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On Nonlinear Ultra –Hyperbolic Wave Operator

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Abstract: In this paper, we study the generalized wave equation of the form

$$Lu = \frac{\partial^2}{\partial t^2}u(x,t) + C^2(\square)^k u(x,t) = f(x,t,u(x,t))$$

with the initial conditions

$$u(x,0) = 0 \text{ and } \frac{\partial u(x,0)}{\partial t} = 0$$

where $(x,t)\epsilon\mathcal{R}^n X[0,\infty)$, $\mathcal{R}^n$ is the n- dimensional Euclidean Space, $\square^k$ is named the ultra- hyperbolic operator iterated k- times, defined by

$$\square^k = \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_p^2} - \frac{\partial^2}{\partial x_{p+1}^2} - \cdots - \frac{\partial^2}{\partial x_{p+q}^2}\right)^k,$$

$p + q = n$, C is a positive constant. We obtain $u(x,t)$as a solution for such equation. Moreover, by $\epsilon-$ approximation the elementary solution $E(x,t) = O\left(\epsilon^{-n/k+1}\right)$ is obtained. Also under certain conditions uniqueness and boundness of the solution is established.

Keywords: Generalized ultra- hyperbolic wave equation, Fourier transform , $\epsilon-$ approximation, asymptotic solution, boundness and uniqueness.

Mathematics Subject Classification: 47F05.

1. Introduction:

It is well known for the n- dimensional wave equation

$$\frac{\partial^2}{\partial t^2}u(x,t) + C^2\Delta u(x,t) = 0 \qquad (1.1)$$

With the initial conditions

$$u(x,0) = f(x) \text{ and } \frac{\partial}{\partial t}u(x,0) = g(x)$$

Where f and g are given functions. By solving the Cauchy problem for such equation, the Fourier transform has been applied and the solution is given by

$$\hat{u}(\xi,t) = \hat{f}(\xi)\cos(2\pi|\xi|)\,t + \hat{g}(\xi)\frac{\sin(2\pi|\xi|)\,t}{2\pi|\xi|}$$

Where $|\xi|^2 = \xi_1^2 + \xi_2^2 + \cdots + \xi_n^2$

(See [1]). By using the inverse Fourier transform, we obtain $u(x,t)$ in the convolution form, that is

$$u(x,t) = f(x) * \psi_t(x) + g(x) * \phi_t(x) \qquad (1.2)$$

Where ϕ_t is an inverse Fourier transform of $\widehat{\phi_t}(\xi) = \frac{\sin(2\pi|\xi|)t}{2\pi|\xi|}$ and ψ_t is an inverse Fourier transform of $\widehat{\psi_t}(\xi) = \cos(2\pi|\xi|)\,t = \frac{\partial}{\partial t}\widehat{\phi_t}(\xi)$.

And the solution, for the equation

$$\frac{\partial^2}{\partial t^2}u(x,t) + C^2(\Delta)^k u(x,t) = 0$$

where

$$\Delta^k = \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_p^2} + \frac{\partial^2}{\partial x_{p+1}^2} + \cdots + \frac{\partial^2}{\partial x_{p+q}^2}\right)^k,$$

was considered (See [2]).

Also the Problem

$$\frac{\partial^2}{\partial t^2}u(x,t) + C^2(\Box)^k u(x,t) = 0$$

with initial conditions

$$u(x,0) = f(x) \text{ and } \frac{\partial}{\partial t}u(x,0) = g(x)$$

was considered,(See [3]).

And for, the problem

$$\frac{\partial^2}{\partial t^2}u(x,t) + C^2(\Box)u(x,t) = f(x,t,u(x,t)) \qquad (1.3)$$

With the initial conditions

$$u(x,0) = f(x) \text{ and } \frac{\partial}{\partial t}u(x,0) = g(x)$$

was considered in [4].

