat{\eta} = 2\mu$ and $r = (M+1)(s_l - s_{l-1})$ that there exists a pair $(y^{\lambda}_l, y^{\mu}_l) \in \Omega(s_{l-1} + (s_l - s_{l-1})) \times \Omega(s_{l-1} + (s_l - s_{l-1})) = \Omega(s_l) \times \Omega(s_l)$ satisfying

$$\|z^{\varepsilon}_{l-1} + (s_l - s_{l-1})A(s_{l-1}, z^{\varepsilon}_{l-1}) - y^{\varepsilon}_l\| \le 2\varepsilon(s_l - s_{l-1}) \quad \text{for } \varepsilon = \lambda, \mu, \qquad (2.22)$$

$$\begin{aligned} V(s_l, y^{\lambda}_l, y^{\mu}_l) &\le \exp\big((s_l - s_{l-1})\overline{\omega}([s_{l-1}, s_l])\big) \\ &\quad \times \big(V(s_{l-1}, z^{\lambda}_{l-1}, z^{\mu}_{l-1}) + 2L(\lambda+\mu)(s_l - s_{l-1})\big). \end{aligned} \qquad (2.23)$$

We define $(z_l^\lambda, z_l^\mu) \in \Omega(s_l) \times \Omega(s_l)$ by

$$z_l^\lambda = \begin{cases} y_l^\lambda \text{ for } s_l < t_i^\lambda, \\ x_i^\lambda \text{ for } s_l = t_i^\lambda \end{cases} \quad \text{and} \quad z_l^\mu = \begin{cases} y_l^\mu \text{ for } s_l < t_j^\mu, \\ x_j^\mu \text{ for } s_l = t_j^\mu. \end{cases}$$

If $s_l = t_i^\lambda$, then by condition (iii) we have

$$\|x_{i-1}^\lambda + (s_l - t_{i-1}^\lambda)A(t_{i-1}^\lambda, x_{i-1}^\lambda) - z_l^\lambda\| \le (s_l - t_{i-1}^\lambda)\lambda/2,$$

while in view of (2.18) and (iv) we find, by applying Lemma 3 (i), with $\varepsilon = 2\lambda$, $\eta = \lambda/4$, $r = (M+1)(t_i^\lambda - t_{i-1}^\lambda)$ and $(t, x) = (t_{i-1}^\lambda, x_{i-1}^\lambda)$, to (2.20) and (2.22), that

$$\|x_{i-1}^\lambda + (s_l - t_{i-1}^\lambda)A(t_{i-1}^\lambda, x_{i-1}^\lambda) - y_l^\lambda\| \le (2\lambda + \lambda/4)(s_l - t_{i-1}^\lambda).$$

These inequalities together yield

$$\begin{aligned} \|z_l^\lambda - y_l^\lambda\| &\le \|x_{i-1}^\lambda + (s_l - t_{i-1}^\lambda)A(t_{i-1}^\lambda, x_{i-1}^\lambda) - y_l^\lambda\| \\ &\quad + \|x_{i-1}^\lambda + (s_l - t_{i-1}^\lambda)A(t_{i-1}^\lambda, x_{i-1}^\lambda) - z_l^\lambda\| \\ &\le (9/4 + 1/2)\lambda(s_l - t_{i-1}^\lambda) \le 3\lambda \sum_{t_i^\lambda = s_l} (t_i^\lambda - t_{i-1}^\lambda). \end{aligned} \tag{2.24}$$

Similarly, we get

$$\|z_l^\mu - y_l^\mu\| \le 3\mu \sum_{t_j^\mu = s_l} (t_j^\mu - t_{j-1}^\mu). \tag{2.25}$$

Combining (2.24) and (2.25) with (2.22), and adding the resulting inequality to the inequality (b) with $k = l - 1$, we conclude that the desired property (b) holds true for $k = l$.

Finally, we show that (c) is true for $k = l$. Using (2.24), (2.25) and $(V1)$ we have

$$\begin{aligned} |V(s_l, z_l^\lambda, z_l^\mu) - V(s_l, y_l^\lambda, y_l^\mu)| &\le L\left(\|z_l^\lambda - y_l^\lambda\| + \|z_l^\mu - y_l^\mu\|\right) \\ &\le 3L\Bigg(\lambda \sum_{t_i^\lambda = s_l} (t_i^\lambda - t_{i-1}^\lambda) + \mu \sum_{t_j^\mu = s_l} (t_j^\mu - t_{j-1}^\mu)\Bigg). \end{aligned}$$

Combining this and (2.23), we obtain

$$V(s_l, z_l^\lambda, z_l^\mu) \le V(s_l, y_l^\lambda, y_l^\mu) + 3L\Big(\lambda \sum_{t_i^\lambda = s_l} (t_i^\lambda - t_{i-1}^\lambda) + \mu \sum_{t_j^\mu = s_l} (t_j^\mu - t_{j-1}^\mu)\Big)$$

$$\le \exp\big((s_l - s_{l-1})\overline{\omega}([s_{l-1}, s_l])\big)\big(V(s_{l-1}, z_{l-1}^\lambda, z_{l-1}^\mu) + 2L(\lambda+\mu)(s_l - s_{l-1})\big)$$

$$+ 3L\Big(\lambda \sum_{t_i^\lambda = s_l} (t_i^\lambda - t_{i-1}^\lambda) + \mu \sum_{t_j^\mu = s_l} (t_j^\mu - t_{j-1}^\mu)\Big)$$

$$\le \exp\big((s_l - \tau)\overline{\omega}([\tau, s_l])\big)\big(2L(\lambda+\mu)(s_l - \tau) + \eta_{l-1}(\lambda,\mu)\big)$$

$$+ 3L\Big(\lambda \sum_{t_i^\lambda = s_l} (t_i^\lambda - t_{i-1}^\lambda) + \mu \sum_{t_j^\mu = s_l} (t_j^\mu - t_{j-1}^\mu)\Big)$$

$$\le \exp\big((s_l - \tau)\overline{\omega}([\tau, s_l])\big)\big(2L(\lambda+\mu)(s_l - \tau) + \eta_l(\lambda,\mu)\big).$$

This means that (c) is true for $k = l$, and the proof is completed. □

The following is a local existence theorem of solutions to (IVP; τ, z).

THEOREM 3. *Let $(\tau, z) \in \Omega$. Let $R > 0$ and $M > 0$ be numbers such that $\tau + R < b$ and $\|A(s,y)\| \le M$ for $(s,y) \in \Omega \cap S_R(\tau, z)$. Let $\sigma \in (0, R/(M+1)]$. Then there exists a solution u to* (IVP; τ, z) *on $[\tau, \tau+\sigma]$ such that*

$$\|u(t) - u(s)\| \le M|t-s| \quad \textit{for } t, s \in [\tau, \tau+\sigma].$$

PROOF. Let $\varepsilon \in (0, 1/2)$. Then, by Proposition 2, there exists a sequence $\{(t_i^\varepsilon, x_i^\varepsilon)\}_{i=0}^\infty$ in Ω satisfying (i)–(iv) of Proposition 4. Let $u^\varepsilon : [\tau, \tau+\sigma) \to X$ be the function defined by $u^\varepsilon(t) = x_i^\varepsilon$ for $t \in [t_i^\varepsilon, t_{i+1}^\varepsilon)$ and $i \ge 0$. We want to prove that the family $\{u^\varepsilon\}$ converges in X uniformly on $[\tau, \tau+\sigma)$ as $\varepsilon \downarrow 0$.

Let $\lambda, \mu \in (0, 1/2)$ and let $\{s_k\}_{k=0}^\infty$ be a sequence defined as in Proposition 4. Then there exists a sequence $\{(z_k^\lambda, z_k^\mu)\}$ in $X \times X$ satisfying $(z_k^\lambda, z_k^\mu) \in \Omega(s_k) \times \Omega(s_k)$ for $k \ge 0$ and (a)–(c) of Proposition 4. We first prove that

$$\sup_{k \ge 0} \|z_k^\lambda - z_k^\mu\| \to 0 \quad \text{as } \lambda, \mu \downarrow 0. \tag{2.26}$$

Assume to the contrary that there exist $\varepsilon_0 > 0$, two null sequences $\{\lambda_n\}$ and $\{\mu_n\}$ of positive numbers, and a sequence $\{k_n\}$ of nonnegative integers such that

$$\|z_{k_n}^{\lambda_n} - z_{k_n}^{\mu_n}\| \ge \varepsilon_0 \quad \text{for } n \ge 1. \tag{2.27}$$

Since the sequence $\{s_{k_n}\}$ is bounded as $n \to \infty$, it has a convergent subsequence $\{s_{k_{n_l}}\}$. Since $(z_{k_{n_l}}^{\lambda_{n_l}}, z_{k_{n_l}}^{\mu_{n_l}}) \in \Omega(s_{k_{n_l}}) \times \Omega(s_{k_{n_l}})$ for $l \ge 1$, and since

$$V(s_{k_{n_l}}, z_{k_{n_l}}^{\lambda_{n_l}}, z_{k_{n_l}}^{\mu_{n_l}}) \le 5L\exp(\sigma\overline{\omega}([\tau, \tau+\sigma]))(\lambda_{n_l} + \mu_{n_l})\sigma \quad \text{for } l \ge 1$$

by Proposition 4 (c), we deduce from condition $(V4)$ that $\lim_{l\to\infty}\|z_{k_{n_l}}^{\lambda_{n_l}} - z_{k_{n_l}}^{\mu_{n_l}}\| = 0$. This is a contradiction to (2.27).

Let $t \in [\tau, \tau+\sigma)$. Let $k \geq 1$ be an integer such that $t \in [s_{k-1}, s_k)$. Let i and j be positive integers such that $t_{i-1}^{\lambda} \leq s_{k-1} < s_k \leq t_i^{\lambda}$ and $t_{j-1}^{\mu} \leq s_{k-1} < s_k \leq t_j^{\mu}$, respectively. Then we have, in a similar way to the derivation of (2.21), $\|z_{k-1}^{\lambda} - x_{i-1}^{\lambda}\| \leq (M+1)(t_i^{\lambda} - t_{i-1}^{\lambda})$ and $\|z_{k-1}^{\mu} - x_{j-1}^{\mu}\| \leq (M+1)(t_j^{\mu} - t_{j-1}^{\mu})$. Since

$$\begin{aligned}\|u^{\lambda}(t) - u^{\mu}(t)\| &\leq \|x_{i-1}^{\lambda} - z_{k-1}^{\lambda}\| + \|z_{k-1}^{\lambda} - z_{k-1}^{\mu}\| + \|z_{k-1}^{\mu} - x_{j-1}^{\mu}\| \\ &\leq (M+1)(\lambda+\mu) + \|z_{k-1}^{\lambda} - z_{k-1}^{\mu}\|,\end{aligned}$$

we observe from (2.26) that the family $\{u^{\varepsilon}(t)\}$ is uniformly Cauchy on $[\tau, \tau+\sigma)$. By Lemma 3 (i) we obtain

$$\|u^{\varepsilon}(t) - u^{\varepsilon}(s)\| \leq (M+\varepsilon/2)(|t-s| + 2\varepsilon) \quad \text{for } t, s \in [\tau, \tau+\sigma)$$

and $\varepsilon \in (0, 1/2)$. These facts imply that there exists a continuous function u defined on $[\tau, \tau+\sigma]$ such that $\sup_{t\in[\tau,\tau+\sigma)}\|u^{\varepsilon}(t) - u(t)\| \to 0$ as $\varepsilon \downarrow 0$. It is clear that $u(\tau) = z$ and $\|u(t) - u(s)\| \leq M|t-s|$ for $t, s \in [\tau, \tau+\sigma]$. Let $\tau^{\varepsilon} : [\tau, \tau+\sigma) \to \mathbb{R}$ be the function defined by $\tau^{\varepsilon}(t) = t_i^{\varepsilon}$ for $t \in [t_i^{\varepsilon}, t_{i+1}^{\varepsilon})$ and $i \geq 0$. Then $\tau \leq \tau^{\varepsilon}(t) \leq t < \tau+\sigma$ and $\lim_{\varepsilon\downarrow 0}\tau^{\varepsilon}(t) = t$ for $t \in [\tau, \tau+\sigma)$. From Proposition 4 (iii) we deduce that

$$\left\| u^{\varepsilon}(t_i^{\varepsilon}) - u^{\varepsilon}(0) - \int_{\tau}^{t_i^{\varepsilon}} A(\tau^{\varepsilon}(s), u^{\varepsilon}(s))\, ds \right\| \leq \varepsilon(t_i^{\varepsilon} - \tau)/2 \leq \varepsilon\sigma/2 \tag{2.28}$$

for $i \geq 0$. Since $(\tau^{\varepsilon}(t), u^{\varepsilon}(t)) \in \Omega$ and $\|A(\tau^{\varepsilon}(t), u^{\varepsilon}(t))\| \leq M$ for $t \in [\tau, \tau+\sigma)$ and since $(\tau^{\varepsilon}(t), u^{\varepsilon}(t)) \to (t, u(t))$, we have $(t, u(t)) \in \Omega$ and $A(\tau^{\varepsilon}(t), u^{\varepsilon}(t)) \to A(t, u(t))$ for $t \in [\tau, \tau+\sigma)$ as $\varepsilon \downarrow 0$, by $(\Omega 2)$ and $(\Omega 1)$ respectively. From (2.28) we obtain

$$u(t) - u(0) = \int_{\tau}^{t} A(s, u(s))\, ds$$

for $t \in [\tau, \tau+\sigma)$. Since $t \to A(t, u(t))$ is continuous on $[\tau, \tau+\sigma]$, u is a solution to (IVP;τ, z) on $[\tau, \tau+\sigma]$. Since the uniqueness follows from Proposition 1, the proof is completed. □

3. Global Existence of Solutions

In this section we investigate the intervals where the solutions to (IVP; τ, z) exist under assumptions $(\Omega 1)$–$(\Omega 4)$. We follow the arguments in [4], [6] and [7].

Proposition 5. *Let $(\tau, z) \in \Omega$. Then there exists $c_0 \in (\tau, b)$ such that for any $c \in (\tau, c_0)$, the following properties are satisfied:*

(i) (IVP; τ, z) *has a solution u on $[\tau, c]$.*

(ii) *For any* $\varepsilon > 0$*, there exists a number* $r \in (0, c-\tau)$ *which satisfies the following:*
  (a) $(\mathrm{IVP}; t, x)$ *has a solution* v *on* $[t, c]$ *for any* $(t, x) \in \Omega \cap S_r(\tau, z)$,
  (b) *if* $(t, x), (\hat{t}, \hat{x}) \in \Omega \cap S_r(\tau, z)$*,* v *and* $\hat{v}$ *are solutions to* $(\mathrm{IVP}; t, x)$ *on* $[t, c]$ *and* $(\mathrm{IVP}; \hat{t}, \hat{x})$ *on* $[\hat{t}, c]$ *respectively, then* $V\big(s, v(s), \hat{v}(s)\big) < \varepsilon$ *for* $s \in [t, c] \cap [\hat{t}, c]$.

Proof. Let $R > 0$ and $M > 0$ be numbers such that $\tau + R < b$ and $\|A(t, x)\| \le M$ for $(t, x) \in \Omega \cap S_R(\tau, z)$, and set $c_0 = \tau + R/(M+1)$. We shall show that for any number $c \in (\tau, c_0)$, the desired properties are satisfied. The first property (i) follows from Theorem 3.

We shall show that such a number c has the second property (ii). Let $\varepsilon > 0$. We take $\delta > 0$ so that $\exp\Big(\int_\tau^s \omega(\theta)d\theta\Big)\delta < \varepsilon$ for any $s \in [a, c]$. Next, we choose $r > 0$ so small that $\tau + r < c \le \tau + (R-r)/(M+1) - r$ and

$$2L(M+1)r \le \exp\Big(\int_\tau^s \omega(\theta)d\theta\Big)\delta \tag{3.1}$$

for $s \in [\tau - r, \tau + r] \cap [a, b)$. To prove (a), let $(t, x) \in \Omega \cap S_r(\tau, z)$. Set $\hat{r} = R - r$. Since $\tau + r < c < \tau + R/(M+1) < \tau + R$, we have $\hat{r} > 0$. Moreover, we have $t + \hat{r} = (t-\tau) + \tau + \hat{r} \le r + \tau + \hat{r} = \tau + R < b$. For $(s, y) \in S_{\hat{r}}(t, x)$, we have

$$|s - \tau| \le |s - t| + |t - \tau| < \hat{r} + r = R$$

and

$$\|y - z\| \le \|y - x\| + \|x - z\| < \hat{r} + r = R.$$

Thus $S_{\hat{r}}(t, x) \subset S_R(\tau, z)$. Since $\|A(s, y)\| \le M$ for $(s, y) \in \Omega \cap S_{\hat{r}}(t, x)$ and $t + \hat{r} < b$, $(\mathrm{IVP}; t, x)$ has a solution v on $[t, t + \hat{r}/(M+1)]$ by Theorem 3. Since $t + \hat{r}/(M+1) > \tau - r + (R-r)/(M+1) \ge c$, we certainly infer that v is defined on $[t, c]$.

To prove (b), let $\hat{v}$ be a solution to $(\mathrm{IVP}; \hat{t}, \hat{x})$ on $[\hat{t}, c]$ with $(\hat{t}, \hat{x}) \in \Omega \cap S_r(\tau, z)$. Assume that $\hat{t} \le t$ without loss of generality. Then

$$\begin{aligned}\|\hat{v}(t) - v(t)\| &= \|\hat{v}(t) - x\| \le \|\hat{v}(t) - \hat{x}\| + \|\hat{x} - z\| + \|z - x\| \\ &\le \|\hat{v}(t) - \hat{v}(\hat{t})\| + 2r \le M(t - \hat{t}) + 2r \\ &= M((t-\tau) + (\tau - \hat{t})) + 2r \le 2(M+1)r.\end{aligned}$$

By Remark 1 and (3.1), we have

$$V\big(t, v(t), \hat{v}(t)\big) \le 2L(M+1)r \le \exp\Big(\int_\tau^t \omega(\theta)d\theta\Big)\delta.$$

Thus, by Proposition 1, we obtain

$$V\big(s, v(s), \hat{v}(s)\big) \le \exp\Big(\int_t^s \omega(\theta)d\theta\Big) V\big(t, v(t), \hat{v}(t)\big) \le \exp\Big(\int_\tau^s \omega(\theta)d\theta\Big)\delta < \varepsilon$$

for $s \in [t, c]$. □

Let $(\tau, z) \in \Omega$ and let u be a solution to (IVP; τ, z) which is noncontinuable to the right. We denote its *final time* by $T(\tau, z)$. It is clear that $\tau < T(\tau, z) \le b$ and u is a solution to (IVP; τ, z) on $[\tau, T(\tau, z))$. Since (IVP; τ, z) has a unique solution, $T(\tau, z) \in (\tau, b]$ is well-defined for every $(\tau, z) \in \Omega$. We consider T as a function from the metric space Ω into the extended real line $\mathbb{R} \cup \{\infty\}$ endowed with the usual topology.

Proposition 6. *Let $(\tau, z) \in \Omega$ and let d be a number such that $\tau < d < T(\tau, z)$. Then there exists a number $r > 0$ with $\tau + r < b$ such that $T(t, x) > d$ for any $(t, x) \in \Omega \cap S_r(\tau, z)$.*

Proof. Let $(\tau, z) \in \Omega$ and let d be a number such that $\tau < d < T(\tau, z)$. Let u be a solution to (IVP; τ, z) on $[\tau, d]$. Since the set $\{\big(s, u(s)\big); s \in [\tau, d]\}$ is compact in Ω and A is continuous on Ω, there exists a number $M > 0$ such that $\|A\big(s, u(s)\big)\| < M$ for $s \in [\tau, d]$.

We first prove that there exists a number $R > 0$ such that $\|A(s, x)\| \le M$ for any $s \in [\tau, d]$ and $x \in \Omega(s)$ satisfying $V\big(s, x, u(s)\big) < R$. Assume to the contrary that for any $n \ge 1$ there exist $s_n \in [\tau, d]$ and $x_n \in \Omega(s_n)$ such that $V\big(s_n, x_n, u(s_n)\big) < 1/n$ and $\|A(s_n, x_n)\| > M$. Since the sequence $\{s_n\}$ is bounded, there exists a convergent subsequence $\{s_{n_k}\}$ converging to some number $s \in [\tau, d]$. Since $V\big(s_{n_k}, x_{n_k}, u(s_{n_k})\big) \to 0$ as $k \to \infty$, we have $\|x_{n_k} - u(s_{n_k})\| \to 0$ as $k \to \infty$ by ($V4$). Since $u(s_{n_k}) \to u(s)$ as $k \to \infty$, we have $(s_{n_k}, x_{n_k}) \to (s, u(s))$ as $k \to \infty$. Thus, by (Ω1), we have $\|A\big(s, u(s)\big)\| \ge M$. This contradicts to the definition of M.

By Proposition 5, we can choose a number c such that $\tau < c < d$ and properties (i) and (ii) in Proposition 5 are satisfied for (τ, z). Let $\varepsilon > 0$ be a number such that $\varepsilon \exp\Big(\int_c^s \omega(\theta)d\theta\Big) \le R$ for $s \in [c, d]$, and then choose $r > 0$ so that $\tau + r < c$ and Proposition 5 (ii) is satisfied for the number ε. Let $(t, x) \in \Omega \cap S_r(\tau, z)$. We want to show that $d < T(t, x)$. To this end, assume to the contrary that $T(t, x) \le d$ and let v be a noncontinuable solution to (IVP; t, x). Note by Proposition 5 (ii) that $[t, c] \subset [t, T(t, x))$ and $V\big(c, v(c), u(c)\big) < \varepsilon$.

By Proposition 1, we have

$$\begin{aligned} V\big(s,v(s),u(s)\big) &\le V\big(c,v(c),u(c)\big)\exp\Big(\int_c^s \omega(\theta)d\theta\Big) \\ &< \varepsilon\exp\Big(\int_c^s \omega(\theta)d\theta\Big) \le R \end{aligned}$$

for $s \in \big[c, T(t,x)\big)$. From the fact proved first, we observe that $\|A\big(s,v(s)\big)\| \le M$ for $s \in \big[c,T(t,x)\big)$. Thus $\|v(t)-v(s)\| \le M|t-s|$ for $t,s \in \big[c,T(t,x)\big)$. Therefore, $w = \lim_{s\uparrow T(t,x)} v(s)$ exists in X and $(T(t,x),w) \in \Omega$ by $(\Omega 2)$. In view of Theorem 3, this contradicts the fact that v is noncontinuable to the right of $T(t,x)$. Hence $T(t,x) > d$. □

Proposition 7. *Let $(\tau,z) \in \Omega$ and let $\{(\tau_n,z_n)\}_{n\ge 1}$ be a sequence in Ω converging to (τ,z) as $n \to \infty$. For $n \ge 1$, let u_n be a noncontinuable solution to* (IVP; τ_n,z_n)*, and let u be a noncontinuable solution to* (IVP; τ,z)*. Assume that $d \in (\tau,b)$ satisfies $d < T(\tau_n,z_n)$ for $n \ge 1$. Then the following assertions hold:*

(i) *$d < T(\tau,z)$.*

(ii) *For any $\sigma \in (\tau,d)$, the sequence $\{u_n\}$ converges to u uniformly on $[\sigma,d]$ as $n \to \infty$.*

Proof. Let $c \in (\tau,d)$ be a number with the properties (i) and (ii) in Proposition 5, and let $\tau < \sigma < c$. We may assume that $\tau_n < \sigma < c < d < T(\tau_n,z_n)$ for $n \ge 1$, because $\lim_{n\to\infty}\tau_n = \tau < d$. Let $\varepsilon > 0$. Let $r \in (0,c-\tau)$ be a number with the property (ii) in Proposition 5 for the number ε. Since $(\tau_n,z_n) \to (\tau,z)$ as $n \to \infty$, there exists an integer $n_0 \ge 1$ such that $(\tau_n,z_n) \in \Omega \cap S_r(\tau,z)$ for $n \ge n_0$. By Proposition 5 (ii-b) we observe that if $n,m \ge n_0$, then $V\big(s,u_m(s),u_n(s)\big) \le \varepsilon$ for $s \in [\sigma,c]$ and

$$\begin{aligned} V\big(t,u_m(t),u_n(t)\big) &\le \exp\Big(\int_c^t \omega(\theta)d\theta\Big)V\big(c,u_m(c),u_n(c)\big) \\ &\le \varepsilon\exp\big((d-c)\overline{\omega}([c,d])\big) \end{aligned}$$

for $t \in [c,d]$. By $(V4)$, the sequence $\{u_n\}$ is uniformly Cauchy on $[\sigma,d]$. Define $\hat{u}(t) = \lim_{n\to\infty} u_n(t)$ for $t \in [\sigma,d]$. Then we observe that $\hat{u}'(t) = A(t,\hat{u}(t))$ for $t \in [\sigma,d]$. By Proposition 5, we observe that if $n \ge n_0$, then $V\big(s,u_n(s),u(s)\big) \le \varepsilon$ for $s \in [\sigma,c]$. Thus, we have $\hat{u}(\sigma) = \lim_{n\to\infty} u_n(\sigma) = u(\sigma)$. Hence $\hat{u}$ is a solution to $\big($IVP; $\sigma,u(\sigma)\big)$ on $[\sigma,d]$. Note that u is a solution to $\big($IVP; $\tau,z\big)$ on $[\tau,\sigma]$. Since the function $v : [\tau,d] \to X$ defined by $v(t) = u(t)$ for $t \in [\tau,\sigma]$ and $v(t) = \hat{u}(t)$ for $t \in [\sigma,d]$ is a solution to $\big($IVP; $\tau,z\big)$ on $[\tau,d]$, we have $T(\tau,z) > d$. Since $v(t) = u(t)$ for $t \in [\tau,d]$ by uniqueness, we observe that the sequence $\{u_n\}$ converges to u uniformly on $[\sigma,d]$ as $n \to \infty$. □