In this paper, we will study equation

$$\frac{\partial^2}{\partial t^2}u(x,t) + C^2(\square)^k u(x,t) = f(x,t,u(x,t)) \tag{1.4}$$

$u(x,0) = 0$ and $\frac{\partial u(x,0)}{\partial t} = 0$

Which is in the form of nonlinear wave equation. Under certain conditions , we obtain

$$u(x,t) = E(x,t) * f(x,t,u(x,t))$$

as a unique solution of (1.4) where $E(x,t)$ is an elementary solution of (1.4).

There are a lot of problems use the ultra –hyperbolic operator, see [5], [6], [7] and [8].

2. Preliminaries:

Definition 2.1. Let $f \epsilon L_1(\mathcal{R}^n)$ – the space of integrable function in $\mathcal{R}^n$.

The Fourier transform of f(x) is defined by

$$\hat{f}(\xi) = \int_{\mathcal{R}^n} e^{-i(\xi,x)} f(x)dx \tag{2.1}$$

Where $\xi = (\xi_1, \xi_2, \dots, \xi_n), x = (x_1, x_2, \dots, x_n)\epsilon\mathcal{R}^n, (\xi, x) = \xi_1 x_1, \xi_2 x_2, \dots, \xi_n x_n$ is the inner product in $\mathcal{R}^n$ and $dx = dx_1 dx_2 \dots dx_n$.

Also, the inverse of Fourier transform is defined by

$$f(\xi) = \frac{1}{(2\pi)^n}\int_{\mathcal{R}^n} e^{i(\xi,x)}\hat{f}(x)dx \tag{2.2}$$

See [9].

Definition 2.2.Let $t > 0$ and p is a real number

$f(t) = O(t^p)$ as $t \to 0 \Leftrightarrow t^{-p}|f(t)|$ is bounded as $t \to 0$

and $f(t) = o(t^p)$ as $t \to 0 \Leftrightarrow t^{-p}|f(t)| \to 0$ as $t \to 0$

Lemma 2.3.Given the function

$$f(x) = \exp[-\sqrt{-\sum_{i=1}^{p} x_i^2 + \sum_{j=p+1}^{p+q} x_j^2}]$$

Where $(x_1, x_2, \ldots, x_n)\epsilon\mathcal{R}^n$, $p + q = n$, $\sum_{i=1}^{p} x_i^2 < \sum_{j=p+1}^{p+q} x_j^2$. Then

$$\left|\int_{\mathcal{R}^n} f(x)dx\right| \leq \frac{\Omega_p\Omega_q}{2}\frac{\Gamma(n)\Gamma(\frac{p}{2})\Gamma(\frac{2-n}{2})}{\Gamma(\frac{2-p}{2})}$$

whereΓ denotes the Gamma function. That is $\int_{\mathcal{R}^n} f(x)dx$ is bounded, (See [4]).

3. Main Results:

Lemma 3.1. Given the operator:

$$L = \frac{\partial^2}{\partial t^2} + C^2(\square)^k \tag{3.1}$$

Where $p + q = n$ is the dimensional Euclidean space$\mathcal{R}^n$, $(x_1, x_2, \ldots, x_n)\epsilon\mathcal{R}^n$, C is a positive constant, k is a non negative integer and $\square^k$ is the ultra- hyperbolic operator iterated k- times.Then we obtain

$$E(x,t) = O(\epsilon^{-n/k+1}) \tag{3.2}$$

Where $E(x,t)$is the elementary solution for the operator L defined by (3.1).

Proof: Using [3]

We have to find function $E(x,t)$ from the equation

$$L\big(E(x,t)\big) = \delta(x,t)$$

Where $\delta(x,t)$ is Dirac delta function for $(x,t)\epsilon\mathcal{R}^n X(0,\infty)$. We can also write

$$\frac{\partial^2}{\partial t^2}E(x,t) + C^2(\square)^k E(x,t) = \delta(x).\delta(t) \tag{3.3}$$

By taking the Fourier transform defined by (2.1) to both sides of (3.3), we obtain

$$\frac{\partial^2}{\partial t^2}\widehat{E}(\xi,t) + C^2\left(\left(\xi_{p+1}^2 + \xi_{p+2}^2 + \cdots + \xi_{p+q}^2\right) - \left(\xi_1^2 + \xi_2^2 + \cdots + \xi_p^2\right)\right)^k \widehat{E}(\xi,t) = \delta(t),$$