PROPOSITION 8. *T is a continuous function from Ω into $\mathbb{R} \cup \{\infty\}$.*

PROOF. Let $(\tau, z) \in \Omega$ and let $\{(t_n, x_n)\}_{n\geq 1}$ be a sequence in Ω converging to (τ, z). Let $\tau < d < T(\tau, z)$. Since $\lim_{n\to\infty}(t_n, x_n) = (\tau, z)$, we deduce from Proposition 6 that $d < T(t_n, x_n)$ for sufficiently large integers n. Thus $d \leq \liminf_{n\to\infty} T(t_n, x_n)$. Since d is arbitrary, we obtain $T(\tau, z) \leq \liminf_{n\to\infty} T(t_n, x_n)$. Note that

$$\tau < T(\tau, z) \leq \liminf_{n\to\infty} T(t_n, x_n) \leq \limsup_{n\to\infty} T(t_n, x_n),$$

and let d satisfy $\tau < d < \limsup_{n\to\infty} T(t_n, x_n)$. Then there exists a subsequence $\{(t_{n_k}, x_{n_k})\}_{k\geq 1}$ of $\{(t_n, x_n)\}_{n\geq 1}$ such that $d < T(t_{n_k}, x_{n_k})$ for $k \geq 1$. Since $(t_{n_k}, x_{n_k}) \to (\tau, z)$ as $k \to \infty$, it follows from Proposition 7 that $d < T(\tau, z)$. Since d is arbitrary chosen, we conclude that $\limsup_{n\to\infty} T(t_n, x_n) \leq T(\tau, z)$. Hence, we obtain $\lim_{n\to\infty} T(t_n, x_n) = T(\tau, z)$. □

A global existence theorem is given as follows.

THEOREM 4. *Let C be a connected component of Ω and set $d = \sup\{t \in [a, b)\,;\, C(t) \neq \emptyset\}$. Then for each $(\tau, z) \in C$, $(\mathrm{IVP}; \tau, z)$ has a unique solution on $[\tau, d)$ and the interval $[\tau, d)$ is the maximal interval of existence of solution. In particular, if Ω is connected, then for $(\tau, z) \in \Omega$, $(\mathrm{IVP}; \tau, z)$ has a unique solution on $[\tau, b)$.*

PROOF. We shall show that $T : \Omega \to \mathbb{R} \cup \{\infty\}$ takes the constant value d on C. To prove that $T(C)$ is a singleton set, let $c, \hat{c} \in T(C) = \{T(t, x)\,;\, (t, x) \in C\}$. Without loss of generality, we assume that $c \leq \hat{c}$, and set

$$C_1 = \{(t, x) \in C\,;\, T(t, x) \leq c\} \quad \text{and} \quad C_2 = \{(t, x) \in C\,;\, T(t, x) > c\}.$$

If $C = C_1$, then $\hat{c} \leq c$, and so $T(C)$ is a singleton set $\{c\}$. To prove that $C = C_1$, we have only to prove that $C_2 = \emptyset$ because C_1 and C_2 are disjoint. To this end, assume to the contrary that C_2 is nonempty. Since T is continuous on C by Proposition 8, C_2 is an open subset of C. Let $\{(t_n, x_n)\}_{n\geq 1}$ be a sequence in C_2 converging to $(t, x) \in C$. By the definition of C_2, we have $c < T(t_n, x_n)$ for $n \geq 1$. Proposition 7 asserts that $c < T(t, x)$. This implies that C_2 is a closed subset of C. It follows that $C = C_1 \cup C_2$, and C_1 and C_2 are disjoint, nonempty and open in C. This is impossible because C is connected, and so we conclude that $C_2 = \emptyset$.

Since $T(C)$ is a singleton set, we can write $T(C) = \{c\}$ for some $c \in \mathbb{R} \cup \{\infty\}$. Since $t < T(t, x) = c$ for $(t, x) \in C$, we obtain $d = \sup\{t\,;\, C(t) \neq \emptyset\} \leq c$. On the other hand, let $s < c$. Note that $c = T(t, x)$ for some $(t, x) \in C$. If $t < s$ then a noncontinuable solution u to $(\mathrm{IVP}; t, x)$ satisfies $(s, u(s)) \in C$, and so $C(s) \neq \emptyset$. This implies that $s \leq d$. If $s \leq t$ then $s \leq t \leq d$ because $C(t) \neq \emptyset$. Since

s is arbitrarily chosen such that $s < c$, we have $c \le d$. Consequently, we get $T(C) = \{d\}$. $\square$

Theorem 1 is a consequence of Proposition 1 and Theorems 3 and 4.

4. Proof of Theorem 2

Proof of the necessity part. Let $(\tau, z) \in \Omega$ and $u(t) = U(t,\tau)z$ for $t \in [\tau, b)$. Let C be a connected component of Ω such that $(\tau, z) \in C$. Since $\{(t, u(t)); t \in [\tau, b)\}$ is a connected set in Ω containing (τ, z), we have $(t, u(t)) \in C$ for $t \in [\tau, b)$ by the maximality of C; hence $C(t) \neq \emptyset$ for $t \in [\tau, b)$. This means that $(\Omega 5)$ holds true. Since $u(\tau + h) \in \Omega(\tau + h)$ for $h \in (0, b - \tau)$, we have

$$\begin{aligned} h^{-1} d(z + hA(\tau, z), \Omega(\tau + h)) &\le h^{-1} \|z + hA(\tau, z) - u(\tau + h)\| \\ &= \|A(\tau, u(\tau)) - h^{-1}(u(\tau + h) - u(\tau))\| \\ &\to \|A(\tau, u(\tau)) - u'(\tau)\| = 0 \end{aligned}$$

as $h \downarrow 0$. Thus, $(\Omega 3)$ also holds true. It remains to show that $(\Omega 4)$ holds true. We set

$$V_0(t, x, y) = \sup_{\sigma \in [t,b)} \left\{ \exp\left(-\int_t^\sigma \omega(\theta)\, d\theta \right) \|U(\sigma, t)x - U(\sigma, t)y\| \right\}$$

for $t \in [a, b)$ and $x,\ y \in \Omega(t)$. From (E1) and (E3) we see that

$$\|x - y\| \le V_0(t, x, y) \le L\|x - y\| \quad \text{for } t \in [a, b) \text{ and } x, y \in \Omega(t). \tag{4.1}$$

For any $x,\ y \in X$, $t \in [a, b)$ and $x',\ y' \in \Omega(t)$, we have

$$\begin{aligned} &V_0(t, x', y') - L\left(\|x - x'\| + \|y - y'\|\right) \\ &\le L\|x' - y'\| - L\left(\|x - x'\| + \|y - y'\|\right) \le L\|x - y\|. \end{aligned}$$

Thus, we can define $V : [a, b) \times X \times X \to [0, \infty)$ by

$$V(t, x, y) = \sup_{(x',y') \in \Omega(t) \times \Omega(t)} \left\{ \max\Big(0,\ V_0(t, x', y') - L\left(\|x - x'\| + \|y - y'\|\right)\Big) \right\}$$

for $(t, x, y) \in [a, b) \times X \times X$. Since

$$\begin{aligned} V_0(t, x', y') &\le V_0(t, x', x) + V_0(t, x, y) + V_0(t, y, y') \\ &\le V_0(t, x, y) + L\left(\|x - x'\| + \|y - y'\|\right) \end{aligned}$$

for $t \in [a, b)$ and $(x, y), (x', y') \in \Omega(t) \times \Omega(t)$, we have $V(t, x, y) \le V_0(t, x, y)$ for $t \in [a, b)$ and $(x, y) \in \Omega(t) \times \Omega(t)$. The converse inequality follows readily from the definition of V. Thus $V(t, x, y) = V_0(t, x, y)$ for $t \in [a, b)$ and $(x, y) \in \Omega(t) \times \Omega(t)$. This combined with (4.1) implies that the functional V satisfies $(V4)'$ and $(V2)$.

Let (x,y), $(\hat{x},\hat{y}) \in X \times X$ and $t \in [a,b)$. For any $(x',y') \in \Omega(t) \times \Omega(t)$, we have

$$\begin{aligned}
&V_0(t,x',y') - L\left(\|x-x'\| + \|y-y'\|\right) \\
&\quad - \Big(V_0(t,x',y') - L\left(\|\hat{x}-x'\| + \|\hat{y}-y'\|\right)\Big) \\
&= L\left(\|\hat{x}-x'\| + \|\hat{y}-y'\|\right) - L\left(\|x-x'\| + \|y-y'\|\right) \\
&\le L\left(\|\hat{x}-x\| + \|\hat{y}-y\|\right),
\end{aligned}$$

which implies that

$$V_0(t,x',y') - L\left(\|x-x'\| + \|y-y'\|\right) \le V(t,\hat{x},\hat{y}) + L(\|\hat{x}-x\| + \|\hat{y}-y\|)$$

and

$$V(t,x,y) \le V(t,\hat{x},\hat{y}) + L(\|\hat{x}-x\| + \|\hat{y}-y\|).$$

Thus, we obtain $(V1)$.

To prove $(V3)$, let $t_n \in [a,b)$ with $t_n \to t \in [a,b)$ as $n \to \infty$ and let $(x_n,y_n) \in \Omega(t_n) \times \Omega(t_n)$ with $(x_n,y_n) \to (x,y) \in \Omega(t) \times \Omega(t)$ as $n \to \infty$. Let $\sigma \in (t,b)$ and N a number such that $\sigma > t_n$ for $n \ge N$. Then we have

$$V_0(t_n,x_n,y_n) \ge \exp\left(-\int_{t_n}^{\sigma} \omega(\theta)\,d\theta\right)\|U(\sigma,t_n)x_n - U(\sigma,t_n)y_n\| \quad \text{for } n \ge N.$$

Taking the inferior limit as $n \to \infty$, we have

$$\liminf_{n\to\infty} V_0(t_n,x_n,y_n) \ge \exp\left(-\int_{t}^{\sigma} \omega(\theta)\,d\theta\right)\|U(\sigma,t)x - U(\sigma,t)y\|.$$

By (4.1), we have $V_0(t_n,x_n,y_n) \ge \|x_n - y_n\|$ for $n \ge 1$. Taking the inferior limit as $n \to \infty$, we see that the above inequality is also valid for $\sigma = t$. Thus, we have

$$\liminf_{n\to\infty} V_0(t_n,x_n,y_n) \ge V_0(t,x,y).$$

Finally, we prove the dissipativity condition

$$D_+V(t,x,y)(A(t,x),A(t,y)) \le \omega(t)V(t,x,y) \quad \text{for } x,y \in \Omega(t) \text{ and } t \in [a,b).$$

For this purpose, let $t \in [a,b)$ and $x,y \in \Omega(t)$. Since

$$\begin{aligned}
&\|U(\sigma,t+h)U(t+h,t)x - U(\sigma,t+h)U(t+h,t)y\| \\
&= \exp\left(\int_t^{\sigma} \omega(\theta)\,d\theta\right) \cdot \exp\left(-\int_t^{\sigma} \omega(\theta)\,d\theta\right)\|U(\sigma,t)x - U(\sigma,t)y\| \\
&\le \exp\left(\int_t^{\sigma} \omega(\theta)\,d\theta\right) V_0(t,x,y) \\
&= \exp\left(\int_t^{t+h} \omega(\theta)\,d\theta\right) \cdot \exp\left(\int_{t+h}^{\sigma} \omega(\theta)\,d\theta\right) V_0(t,x,y)
\end{aligned}$$

for $h \in (0, b-t)$ and $\sigma \in [t+h, b)$, we have

$$V_0(t+h, U(t+h,t)x, U(t+h,t)y) \le \exp\biggl(\int_t^{t+h} \omega(\theta)\, d\theta\biggr) V_0(t,x,y) \tag{4.2}$$

for $h \in (0, b-t)$. Since $V(t,x,y) = V_0(t,x,y)$ for $t \in [a,b)$ and $x, y \in \Omega(t)$ and since $V(t,\cdot,\cdot)$ is Lipschitz continuous on $X \times X$ with Lipschitz constant L, by (4.2) we have

$$\begin{aligned}
&(V(t+h, x+hA(t,x), y+hA(t,y)) - V(t,x,y))/h \\
&\le (V(t+h, U(t+h,t)x, U(t+h,t)y) - V(t,x,y))/h \\
&\quad + L(\|x+hA(t,x) - U(t+h,t)x\| + \|y+hA(t,y) - U(t+h,t)y\|)/h \\
&\le \frac{1}{h}\left(\exp\biggl(\int_t^{t+h} \omega(\theta)\, d\theta\biggr) - 1\right) V(t,x,y) \\
&\quad + L(\|x+hA(t,x) - U(t+h,t)x\| + \|y+hA(t,y) - U(t+h,t)y\|)/h \\
&\to \omega(t)V(t,x,y) \quad \text{as } h \downarrow 0.
\end{aligned}$$