We consider also

$$\frac{\partial^2}{\partial t^2}\hat{u}(\xi,t) + C^2\left(-\xi_1^2 - \xi_2^2 - \cdots - \xi_p^2 + \xi_{p+1}^2 + \xi_{p+2}^2 + \cdots + \xi_{p+q}^2\right)^k \hat{u}(\xi,t) = 0,$$

With initial conditions

$$\hat{u}(\xi,0) = 0 \quad \text{and} \quad \frac{\partial\hat{u}}{\partial t}(\xi,0) = 1$$

And let $s > r$. Thus we have

$$\frac{\partial^2}{\partial t^2}\hat{u}(\xi,t) + C^2(s^2 - r^2)^k\hat{u}(\xi,t) = 0, \tag{3.4}$$

$$\hat{u}(\xi,0) = 0 \quad \text{and} \quad \frac{\partial \hat{u}}{\partial t}(\xi,0) = 1$$

Where $r^2 = \xi_1^2 + \xi_2^2 + \cdots + \xi_p^2$ and $\quad s^2 = \xi_{p+1}^2 + \xi_{p+2}^2 + \cdots + \xi_{p+q}^2$.

Then, we get

$$\hat{u}(\xi,t) = \frac{\operatorname{sinc}\left(\sqrt{s^2-r^2}\right)^k t}{c\left(\sqrt{s^2-r^2}\right)^k} \tag{3.5}$$

Thus(See [10]) , we have

$$\widehat{E}(\xi,t) = H\,\hat{u}(\xi,t) = H(t)\left(\frac{\operatorname{sinc}\left(\sqrt{s^2-r^2}\right)^k t}{c\left(\sqrt{s^2-r^2}\right)^k}\right) \tag{3.6}$$

Where $H(t)$is a Heaviside function.

By applying the inverse Fourier transform to (3.4), we obtain the solution $E(\xi,t)$ in the form

$$E(\xi,t) = \phi_t(x) \tag{3.7}$$

Where $\phi_t(x)$ is the inverse transform of $\widehat{\phi_t}(\xi) = \frac{\operatorname{sinc}\left(\sqrt{s^2-r^2}\right)^k t}{c\left(\sqrt{s^2-r^2}\right)^k}$

It is tempered distributions but it is not $L_1(\mathcal{R}^n)$ the space of integrable function. So we cannot compute the inverse Fourier transform $\phi_t(x)$ directly.

Thus we compute the inverse $\phi_t(x)$ by using the method of ϵ - approximation.

Let us defined

$$\widehat{\phi_t^\epsilon}(\xi) = e^{-\epsilon c\left(\sqrt{s^2-r^2}\right)^k}\widehat{\phi_t}(\xi) = e^{-\epsilon c\left(\sqrt{s^2-r^2}\right)^k}\frac{\operatorname{sinc}\left(\sqrt{s^2-r^2}\right)^k t}{c\left(\sqrt{s^2-r^2}\right)^k} \text{ for } \epsilon > 0 \tag{3.8}$$

We see that $\phi_t^\epsilon(x) \epsilon L_1(\mathcal{R}^n)$ and $\widehat{\phi_t^\epsilon}(x) \to \widehat{\phi_t}$ uniformly as $\epsilon \to 0$

So that $\phi_t(x)$ will be limit in the topology of tempered distribution of $\phi_t^\epsilon(x)$. Now

$$\phi_t^\epsilon(x) = \frac{1}{(2\pi)^n}\int_{\mathcal{R}^n} e^{i(\xi,x)}\widehat{\phi_t^\epsilon}(\xi)d\xi = \frac{1}{(2\pi)^n}\int_{\mathcal{R}^n} e^{i(\xi,x)}e^{-\epsilon c\left(\sqrt{s^2-r^2}\right)^k}\frac{\operatorname{sinc}\left(\sqrt{s^2-r^2}\right)^k t}{c\left(\sqrt{s^2-r^2}\right)^k}d\xi$$

$$|\phi_t^\epsilon(x)| \le \frac{1}{(2\pi)^n}\int_{\mathcal{R}^n}\frac{e^{-\epsilon c\left(\sqrt{s^2-r^2}\right)^k}}{c\left(\sqrt{s^2-r^2}\right)^k}d\xi \tag{3.9}$$