This means that the desired dissipativity condition holds true. □

Proof of the sufficiency part. By condition (Ω5), Theorem 4 asserts that for any $(\tau, z) \in \Omega$, there exists a unique global solution $u = u(\cdot;\tau,z)$ to (IVP;τ, z) on $[\tau, b)$. Define $\{U(t,\tau)\}_{(t,\tau)\in\Delta}$ by $U(t,\tau)z = u(t;\tau,z)$ for $(\tau,z) \in \Omega$ and $t \in [\tau, b)$. Then we see that for each $(t,\tau) \in \Delta$, $U(t,\tau)$ maps $\Omega(\tau)$ to $\Omega(t)$. We immediately obtain (E1) from the uniqueness of solutions to initial value problem (IVP;τ, z). By Proposition 1, we find, noting $(V4)'$, that

$$\begin{aligned}
&\|U(t,\tau)z - U(t,\tau)\hat{z}\| \le V(t, U(t,\tau)z, U(t,\tau)\hat{z}) \\
&\le \exp\biggl(\int_\tau^t \omega(\theta) d\theta\biggr) V(\tau, z, \hat{z}) \le L \exp\biggl(\int_\tau^t \omega(\theta) d\theta\biggr) \|z - \hat{z}\|
\end{aligned}$$

for $z, \hat{z} \in \Omega(\tau)$ and $(t,\tau) \in \Delta$, namely, (E3) holds true.

It remains to show that (E2) holds true. Let $(t_n, \tau_n), (t,\tau) \in \Delta$, $z_n \in \Omega(\tau_n)$ and $z \in \Omega(\tau)$ and suppose that $(t_n, \tau_n) \to (t,\tau)$ and $z_n \to z$ as $n \to \infty$. We have to show that $u(t_n; \tau_n, z_n) = U(t_n, \tau_n)z_n \to u(t;\tau,z) = U(t,\tau)z$ as $n \to \infty$. First, we assume that $t > \tau$. Let $d \in (\tau, b)$ be a number such that $t < d$ and take $\sigma \in (\tau, t)$. Since $t_n \to t$ as $n \to \infty$, we may assume that $t_n \in [\sigma, d]$ for $n \ge 1$. Then, we deduce from Proposition 7 that $\lim_{n\to\infty} u(\cdot;\tau_n, z_n) = u(\cdot;\tau,z)$ uniformly on $[\sigma, d]$, and hence $u(t_n;\tau_n,z_n) \to u(t;\tau,z)$ as $n \to \infty$. Next, we assume that $t = \tau$. Since $u(t;\tau,z) = U(t,\tau)z = z$, we need to show that $u(t_n;\tau_n,z_n) \to z$ as $n \to \infty$. To this end, let $M > 0$ and $R > 0$ be numbers such that $\tau + R < b$ and $\|A(s,y)\| \le M$ for $(s,y) \in \Omega \cap S_R(\tau,z)$. Since $(\tau_n, z_n) \to (\tau, z)$ as $n \to \infty$, there exists an integer $N \ge 1$ such that $\tau_n + R/2 < b$ and $(\tau_n, z_n) \in S_{R/2}(\tau, z)$ for $n \ge N$. Take $r = R/2$. Thus, we observe that if $n \ge N$, then

$S_r(\tau_n, z_n) \subset S_R(\tau, z)$ and $\|A(s,y)\| \le M$ for $(s,y) \in \Omega \cap S_r(\tau_n, z_n)$. Let $\sigma \in (0, r/(M+1))$. Thus, we deduce from Theorem 3 that if $n \ge N$ then

$$\|u(s; \tau_n, z_n) - u(\hat{s}; \tau_n, z_n)\| \le M|s - \hat{s}|$$

for $s, \hat{s} \in [\tau_n, \tau_n + \sigma]$. Since $\tau_n \to \tau$ and $t_n \to t = \tau$ as $n \to \infty$, we find that $t_n \in [\tau_n, \tau_n + \sigma]$ for sufficient large n, and so the above inequality implies that

$$\|u(t_n; \tau_n, z_n) - z_n\| \le M|t_n - \tau_n|$$

for sufficient large n. Since $z_n \to z$ as $n \to \infty$, we conclude that $u(t_n; \tau_n, z_n) \to z$ as $n \to \infty$. □

5. Application to Wave Equations

In this section, we apply Theorem 1 to the initial value problem for nonlinear wave equation with dissipation:

$$\begin{cases} \partial_t u = \partial_x v, \quad \partial_t v = \partial_x \sigma(t,u) - \gamma v, \\ u(0,x) = u_0(x), \; v(0,x) = v_0(x) \quad \text{for } x \in \mathbb{R} \text{ and } t \in [0,\infty). \end{cases} \tag{5.1}$$

Here γ is a positive constant and $\sigma(\cdot,\cdot)$ a real-valued smooth function on $[0,\infty) \times \mathbb{R}$ satisfying $\sigma(t,0) = 0$ for $t \in [0,\infty)$. We make the following assumptions on the function σ.

(i) There exists a positive constant δ_0 such that $\sigma_r(t,r) \ge \delta_0$ for $(t,r) \in [0,\infty) \times \mathbb{R}$.

(ii) There exists a constant $L_0 > 0$ such that

$$\|\sigma_r(t,\cdot)\|_{L^\infty} \le L_0, \quad \|\sigma_{rr}(t,\cdot)\|_{L^\infty} \le L_0$$
$$\text{and} \quad \|\sigma_{rrr}(t,\cdot)\|_{L^\infty} \le L_0 \quad \text{for} \quad t \in [0,\infty).$$

(iii) There exists a continuous integrable function $h : [0,\infty) \to [0,\infty)$ such that

$$\|\sigma_{tr}(t,\cdot)\|_{L^\infty} \le h(t) \quad \text{for } t \in [0,\infty).$$

Let $X = L^2(\mathbb{R}) \times L^2(\mathbb{R})$ with the standard norm $\|(u,v)\| = (\|u\|_{L^2}^2 + \|v\|_{L^2}^2)^{1/2}$, and define $H : [0,\infty) \times H^2(\mathbb{R}) \times H^2(\mathbb{R}) \to \mathbb{R}$ by

$$\begin{aligned} H(t,u,v) &= H^{(0)}(t,u,v) + H^{(1)}(t,u,v) + H^{(2)}(t,u,v) \\ &= \int_{-\infty}^{\infty} \Big(\int_0^u \sigma(t,r)dr + \frac{1}{2}v^2 \Big) dx \\ &\quad + \frac{1}{2} \int_{-\infty}^{\infty} \big(\sigma_r(t,u)(\partial_x u)^2 + (\gamma u + \partial_x v)^2 \big) dx \\ &\quad + \frac{1}{2} \int_{-\infty}^{\infty} \big(\sigma_r(t,u)(\partial_x^2 u)^2 + (\gamma \partial_x u + \partial_x^2 v)^2 \big) dx \end{aligned}$$

for $(u,v) \in H^2(\mathbb{R}) \times H^2(\mathbb{R})$ and $t \in [0,\infty)$. The assumptions imply that there exist constants $C_0 \geq c_0 > 0$ such that

$$c_0\|(u,v)\|^2_{H^2\times H^2} \leq H(t,u,v) \leq C_0\|(u,v)\|^2_{H^2\times H^2} \tag{5.2}$$

for $(u,v) \in H^2(\mathbb{R})\times H^2(\mathbb{R})$ and $t \in [0,\infty)$. The following proposition will be used in order to convert the problem (5.1) into the initial value problem for a continuous mapping $A : \Omega\ (\subset [0,\infty) \times X) \to X$.

Proposition 9. *Let $t \in [0,\infty)$ and $(u_0,v_0) \in H^2(\mathbb{R}) \times H^2(\mathbb{R})$. Then there exists $\lambda_0 > 0$ such that for any $\lambda \in (0,\lambda_0]$, the problem*

$$(u_\lambda - u_0)/\lambda = \partial_x v_\lambda, \tag{5.3}$$

$$(v_\lambda - v_0)/\lambda = \sigma_r(t,u_0)\partial_x u_\lambda - \gamma v_\lambda \tag{5.4}$$

has a solution $(u_\lambda, v_\lambda) \in H^3(\mathbb{R}) \times H^3(\mathbb{R})$ satisfying the following properties:

(i) *The family $\{(u_\lambda,v_\lambda)\}$ converges to (u_0,v_0) in $H^2(\mathbb{R})\times H^2(\mathbb{R})$ as $\lambda \downarrow 0$.*

(ii) *There exists a nondecreasing continuous function $g : [0,\infty) \to [0,\infty)$ with $g(0) = 0$, depending only γ and $\sigma(\cdot,\cdot)$, such that*

$$\begin{aligned}&\frac{1}{\lambda}\big(H(t+\lambda,u_\lambda,v_\lambda) - H(t,u_0,v_0)\big)\\ &\quad\leq \frac{1}{2\lambda}\left(\int_t^{t+\lambda} h(s)ds\right)\|u_\lambda\|^2_{H^2} - \gamma\delta_0\|\partial_x u_\lambda\|^2_{H^1}\\ &\qquad + (1+\lambda^2)g(\|(u_0,v_0)\|_{H^2\times H^2} \vee \|(u_\lambda,v_\lambda)\|_{H^2\times H^2})\\ &\qquad\qquad \times (\|\partial_x u_0\|_{H^1} \vee \|\partial_x u_\lambda\|_{H^1})^2\end{aligned} \tag{5.5}$$

for $\lambda \in (0,\lambda_0]$.

Here and subsequently, we use notation $a \vee b = \max\{a,b\}$ for $a,b \in \mathbb{R}$.

Proof. Let $t \in [0,\infty)$ and $(u_0,v_0) \in H^2(\mathbb{R}) \times H^2(\mathbb{R})$. Define $D(L(t)) = H^1(\mathbb{R}) \times H^1(\mathbb{R})$ and

$$L(t)(u,v) = (\partial_x v, \sigma_r(t,u_0)\partial_x u - \gamma v)$$

for $(u,v) \in D(L(t))$. Let β_0 be a positive number such that $\beta_0 \geq L_0\|\partial_x u_0\|_{L^\infty}\ /(2\sqrt{\delta_0})$. Since

$$\frac{\|\partial_x\big(\sigma_r(t,u_0)\big)\|_{L^\infty}}{2\sqrt{\delta_0}} = \frac{\|\sigma_{rr}(t,u_0)\partial_x u_0\|_{L^\infty}}{2\sqrt{\delta_0}} \leq \beta_0,$$

we deduce from [8, Proposition 5.7] that $L(t) - \beta_0 I$ is m-dissipative in $X = L^2(\mathbb{R})\times L^2(\mathbb{R})$ with inner product $\big((u,v),(\hat{u},\hat{v})\big) = (\int_{-\infty}^{\infty} \sigma_r(t,u_0)u\hat{u} + v\hat{v}dx)^{1/2}$ for $(u,v),(\hat{u},\hat{v}) \in X$. Choose $\lambda_0 > 0$ so that $\lambda_0\beta_0 < 1$. Then, for $\lambda \in (0,\lambda_0]$, $(u_\lambda,v_\lambda) := \big(I - \lambda L(t)\big)^{-1}(u_0,v_0)$ satisfies (5.3) and (5.4). Note that $D(L(t)^k) = H^k(\mathbb{R}) \times H^k(\mathbb{R})$ for $k = 2,3$. It follows from the proof of [8, Proposition 5.7] that $(u_\lambda,v_\lambda) \in D\big(L(t)^3\big)$

and $L(t)^k(u_\lambda, v_\lambda) = (I - \lambda L(t))^{-1} L(t)^k(u_0, v_0)$ for $k = 0, 1, 2$ and that the family $\{L(t)^k(u_\lambda, v_\lambda)\}$ converges to $L(t)^k(u_0, v_0)$ in X as $\lambda \downarrow 0$, for $k = 0, 1, 2$. Hence the family $\{(u_\lambda, v_\lambda)\}$ converges to (u_0, v_0) in $H^2(\mathbb{R}) \times H^2(\mathbb{R})$ as $\lambda \downarrow 0$.

We shall show (ii). Since $\sigma(t, 0) = 0$, we have $\sigma(t, u_\lambda) \in H^1(\mathbb{R})$ and $\partial_x \sigma(t, u_\lambda) = \sigma_r(t, u_\lambda) \partial_x u_\lambda$. By (5.4), we get

$$\frac{1}{\lambda}(v_\lambda - v_0) = \partial_x \sigma(t, u_\lambda) - \gamma v_\lambda + \big(\sigma_r(t, u_0) - \sigma_r(t, u_\lambda)\big) \partial_x u_\lambda.$$

We multiply this equality and (5.3) by v_λ and $\sigma(t, u_\lambda)$, respectively. The sum of these two equations gives us

$$\begin{aligned} &\frac{1}{\lambda}\sigma(t, u_\lambda)(u_\lambda - u_0) + \frac{1}{\lambda} v_\lambda (v_\lambda - v_0) \\ &= \partial_x\big(v_\lambda \sigma(t, u_\lambda)\big) - \gamma v_\lambda^2 + v_\lambda\big(\sigma_r(t, u_0) - \sigma_r(t, u_\lambda)\big)\partial_x u_\lambda. \end{aligned}$$

Integrating this equality, we have

$$\begin{aligned} &\frac{1}{\lambda}\int_{-\infty}^{\infty} \sigma(t, u_\lambda)(u_\lambda - u_0)dx + \frac{1}{\lambda}\int_{-\infty}^{\infty} v_\lambda(v_\lambda - v_0)dx \\ &= -\gamma \int_{-\infty}^{\infty} v_\lambda^2 dx + \int_{-\infty}^{\infty} v_\lambda\big(\sigma_r(t, u_0) - \sigma_r(t, u_\lambda)\big)\partial_x u_\lambda dx \\ &\le \frac{1}{4\gamma}\int_{-\infty}^{\infty}\big(\sigma_r(t, u_0) - \sigma_r(t, u_\lambda)\big)^2(\partial_x u_\lambda)^2 dx \\ &\le \frac{L_0^2}{4\gamma}\int_{-\infty}^{\infty}(u_0 - u_\lambda)^2(\partial_x u_\lambda)^2 dx = \frac{\lambda^2 L_0^2}{4\gamma}\int_{-\infty}^{\infty}(\partial_x v_\lambda)^2(\partial_x u_\lambda)^2 dx \\ &\le \frac{\lambda^2 L_0^2}{4\gamma}\|\partial_x v_\lambda\|_{H^1}^2 \int_{-\infty}^{\infty}(\partial_x u_\lambda)^2 dx. \end{aligned}$$

Since the function $r \to \sigma(t, r)$ is nondecreasing, we have

$$\begin{aligned} \frac{1}{\lambda}\int_{-\infty}^{\infty}\bigg(\int_{u_0}^{u_\lambda}\sigma(t, r)dr\bigg)dx + \frac{1}{2\lambda}\int_{-\infty}^{\infty}(v_\lambda^2 - v_0^2)dx \\ \le \frac{\lambda^2 L_0^2}{4\gamma}\|\partial_x v_\lambda\|_{H^1}^2\int_{-\infty}^{\infty}(\partial_x u_\lambda)^2 dx, \end{aligned}$$

or

$$\begin{aligned} &\frac{1}{\lambda}\big(H^{(0)}(t + \lambda, u_\lambda, v_\lambda) - H^{(0)}(t, u_0, v_0)\big) \\ &\le \frac{1}{\lambda}\int_{-\infty}^{\infty}\bigg(\int_0^{u_\lambda}\big(\sigma(t + \lambda, r) - \sigma(t, r)\big)dr\bigg)dx \\ &\quad + \frac{\lambda^2 L_0^2}{4\gamma}\|\partial_x v_\lambda\|_{H^1}^2\int_{-\infty}^{\infty}(\partial_x u_\lambda)^2 dx. \end{aligned}$$

The first term on the right-hand side is estimated as follows:

$$
\begin{aligned}
&\frac{1}{\lambda}\int_{-\infty}^{\infty}\Big(\int_0^{u_\lambda}\big(\sigma(t+\lambda,r)-\sigma(t,r)\big)dr\Big)dx\\
&=\frac{1}{\lambda}\int_t^{t+\lambda}\Big(\int_{-\infty}^{\infty}\Big(\int_0^{u_\lambda}\sigma_t(s,r)dr\Big)dx\Big)\,ds\\
&=\frac{1}{\lambda}\int_t^{t+\lambda}\Big(\int_{-\infty}^{\infty}\Big(\int_0^{u_\lambda}\Big(\int_0^1\sigma_{tr}(s,\theta r)\,d\theta\Big)\,rdr\Big)dx\Big)\,ds\\
&\le\frac{1}{2\lambda}\Big(\int_t^{t+\lambda}h(s)ds\Big)\,\|u_\lambda\|_{L^2}^2.
\end{aligned}
$$

Hence

$$
\begin{aligned}
&\frac{1}{\lambda}\big(H^{(0)}(t+\lambda,u_\lambda,v_\lambda)-H^{(0)}(t,u_0,v_0)\big)\\
&\le\frac{1}{2\lambda}\Big(\int_t^{t+\lambda}h(s)ds\Big)\,\|u_\lambda\|_{L^2}^2+\frac{\lambda^2}{4\gamma}L_0^2\|\partial_x v_\lambda\|_{H^1}^2\|\partial_x u_\lambda\|_{L^2}^2. \qquad (5.6)
\end{aligned}
$$

Differentiating (5.3) and (5.4), we have

$$
\frac{1}{\lambda}(\partial_x u_\lambda-\partial_x u_0)=\partial_x(\partial_x v_\lambda), \qquad (5.7)
$$

$$
\frac{1}{\lambda}\big((\gamma u_\lambda+\partial_x v_\lambda)-(\gamma u_0+\partial_x v_0)\big)=\partial_x(\sigma_r(t,u_0)\partial_x u_\lambda). \qquad (5.8)
$$

We multiply (5.7) and (5.8) by $\sigma_r(t,u_0)\partial_x u_\lambda$ and $\gamma u_\lambda+\partial_x v_\lambda$, respectively. The sum of these two equations gives us

$$
\begin{aligned}
&\frac{1}{2\lambda}\sigma_r(t,u_0)\big((\partial_x u_\lambda)^2-(\partial_x u_0)^2\big)+\frac{1}{2\lambda}\big((\gamma u_\lambda+\partial_x v_\lambda)^2-(\gamma u_0+\partial_x v_0)^2\big)\\
&\le\partial_x\big(\sigma_r(t,u_0)\partial_x u_\lambda\partial_x v_\lambda\big)+\gamma u_\lambda\partial_x(\sigma_r(t,u_0)\partial_x u_\lambda).
\end{aligned}
$$

Integrating this equality, we have

$$
\begin{aligned}
&\frac{1}{2\lambda}\int_{-\infty}^{\infty}\sigma_r(t,u_0)\big((\partial_x u_\lambda)^2-(\partial_x u_0)^2\big)dx\\
&\quad+\frac{1}{2\lambda}\int_{-\infty}^{\infty}\big((\gamma u_\lambda+\partial_x v_\lambda)^2-(\gamma u_0+\partial_x v_0)^2\big)dx\\
&\le-\gamma\int_{-\infty}^{\infty}(\partial_x u_\lambda)(\sigma_r(t,u_0)\partial_x u_\lambda)dx.
\end{aligned}
$$

Thus

$$
\begin{aligned}
&\frac{1}{\lambda}\big(H^{(1)}(t+\lambda,u_\lambda,v_\lambda)-H^{(1)}(t,u_0,v_0)\big)\\
&\le\frac{1}{2\lambda}\int_{-\infty}^{\infty}\big(\sigma_r(t+\lambda,u_\lambda)-\sigma_r(t,u_0)\big)(\partial_x u_\lambda)^2\,dx-\gamma\int_{-\infty}^{\infty}\sigma_r(t,u_0)(\partial_x u_\lambda)^2dx.
\end{aligned}
$$

Since

$$\begin{aligned}&\bigl|\sigma_r(t+\lambda,u_\lambda)-\sigma_r(t,u_0)\bigr|\le\bigl|\sigma_r(t+\lambda,u_\lambda)-\sigma_r(t,u_\lambda)\bigr|+\bigl|\sigma_r(t,u_\lambda)-\sigma_r(t,u_0)\bigr|\\&\quad\le\left|\int_t^{t+\lambda}\sigma_{tr}(s,u_\lambda)\,ds\right|+L_0\bigl|u_\lambda-u_0\bigr|\le\int_t^{t+\lambda}h(s)\,ds+\lambda L_0|\partial_x v_\lambda|,\end{aligned}\tag{5.9}$$

we have

$$\begin{aligned}&\frac{1}{\lambda}\bigl(H^{(1)}(t+\lambda,u_\lambda,v_\lambda)-H^{(1)}(t,u_0,v_0)\bigr)\\&\le\frac{1}{2\lambda}\left(\int_t^{t+\lambda}h(s)ds\right)\|\partial_x u_\lambda\|_{L^2}^2+\frac{1}{2}L_0\|\partial_x v_\lambda\|_{H^1}\|\partial_x u_\lambda\|_{L^2}^2\\&\quad-\gamma\delta_0\|\partial_x u_\lambda\|_{L^2}^2.\end{aligned}\tag{5.10}$$

Differentiating (5.7) and (5.8), we have

$$\frac{1}{\lambda}(\partial_x^2 u_\lambda-\partial_x^2 u_0)=\partial_x(\partial_x^2 v_\lambda),\tag{5.11}$$

$$\begin{aligned}&\frac{1}{\lambda}\bigl((\gamma\partial_x u_\lambda+\partial_x^2 v_\lambda)-(\gamma\partial_x u_0+\partial_x^2 v_0)\bigr)\\&=\partial_x(\sigma_{rr}(t,u_0)\partial_x u_0\partial_x u_\lambda+\sigma_r(t,u_0)\partial_x^2 u_\lambda).\end{aligned}\tag{5.12}$$

We multiply (5.11) and (5.12) by $\sigma_r(t,u_0)\partial_x^2 u_\lambda$ and $\gamma\partial_x u_\lambda+\partial_x^2 v_\lambda$, respectively. The sum of these two equations gives us

$$\begin{aligned}&\frac{1}{2\lambda}\sigma_r(t,u_0)\bigl((\partial_x^2 u_\lambda)^2-(\partial_x^2 u_0)^2\bigr)+\frac{1}{2\lambda}\bigl((\gamma\partial_x u_\lambda+\partial_x^2 v_\lambda)^2-(\gamma\partial_x u_0+\partial_x^2 v_0)^2\bigr)\\&\le\partial_x(\sigma_r(t,u_0)\partial_x^2 u_\lambda\partial_x^2 v_\lambda)+\gamma\partial_x u_\lambda\partial_x(\sigma_r(t,u_0)\partial_x^2 u_\lambda)\\&\quad+(\partial_x^2 v_\lambda+\gamma\partial_x u_\lambda)\partial_x(\sigma_{rr}(t,u_0)\partial_x u_0\partial_x u_\lambda).\end{aligned}$$

Integrating this equality, we have

$$\begin{aligned}&\frac{1}{2\lambda}\int_{-\infty}^{\infty}\sigma_r(t,u_0)\bigl((\partial_x^2 u_\lambda)^2-(\partial_x^2 u_0)^2\bigr)dx\\&\quad+\frac{1}{2\lambda}\int_{-\infty}^{\infty}\bigl((\gamma\partial_x u_\lambda+\partial_x^2 v_\lambda)^2-(\gamma\partial_x u_0+\partial_x^2 v_0)^2\bigr)dx\\&\le-\gamma\int_{-\infty}^{\infty}\sigma_r(t,u_0)(\partial_x^2 u_\lambda)^2\,dx+\int_{-\infty}^{\infty}(\gamma\partial_x u_\lambda+\partial_x^2 v_\lambda)\partial_x(\sigma_{rr}(t,u_0)\partial_x u_0\partial_x u_\lambda)dx\\&=-\gamma\int_{-\infty}^{\infty}\sigma_r(t,u_0)(\partial_x^2 u_\lambda)^2\,dx-\gamma\int_{-\infty}^{\infty}\partial_x^2 u_\lambda(\sigma_{rr}(t,u_0)\partial_x u_0\partial_x u_\lambda)\,dx\\&\quad+\int_{-\infty}^{\infty}(\partial_x^2 v_\lambda)\partial_x(\sigma_{rr}(t,u_0)\partial_x u_0\partial_x u_\lambda)dx.\end{aligned}$$

Hence

$$\begin{aligned}
&\frac{1}{\lambda}\big(H^{(2)}(t+\lambda,u_\lambda,v_\lambda)-H^{(2)}(t,u_0,v_0)\big)\\
&\le \frac{1}{2\lambda}\int_{-\infty}^{\infty}\big(\sigma_r(t+\lambda,u_\lambda)-\sigma_r(t,u_0)\big)(\partial_x^2u_\lambda)^2dx-\gamma\int_{-\infty}^{\infty}\sigma_r(t,u_0)(\partial_x^2u_\lambda)^2dx\\
&\quad-\gamma\int_{-\infty}^{\infty}\partial_x^2u_\lambda(\sigma_{rr}(t,u_0)(\partial_xu_0)\partial_xu_\lambda)dx\\
&\quad+\int_{-\infty}^{\infty}(\partial_x^2v_\lambda)\partial_x(\sigma_{rr}(t,u_0)\partial_xu_0\partial_xu_\lambda)dx. \qquad (5.13)
\end{aligned}$$

The third term on the right-hand side is estimated by

$$\begin{aligned}
&-\gamma\int_{-\infty}^{\infty}\partial_x^2u_\lambda(\sigma_{rr}(t,u_0)(\partial_xu_0)\partial_xu_\lambda)dx\\
&\le \gamma L_0\|\partial_x^2u_\lambda\|_{L^2}\|\partial_xu_0\|_{L^\infty}\|\partial_xu_\lambda\|_{L^2}\le \gamma L_0\|u_0\|_{H^2}\|\partial_xu_\lambda\|_{H^1}^2.
\end{aligned}$$

Since

$$\begin{aligned}
&\partial_x(\sigma_{rr}(t,u_0)\partial_xu_0\partial_xu_\lambda)\\
&=\sigma_{rrr}(t,u_0)(\partial_xu_0)^2\partial_xu_\lambda+\sigma_{rr}(t,u_0)\partial_x^2u_0\partial_xu_\lambda+\sigma_{rr}(t,u_0)\partial_xu_0\partial_x^2u_\lambda,
\end{aligned}$$

we have

$$\begin{aligned}
&\int_{-\infty}^{\infty}(\partial_x^2v_\lambda)\partial_x(\sigma_{rr}(t,u_0)\partial_xu_0\partial_xu_\lambda)dx\\
&\le L_0\|\partial_x^2v_\lambda\|_{L^2}\big(\|\partial_xu_0\|_{L^\infty}^2\|\partial_xu_\lambda\|_{L^2}\\
&\qquad+\|\partial_x^2u_0\|_{L^2}\|\partial_xu_\lambda\|_{L^\infty}+\|\partial_xu_0\|_{L^\infty}\|\partial_x^2u_\lambda\|_{L^2}\big)\\
&\le L_0\|v_\lambda\|_{H^2}\big(\|u_0\|_{H^2}\|\partial_xu_0\|_{H^1}\|\partial_xu_\lambda\|_{L^2}\\
&\qquad+\|\partial_xu_0\|_{H^1}\|\partial_xu_\lambda\|_{H^1}+\|\partial_xu_0\|_{H^1}\|\partial_x^2u_\lambda\|_{L^2}\big)\\
&\le L_0\|v_\lambda\|_{H^2}\big(\|u_0\|_{H^2}+2\big)\|\partial_xu_0\|_{H^1}\|\partial_xu_\lambda\|_{H^1}.
\end{aligned}$$

We estimate the first term on the right-hand side of (5.13) by (5.9), and combine the resulting inequality and the inequalities obtained above. This yields

$$\begin{aligned}
&\frac{1}{\lambda}\big(H^{(2)}(t+\lambda,u_\lambda,v_\lambda)-H^{(2)}(t,u_0,v_0)\big)\\
&\le\frac{1}{2\lambda}\left(\int_t^{t+\lambda}h(s)ds\right)\|\partial_x^2u_\lambda\|_{L^2}^2+\frac{L_0}{2}\|\partial_xv_\lambda\|_{H^1}\|\partial_x^2u_\lambda\|_{L^2}^2-\gamma\delta_0\|\partial_x^2u_\lambda\|_{L^2}^2\\
&\quad+L_0\Big(\gamma\|u_0\|_{H^2}+\|v_\lambda\|_{H^2}\big(\|u_0\|_{H^2}+2\big)\Big)(\|\partial_xu_0\|_{H^1}\vee\|\partial_xu_\lambda\|_{H^1})^2.
\end{aligned}$$

Combining this inequality with (5.6) and (5.10) we observe that the desired inequality (5.5) is satisfied for the function

$$g(r)=L_0r\left\{\left(\frac{L_0r}{4\gamma}\right)\vee\Big(3+\gamma+r\Big)\right\}\quad\text{for}\quad r\ge 0.$$

□

Let c_0 be the constant in (5.2), and define $\hat{H} : [0,\infty) \times H^2(\mathbb{R}) \times H^2(\mathbb{R}) \to \mathbb{R}$ by

$$\hat{H}(t,u,v) = \exp\Big(-\frac{1}{c_0}\int_0^t h(s)ds\Big) H(t,u,v)$$

for $(t,u,v) \in [0,\infty) \times H^2(\mathbb{R}) \times H^2(\mathbb{R})$. Then we have

$$\hat{H}(t,u,v) \le H(t,u,v) \le \exp\Big(\frac{1}{c_0}\int_0^\infty h(s)ds\Big)\hat{H}(t,u,v) \qquad (5.14)$$

for $(t,u,v) \in [0,\infty) \times H^2(\mathbb{R}) \times H^2(\mathbb{R})$. Since g is continuous and $g(0) = 0$, we choose a number $R_0 > 0$ so small that

$$\text{if } r \ge 0 \text{ and } r^2 \le \frac{R_0}{c_0}\exp\Big(\frac{1}{c_0}\int_0^\infty h(s)\,ds\Big) \text{ then } g(r) < \gamma\delta_0, \quad (5.15)$$

and define a subset Ω of $[0,\infty) \times X$ by

$$\Omega = \{(t,(u,v)) \in [0,\infty) \times (H^2(\mathbb{R}) \times H^2(\mathbb{R}))\,;\, \hat{H}(t,u,v) \le R_0\}.$$

Let $r_0 = \sqrt{R_0/C_0}$, where C_0 is the constant in (5.2). Then, by (5.2) we have

$$S_0 := \{(u,v) \in H^2(\mathbb{R}) \times H^2(\mathbb{R})\,;\, \|(u,v)\|_{H^2\times H^2} \le r_0\} \subset \Omega(t) \qquad (5.16)$$

for any $t \in [0,\infty)$, and there exists a connected component C of Ω such that $[0,\infty) \times S_0 \subset C \subset \Omega$. Let R_0' be the positive number such that $(R_0')^2 = \dfrac{R_0}{c_0}\exp\left(\dfrac{1}{c_0}\displaystyle\int_0^\infty h(s)\,ds\right)$. Then, by (5.2) and (5.14) we have

$$\Omega(t) \subset S_0' := \{(u,v) \in H^2(\mathbb{R}) \times H^2(\mathbb{R})\,;\, \|(u,v)\|_{H^2\times H^2} \le R_0'\} \qquad (5.17)$$

for any $t \in [0,\infty)$. Let V be the functional on $[0,\infty)\times X \times X$ defined by

$$V\big(t,(u,v),(\hat{u},\hat{v})\big) = \left(\int_{-\infty}^\infty (\hat{v}-v)^2 + \Big(\int_u^{\hat{u}} \sqrt{\sigma_r(t,r)}dr\Big)^2 dx\right)^{\frac{1}{2}}$$

for $(u,v),(\hat{u},\hat{v}) \in X$ and $t \in [0,\infty)$. It is easily seen that conditions $(V1)$–$(V4)$ are satisfied. In particular, we see that for each $t \in [0,\infty)$, $V(t,\cdot,\cdot)$ is a metric on X and

$$\begin{aligned}\min\{1,\sqrt{\delta_0}\}\|(u,v)-(\hat{u},\hat{v})\| &\le V\big(t,(u,v),(\hat{u},\hat{v})\big)\\ &\le (1 \vee \sqrt{L_0})\|(u,v)-(\hat{u},\hat{v})\|\end{aligned}$$

for $(u,v),(\hat{u},\hat{v}) \in X$. Consider the operator $A : \Omega \to X$ defined by

$$A(t,(u,v)) = (\partial_x v, \partial_x\sigma(t,u) - \gamma v)$$

for $(t,(u,v)) \in \Omega$. Then the nonlinear wave equation with dissipation (5.1) is converted into the initial value problem for A. We can prove that the initial value problem for A is globally well-posed, by Theorem 1 combined with the following theorem which will be proved by a sequence of propositions.

THEOREM 5. *The operator A satisfies* (Ω1)–(Ω4).

In view of (5.16) and (5.17), we are in a position to state the global solvability of the nonlinear wave equation with dissipation (5.1).

COROLLARY 1. *For any (u_0, v_0) such that $\|(u_0, v_0)\|_{H^2\times H^2} \le r_0$, there exists a unique time global solution $(u(\cdot), v(\cdot))$ to* (5.1) *such that*

$$(u(\cdot), v(\cdot)) \in C^1([0,\infty); L^2(\mathbb{R})\times L^2(\mathbb{R}))\cap L^\infty(0,\infty; H^2(\mathbb{R})\times H^2(\mathbb{R})).$$

REMARK 2. *Similar results are obtained in Yamada* [23] *and Matsumura* [14].

For the proof of Theorem 5 we follow the argument in [8]. We note here that

$$\|\partial_x w\|_{L^2}^2 \le \|w\|_{L^2}\|\partial_x^2 w\|_{L^2} \quad \text{for} \quad w\in H^2(\mathbb{R}). \tag{5.18}$$

PROPOSITION 10. *The operator A is continuous on Ω.*

PROOF. Let $\big(t,(u,v)\big), \big(\hat t,(\hat u,\hat v)\big) \in \Omega$. Since $\sigma(t,0)=0$, we have

$$\sigma(t,u(x)) - \sigma(\hat t, u(x)) = u(x)\int_0^1 \Big(\sigma_r(t,\hat\theta u(x)) - \sigma_r(\hat t,\hat\theta u(x))\Big)\, d\hat\theta$$

and

$$\begin{aligned}
&\|\sigma(t,u)-\sigma(\hat t,u)\|_{L^2}^2\\
&= \int_{-\infty}^{\infty}\left((t-\hat t)u(x)\int_0^1\int_0^1 \sigma_{tr}(\hat t+\theta(t-\hat t),\hat\theta u(x))\,d\theta\,d\hat\theta\right)^2 dx\\
&\le \int_{-\infty}^{\infty}\left(|t-\hat t|\cdot|u(x)|\int_0^1 h(\hat t+\theta(t-\hat t))\,d\theta\right)^2 dx\\
&= \left(\int_{\hat t}^t h(s)\,ds\right)^2 \|u\|_{L^2}^2.
\end{aligned}$$

Since $\|u\|_{L^2}\le R_0'$ by (5.17) and $\|\sigma_r(\hat t,\cdot)\|_{L^\infty}\le L_0$, we get

$$\begin{aligned}
\|\sigma(t,u)-\sigma(\hat t,\hat u)\|_{L^2} &\le \|\sigma(t,u)-\sigma(\hat t,u)\|_{L^2} + \|\sigma(\hat t,u)-\sigma(\hat t,\hat u)\|_{L^2}\\
&\le \left|\int_{\hat t}^t h(s)\,ds\right|\|u\|_{L^2} + L_0\|u-\hat u\|_{L^2}\\
&\le R_0'\left|\int_{\hat t}^t h(s)\,ds\right| + L_0\|u-\hat u\|_{L^2}.
\end{aligned}$$

By (5.17) we have $\|\partial_x^2(v-\hat{v})\|_{L^2} \le \|\partial_x^2 v\|_{L^2} + \|\partial_x^2 \hat{v}\|_{L^2} \le 2R_0'$. Since

$$\begin{aligned}&\partial_x^2\sigma(t,u(x)) = \partial_x(\sigma_r(t,u(x))\partial_x u(x))\\&= \sigma_{rr}(t,u(x))(\partial_x u(x))^2 + \sigma_r(t,u(x))\partial_x^2 u(x),\end{aligned}$$

we get, by using the inequality $\|w\|_{L^\infty} \le \|w\|_{H^1}$ for $w \in H^1(\mathbb{R})$,

$$\begin{aligned}&\|\partial_x^2\big(\sigma(t,u)-\sigma(\hat{t},\hat{u})\big)\|_{L^2} \le \|\partial_x^2\sigma(t,u)\|_{L^2} + \|\partial_x^2\sigma(\hat{t},\hat{u})\|_{L^2}\\&\le L_0(\|(\partial_x u)^2\|_{L^2} + \|(\partial_x\hat{u})^2\|_{L^2}) + L_0(\|\partial_x^2 u\|_{L^2} + \|\partial_x^2\hat{u}\|_{L^2})\\&\le L_0(\|\partial_x u\|_{L^\infty}\|\partial_x u\|_{L^2} + \|\partial_x\hat{u}\|_{L^\infty}\|\partial_x\hat{u}\|_{L^2}) + 2L_0R_0'\\&\le 2L_0(R_0')^2 + 2L_0R_0'.\end{aligned}$$

Thus, using (5.18), we have

$$\begin{aligned}&\|A(t,(u,v)) - A(\hat{t},(\hat{u},\hat{v}))\|^2\\&\le \|\partial_x(v-\hat{v})\|_{L^2}^2 + \|\partial_x\big(\sigma(t,u)-\sigma(\hat{t},\hat{u})\big) - \gamma(v-\hat{v})\|_{L^2}^2\\&\le \|\partial_x(v-\hat{v})\|_{L^2}^2 + 2\|\partial_x\big(\sigma(t,u)-\sigma(\hat{t},\hat{u})\big)\|_{L^2}^2 + 2\gamma^2\|v-\hat{v}\|_{L^2}^2\\&\le \|v-\hat{v}\|_{L^2}\|\partial_x^2(v-\hat{v})\|_{L^2} + 2\gamma^2\|v-\hat{v}\|_{L^2}^2\\&\qquad + 2\|\sigma(t,u)-\sigma(\hat{t},\hat{u})\|_{L^2}\|\partial_x^2\big(\sigma(t,u)-\sigma(\hat{t},\hat{u})\big)\|_{L^2}\\&\le 2R_0'\|v-\hat{v}\|_{L^2} + 2\gamma^2\|v-\hat{v}\|_{L^2}^2\\&\qquad + 4L_0R_0'\big(1+R_0'\big)\left(R_0'\left|\int_{\hat{t}}^{t} h(s)\,ds\right| + L_0\|u-\hat{u}\|_{L^2}\right),\end{aligned}$$

which implies the continuity of A on Ω. □

PROPOSITION 11. *Condition* (Ω2) *is satisfied for the set* Ω.

PROOF. Let $t_n \in [0,\infty)$ with $t_n \uparrow t \in [0,\infty)$ as $n \to \infty$. Let $(u,v) \in X$ and let $\{(u_n,v_n)\}$ be a sequence in X such that $(u_n,v_n) \in \Omega(t_n)$ for $n \ge 1$ and $(u_n,v_n) \to (u,v)$ in X as $n \to \infty$. We have to show that $(u,v) \in \Omega(t)$. Since the sequence $\{(u_n,v_n)\}$ is bounded in $H^2(\mathbb{R}) \times H^2(\mathbb{R})$ it follows that $(u,v) \in H^2(\mathbb{R}) \times H^2(\mathbb{R})$ and the sequence $\{(u_n,v_n)\}$ converges weakly to (u,v) in $H^2(\mathbb{R}) \times H^2(\mathbb{R})$ as $n \to \infty$. By (5.18), we see that the sequence $\{(u_n,v_n)\}$ converges to (u,v) in $H^1(\mathbb{R}) \times H^1(\mathbb{R})$ as $n \to \infty$. Moreover, $\{(u_n,v_n)\}$ converges to (u,v) in $L^\infty(\mathbb{R}) \times L^\infty(\mathbb{R})$ as $n \to \infty$. Since $\hat{H}(t_n,u_n,v_n) \le R_0$ for

$n \geq 1$, we have

$$
\begin{aligned}
& R_0 \exp\Big(\frac{1}{c_0}\int_0^{t_n} h(s)\,ds\Big) \\
\geq & \int_{-\infty}^{\infty}\Big(\int_0^{u_n}\sigma(t_n,r)dr + \frac{1}{2}v_n^2\Big)dx \\
& + \frac{1}{2}\int_{-\infty}^{\infty}\big(\sigma_r(t_n,u_n)(\partial_x u_n)^2 + (\gamma u_n + \partial_x v_n)^2\big)dx \\
& + \frac{1}{2}\int_{-\infty}^{\infty}\big(\sigma_r(t_n,u_n)(\partial_x^2 u_n)^2 + (\gamma \partial_x u_n + \partial_x^2 v_n)^2\big)dx \\
= & \int_{-\infty}^{\infty}\Big(\int_0^{u_n}\sigma(t,r)dr + \frac{1}{2}v_n^2\Big)dx + \frac{1}{2}\int_{-\infty}^{\infty}\big\{\sigma_r(t,u)\big((\partial_x u_n)^2 \\
& + (\partial_x^2 u_n)^2\big) + (\gamma u_n + \partial_x v_n)^2 + (\gamma\partial_x u_n + \partial_x^2 v_n)^2\big\}dx \\
& + \int_{-\infty}^{\infty}\Big(\int_0^{u_n}(\sigma(t_n,r)-\sigma(t,r))dr\Big)dx \\
& + \frac{1}{2}\int_{-\infty}^{\infty}\Big\{\big(\sigma_r(t_n,u_n)-\sigma_r(t,u)\big)\big((\partial_x u_n)^2 + (\partial_x^2 u_n)^2\big)\Big\} \quad \text{for } n \geq 1.
\end{aligned}
\tag{5.19}
$$

Since

$$
\begin{aligned}
& \Big|\int_{-\infty}^{\infty}\Big(\int_0^{u_n}(\sigma(t_n,r)-\sigma(t,r))dr\Big)dx\Big| \\
& = \Big|\int_{-\infty}^{\infty}(t_n - t)\Big(\int_0^{u_n}\Big(\int_0^1\int_0^1 \sigma_{tr}(t+\theta(t_n-t),\hat{\theta}r)\,d\theta\,d\hat{\theta}\Big)r\,dr\Big)dx\Big| \\
& \leq \Big|\int_{-\infty}^{\infty}(t_n - t)\Big(\int_0^{u_n}\Big(\int_0^1 h(t+\theta(t_n-t))\,d\theta\Big)r\,dr\Big)dx\Big| \\
& = \frac{\|u_n\|_{L^2}^2}{2}\Big|\int_t^{t_n} h(s)\,ds\Big|
\end{aligned}
$$

and

$$
\Big|\sigma_r(t_n,u_n)-\sigma_r(t,u)\Big| \leq \Big|\sigma_r(t_n,u_n)-\sigma_r(t_n,u)\Big| + \Big|\sigma_r(t_n,u)-\sigma_r(t,u)\Big|
$$

$$
\leq L_0\|u_n - u\|_{L^\infty} + \Big|\int_t^{t_n} h(s)\,ds\Big|
$$

for $n \geq 1$, we have $R_0 \geq \hat{H}(t,u,v)$ by taking the inferior limit in (5.19) as $n \to \infty$. □

Proposition 12. *There exists a real-valued continuous function ω defined on $[0,\infty)$ such that*

$$
D_+V\big(t,(u,v),(\hat{u},\hat{v})\big)(A(t,(u,v)),A(t,(\hat{u},\hat{v})) \leq \omega(t)V\big(t,(u,v),(\hat{u},\hat{v})\big)
$$

for $(u,v),(\hat{u},\hat{v}) \in \Omega(t)$ and $t \in [0,\infty)$.

PROOF. Let $(u, v), (\hat{u}, \hat{v}) \in \Omega(t)$ for $t \in [0, \infty)$. Let $(\xi, \eta), (\hat{\xi}, \hat{\eta}) \in X$. Then we get

$$\begin{aligned}
&2D_+V\big(t,(u,v),(\hat{u},\hat{v})\big)\big((\xi,\eta),(\hat{\xi},\hat{\eta})\big)V\big(t,(u,v),(\hat{u},\hat{v})\big)\\
&=\liminf_{h\downarrow 0}\frac{1}{h}\Big(V\big(t+h,(u,v)+h(\xi,\eta),(\hat{u},\hat{v})+h(\hat{\xi},\hat{\eta})\big)^2-V\big(t,(u,v),(\hat{u},\hat{v})\big)^2\Big)\\
&=\liminf_{h\downarrow 0}\frac{1}{h}\bigg\{\int_{-\infty}^{\infty}\Big((\hat{v}+h\hat{\eta}-(v+h\eta))^2-(\hat{v}-v)^2\Big)\,dx\\
&\quad+\int_{-\infty}^{\infty}\bigg(\Big(\int_{u+h\xi}^{\hat{u}+h\hat{\xi}}\sqrt{\sigma_r(t+h,r)}\,dr\Big)^2-\Big(\int_u^{\hat{u}}\sqrt{\sigma_r(t,r)}\,dr\Big)^2\bigg)\,dx\bigg\}\\
&=\int_{-\infty}^{\infty}\bigg(2(\hat{v}-v)(\hat{\eta}-\eta)+2\int_u^{\hat{u}}\sqrt{\sigma_r(t,r)}\,dr\Big\{\Big(\hat{\xi}\sqrt{\sigma_r(t,\hat{u})}-\xi\sqrt{\sigma_r(t,u)}\Big)\\
&\quad+\int_u^{\hat{u}}\frac{\sigma_{tr}(t,r)}{2\sqrt{\sigma_r(t,r)}}\,dr\Big\}\bigg)\,dx. \qquad (5.20)
\end{aligned}$$

Substituting $(\xi, \eta) = A(t, (u, v))$ and $(\hat{\xi}, \hat{\eta}) = A(t, (\hat{u}, \hat{v}))$ into (5.20) yields

$$\begin{aligned}
&D_+V\big(t,(u,v),(\hat{u},\hat{v})\big)\big(A(t,(u,v)),A(t,(\hat{u},\hat{v})\big)V\big(t,(u,v),(\hat{u},\hat{v})\big)\\
&=\int_{-\infty}^{\infty}\bigg((\hat{v}-v)\Big(\partial_x(\sigma(t,\hat{u})-\sigma(t,u))-\gamma(\hat{v}-v)\Big)\\
&\quad+\int_u^{\hat{u}}\sqrt{\sigma_r(t,r)}\,dr\Big(\big(\partial_x\hat{v}\sqrt{\sigma_r(t,\hat{u})}-\partial_x v\sqrt{\sigma_r(t,u)}\big)\\
&\quad+\int_u^{\hat{u}}\frac{\sigma_{tr}(t,r)}{2\sqrt{\sigma_r(t,r)}}\,dr\Big)\bigg)\,dx\\
&=-\gamma\int_{-\infty}^{\infty}(\hat{v}-v)^2\,dx-\int_{-\infty}^{\infty}\partial_x(\hat{v}-v)(\sigma(t,\hat{u})-\sigma(t,u))\,dx\\
&\quad+\int_{-\infty}^{\infty}\bigg(\int_u^{\hat{u}}\sqrt{\sigma_r(t,r)}\,dr\Big(\big(\partial_x\hat{v}\sqrt{\sigma_r(t,\hat{u})}-\partial_x v\sqrt{\sigma_r(t,u)}\big)\Big)\bigg)\,dx\\
&\quad+\int_{-\infty}^{\infty}\bigg(\int_u^{\hat{u}}\sqrt{\sigma_r(t,r)}\,dr\int_u^{\hat{u}}\frac{\sigma_{tr}(t,r)}{2\sqrt{\sigma_r(t,r)}}\,dr\bigg)\,dx\\
&=-\gamma\int_{-\infty}^{\infty}(\hat{v}-v)^2\,dx+\int_{-\infty}^{\infty}\bigg(\int_u^{\hat{u}}\sqrt{\sigma_r(t,r)}\,dr\int_u^{\hat{u}}\frac{\sigma_{tr}(t,r)}{2\sqrt{\sigma_r(t,r)}}\,dr\bigg)\,dx\\
&\quad+\int_{-\infty}^{\infty}\partial_x\hat{v}\int_u^{\hat{u}}\Big(\sqrt{\sigma_r(t,r)}\sqrt{\sigma_r(t,\hat{u})}-\sigma_r(t,r)\Big)dr\,dx\\
&\quad+\int_{-\infty}^{\infty}\partial_x v\int_{\hat{u}}^{u}\Big(\sqrt{\sigma_r(t,r)}\sqrt{\sigma_r(t,u)}-\sigma_r(t,r)\Big)dr\,dx.
\end{aligned}$$

The second term on the right-hand side is estimated as follows:

$$\left|\int_{-\infty}^{\infty}\left(\int_u^{\hat u}\sqrt{\sigma_r(t,r)}\,dr\int_u^{\hat u}\frac{\sigma_{tr}(t,r)}{2\sqrt{\sigma_r(t,r)}}\,dr\right)dx\right|\le\frac{\sqrt{L_0}h(t)}{2\sqrt{\delta_0}}\int_{-\infty}^{\infty}(\hat u-u)^2\,dx.$$

The third and fourth terms are estimated as follows:

$$\begin{aligned}&\left|\int_{-\infty}^{\infty}\partial_x\hat v\int_u^{\hat u}\left(\sqrt{\sigma_r(t,r)}\sqrt{\sigma_r(t,\hat u)}-\sigma_r(t,r)\right)dr\,dx\right|\\&\le\|\partial_x\hat v\|_{L^\infty}\int_{-\infty}^{\infty}\left|\int_u^{\hat u}\frac{\sqrt{\sigma_r(t,r)}(\sigma_r(t,\hat u)-\sigma_r(t,r))}{\sqrt{\sigma_r(t,\hat u)}+\sqrt{\sigma_r(t,r)}}\,dr\right|dx\\&\le L_0\|\hat v\|_{H^2}\int_{-\infty}^{\infty}\left|\int_u^{\hat u}|\hat u-r|\,dr\right|dx=L_0\|\hat v\|_{H^2}\|\hat u-u\|^2/2\end{aligned}$$

and

$$\left|\int_{-\infty}^{\infty}\partial_x v\int_{\hat u}^{u}\left(\sqrt{\sigma_r(t,r)}\sqrt{\sigma_r(t,u)}-\sigma_r(t,r)\right)dr\,dx\right|\le L_0\|v\|_{H^2}\|\hat u-u\|^2/2.$$

Setting $\omega(t)=C_0'(1+h(t))$ for a suitable positive number C_0', we conclude that

$$D_+V\big(t,(u,v),(\hat u,\hat v)\big)\big(A(t,(u,v)),A(t,(\hat u,\hat v))\big)\le\omega(t)V\big(t,(u,v),(\hat u,\hat v)\big)$$

for $(u,v),(\hat u,\hat v)\in\Omega(t)$ and $t\in[0,\infty)$. □

PROPOSITION 13. *For any $t\in[0,\infty)$ and $(u_0,v_0)\in\Omega(t)$,*

$$\liminf_{\lambda\downarrow 0}\frac{1}{\lambda}d\big((u_0,v_0)+\lambda A(t,(u_0,v_0)),\Omega(t+\lambda)\big)=0. \tag{5.21}$$

PROOF. Let $t\in[0,\infty)$ and $(u_0,v_0)\in\Omega(t)$. By (5.15) and (5.17), we note that

$$-\gamma\delta_0+g(\|(u_0,v_0)\|_{H^2\times H^2})<0. \tag{5.22}$$

By Proposition 9, there exists $\lambda_0>0$ such that for any $\lambda\in(0,\lambda_0]$, the problem

$$\begin{cases}(u_\lambda-u_0)/\lambda=\partial_x v_\lambda,\\(v_\lambda-v_0)/\lambda=\sigma_r(t,u_0)\partial_x u_\lambda-\gamma v_\lambda\end{cases}$$

has a solution $(u_\lambda,v_\lambda)\in H^3(\mathbb{R})\times H^3(\mathbb{R})$ satisfying the properties (i) and (ii) in Proposition 9. If it is proved that $(u_\lambda,v_\lambda)\in\Omega(t+\lambda)$ for sufficiently small $\lambda>0$, then the subtangential condition (5.21) is shown to be satisfied by using the property (i) in Proposition 9.

We shall prove that $(u_\lambda,v_\lambda)\in\Omega(t+\lambda)$ for sufficiently small $\lambda>0$. By (5.2) and (5.5), we have

$$\begin{aligned}&\frac{1}{\lambda}\left(\left(1-\frac{1}{2c_0}\int_t^{t+\lambda}h(s)ds\right)H(t+\lambda,u_\lambda,v_\lambda)-H(t,u_0,v_0)\right)\\&\le(1+\lambda^2)g(\|(u_0,v_0)\|_{H^2\times H^2}\vee\|(u_\lambda,v_\lambda)\|_{H^2\times H^2})(\|\partial_x u_0\|_{H^1}\vee\|\partial_x u_\lambda\|_{H^1})^2\\&\quad-\gamma\delta_0\|\partial_x u_\lambda\|_{H^1}^2\end{aligned} \tag{5.23}$$

for $\lambda \in (0, \lambda_0]$. Choose $\lambda_1 \in (0, \lambda_0]$ so that $\frac{1}{c_0}\int_t^{t+\lambda} h(s)ds \leq 1$ for $\lambda \in (0, \lambda_1]$ and $t \in [0, \infty)$. Noting that $e^{-2r} \leq 1 - r$ for $0 \leq r \leq 1/2$, we have

$$\exp\Big(-\frac{1}{c_0}\int_t^{t+\lambda} h(s)ds\Big) \leq 1 - \frac{1}{2c_0}\int_t^{t+\lambda} h(s)ds$$

for $\lambda \in (0, \lambda_1]$. Hence

$$\begin{aligned}\frac{1}{\lambda}(\hat{H}(t+\lambda, u_\lambda, v_\lambda) - \hat{H}(t, u_0, v_0)) &\leq \exp\Big(-\frac{1}{c_0}\int_0^t h(s)ds\Big)\Big(-\gamma\delta_0\|\partial_x u_\lambda\|_{H^1}^2 \\ &+ (1+\lambda^2)g(\|(u_0,v_0)\|_{H^2\times H^2} \vee \|(u_\lambda,v_\lambda)\|_{H^2\times H^2})(\|\partial_x u_0\|_{H^1} \vee \|\partial_x u_\lambda\|_{H^1})^2\Big)\end{aligned} \tag{5.24}$$

for $\lambda \in (0, \lambda_1]$. Since $(u_\lambda, v_\lambda) \to (u_0, v_0)$ in $H^2(\mathbb{R}) \times H^2(\mathbb{R})$ as $\lambda \downarrow 0$, we have

$$\begin{aligned}&\limsup_{\lambda\downarrow 0} \frac{1}{\lambda}\big(\hat{H}(t+\lambda, u_\lambda, v_\lambda) - \hat{H}(t, u_0, v_0)\big) \\ &\leq \exp\Big(-\frac{1}{c_0}\int_0^t h(s)ds\Big)\big(-\gamma\delta_0 + g(\|(u_0,v_0)\|_{H^2\times H^2})\big)\|\partial_x u_0\|_{H^1}^2.\end{aligned} \tag{5.25}$$

If $\|\partial_x u_0\|_{H^1} \neq 0$, then we have $(-\gamma\delta_0 + g(\|(u_0,v_0)\|_{H^2\times H^2}))\|\partial_x u_0\|_{H^2} < 0$ by (5.22). Hence (5.25) implies that $\hat{H}(t+\lambda, u_\lambda, v_\lambda) < \hat{H}(t, u_0, v_0) \leq R_0$ and $(u_\lambda, v_\lambda) \in \Omega(t+\lambda)$ for sufficiently small $\lambda > 0$. If $\|\partial_x u_0\|_{H^1} = 0$, then (5.24) implies that

$$\begin{aligned}\frac{1}{\lambda}(\hat{H}(t+\lambda, u_\lambda, v_\lambda) - \hat{H}(t, u_0, v_0)) &\leq \exp\Big(-\frac{1}{c_0}\int_0^t h(s)ds\Big) \\ \times \Big(-\gamma\delta_0 + (1+\lambda^2)g(\|(u_0,v_0)\|_{H^2\times H^2} &\vee \|(u_\lambda,v_\lambda)\|_{H^2\times H^2})\Big)\|\partial_x u_\lambda\|_{H^1}^2\end{aligned}$$

for $\lambda \in (0, \lambda_1]$. Since

$$\begin{aligned}&\lim_{\lambda\downarrow 0}\Big(-\gamma\delta_0 + (1+\lambda^2)g(\|(u_0,v_0)\|_{H^2\times H^2} \vee \|(u_\lambda,v_\lambda)\|_{H^2\times H^2})\Big) \\ &= -\gamma\delta_0 + g(\|(u_0,v_0)\|_{H^2\times H^2}) < 0,\end{aligned}$$

the right-hand side of the above inequality is less than or equal to zero for sufficient small $\lambda > 0$; hence $\hat{H}(t+\lambda, u_\lambda, v_\lambda) \leq \hat{H}(t, u_0, v_0) \leq R_0$ and $(u_\lambda, v_\lambda) \in \Omega(t+\lambda)$ for sufficient small $\lambda > 0$. □

Acknowledgement

The authors would like to thank the anonymous referees for their valuable comments and suggestions to improve the quality of the paper.

References

[1] R.P. Agarwal and V. Lakshmikantham, Uniqueness and nonuniqueness criteria for ordinary differential equations, Series in Real Analysis 6, World Scientific Publishing Co., Inc., River Edge, NJ, 1993.

[2] A. Ardito, P. Ricciardi and L. Tubaro, Dissipative Lyapunov functions and differential equations in a Banach space, Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur., (8) **57** (1974), no. 6, 511–518.

[3] A. Bacciotti and L. Rosier, Liapunov Functions and Stability in Control Theory, Second edition, Communications and Control Engineering Series, Springer-Verlag, Berlin, 2005.

[4] T. Iwamiya, Global existence of solutions to nonautonomous differential equations in Banach spaces, Hiroshima Math. J., **13** (1983), no. 1, 65–81.

[5] S. Kato, On existence and uniqueness conditions for nonlinear ordinary differential equations in Banach spaces, Funkcial. Ekvac., **19** (1976), no. 3, 239–245.

[6] S. Kato, Some remarks on nonlinear ordinary differential equations in a Banach space, Nonlinear Anal., **5** (1981), no. 1, 81–93.

[7] N. Kenmochi and T. Takahashi, Nonautonomous differential equations in Banach spaces, Nonlinear Anal., **4** (1980), no. 6, 1109–1121.

[8] Y. Kobayashi and N. Tanaka, Semigroups of Lipschitz operators, Adv. Differential Equations, **6** (2001), no. 5, 613–640.

[9] V. Lakshmikantham and S. Leela, Differential and integral inequalities: Theory and applications: Vol. I: Ordinary differential equations, Mathematics in Science and Engineering, Vol. 55-I, Academic Press, New York-London, 1969.

[10] V. Lakshmikantham, A.R. Mitchell and R.W. Mitchell, Differential equations on closed subsets of a Banach space, Trans. Amer. Math. Soc., **220** (1976), 103–113.

[11] R.H. Martin Jr., Differential equations on closed subsets of a Banach space, Trans. Amer. Math. Soc., **179** (1973), 399–414.

[12] R.H. Martin Jr., Lyapunov functions and autonomous differential equations in a Banach space, Math. Systems Theory, **7** (1973), 66–72.

[13] R.H. Martin Jr., Nonlinear operators and differential equations in Banach spaces, Pure and Applied Mathematics, Wiley-Interscience [JohnWiley & Sons], New York-London-Sydney, 1976.

[14] A. Matsumura, Global existence and asymptotics of the solutions of the second-order quasilinear hyperbolic equations with the first-order dissipation, Publ. Res. Inst. Math. Sci., **13** (1977/78), no. 2, 349–379.

[15] H. Murakami, On nonlinear ordinary and evolution equations, Funkcialaj Ekvacioj, **9** (1966), 151–162.

[16] M. Nagumo, Über die Lage der Integralkurvengewöhnlicher Differentialgleichungen, Proc. Phys. -Math. Soc. Japan, (3) **24** (1942), 551–559.

[17] H. Okamura, Condition ńecessaire et suffisante remplie par les équations différentielles ordinaires sans points de Peano, Mem. Coll. Sci. Kyoto Imp., Univ. Ser. A., **24** (1942), 21–28.

[18] N.H. Pavel, Differential equations, flow invariance and applications Research Notes in Mathematics, 113, Pitman (Advanced Publishing Program), Boston, MA, 1984.

[19] N.H. Pavel and I.I. Vrabie, Differential equations associated with continuous and dissipative time-dependent domain operators, in “Volterra equations” (Proc. Helsinki Sympos. Integral Equations, Otaniemi, 1978), pp. 236–250, Lecture Notes in Math., 737, Springer, Berlin, 1979.

[20] P. Ricciardi and L. Tubaro, Local existence for differential equations in Banach space, Boll. Un. Mat. Ital., (4) **8** (1973), 306–316.
[21] J. Szarski, Differential inequalities, Monografie Matematyczne, 43, Państwowe Wydawnictwo Naukowe, Warsaw, 1965.
[22] O. Cârjă, M. Necula and I.I. Vrabie, Viability, invariance and applications, North-Holland Mathematics Studies, 207, Elsevier Science B. V., Amsterdam, 2007.
[23] Y. Yamada, Quasilinear wave equations and related nonlinear evolution equations, Nagoya Math. J., **84** (1981), 31–83.
[24] T. Yoshizawa, Stability theory by Liapunov's second method, Publications of the Mathematical Society of Japan, No. 9, The Mathematical Society of Japan, Tokyo 1966.

Yoshikazu Kobayashi
Department of Mathematics
Faculty of Science and Engineering
Chuo University
Tokyo 112-8551 JAPAN
E-mail: kobayashi@math.chuo-u.ac.jp

Naoki Tanaka
Department of Mathematics
Faculty of Science
Shizuoka University
Shizuoka 422-8529 JAPAN
E-mail: tanaka.naoki@shizuoka.ac.jp.

Yukino Tomizawa
Department of Mathematics
Faculty of Science and Engineering
Chuo University
Tokyo 112-8551 JAPAN
E-mail: tomizawa@gug.math.chuo-u.ac.jp