By changing to bipolar coordinates. Now, put

$\xi_1 = r\omega_1, \xi_2 = r\omega_2, \ldots, \xi_p = r\omega_p$ $\quad ,d\xi_1 = rd\omega_1, d\xi_2 = rd\omega_2, \ldots, d\xi_p = rd\omega_p$;

$\xi_{p+1} = s\omega_{p+1}, \xi_{p+2} = s\omega_{p+2}, \ldots, \xi_{p+q} = s\omega_{p+q}$ $,d\xi_{p+1} = sd\omega_{p+1}, d\xi_{p+2} = sd\omega_{p+2}, \ldots, d\xi_{p+q} = sd\omega_{p+q}$ and $p + q = n$.

Where $\omega_1^2 + \omega_2^2 + \cdots + \omega_p^2 = 1$ and $\omega_{p+1}^2 + \omega_{p+2}^2 + \cdots + \omega_{p+q}^2 = 1$

$$|\phi_t^\epsilon(x)| \le \frac{1}{(2\pi)^n}\int_{\mathcal{R}^n} \frac{e^{-\epsilon c\left(\sqrt{s^2-r^2}\right)^k}}{c\left(\sqrt{s^2-r^2}\right)^k} r^{p-1}s^{q-1}drdsd\Omega_p d\,\Omega_q,$$

Where $d\xi = r^{p-1}s^{q-1}drdsd\Omega_p d\Omega_q$, where $d\Omega_p$and $d\Omega_q$ are the elements of surface area on the unit sphere in $\mathcal{R}^p$ and $\mathcal{R}^q$ respectively, where$\Omega_p = \frac{2\pi^{\frac{p}{2}}}{\Gamma(\frac{p}{2})}$ and $\Omega_q = \frac{2\pi^{\frac{q}{2}}}{\Gamma(\frac{q}{2})}$.

So,

$$|\phi_t^\epsilon(x)| \le \frac{\Omega_p\Omega_q}{(2\pi)^n}\int_0^\infty\int_0^s \frac{e^{-\epsilon c\left(\sqrt{s^2-r^2}\right)^k}}{c\left(\sqrt{s^2-r^2}\right)^k} r^{p-1}s^{q-1}drds,$$

Put $r = s\sin\theta$, $dr = s\cos\theta\, d\theta$and $0 \le \theta \le \frac{\pi}{2}$,

$$|\phi_t^\epsilon(x)| \le \frac{\Omega_p\Omega_q}{(2\pi)^n}\int_0^\infty\int_0^{\frac{\pi}{2}} \frac{e^{-\epsilon c\left(\sqrt{s^2-s^2\sin\theta^2}\right)^k}}{c\left(\sqrt{s^2-s^2\sin\theta^2}\right)^k} (s\sin\theta)^{p-1}s^{q-1}s\cos\theta\, d\theta ds,$$

$$= \frac{\Omega_p\Omega_q}{c(2\pi)^n}\int_0^\infty\int_0^{\frac{\pi}{2}} \frac{e^{-\epsilon c(s\cos\theta)^k}}{(s\cos\theta)^k} (s)^{p-1}(\sin\theta)^{p-1}s^{q-1}s\cos\theta\, d\theta ds.$$

Put $y = \epsilon c(s\cos\theta)^k = \epsilon c s^k(\cos\theta)^k$, $s^k = \frac{y}{\epsilon c(\cos\theta)^k}$ $ds = \frac{dy}{cks^{k-1}\epsilon(\cos\theta)^k} = \frac{sdy}{ky}$,

Thus $|\phi_t^\epsilon(x)| \le \frac{\Omega_p\Omega_q}{c(2\pi)^n}\int_0^{\frac{\pi}{2}}\int_0^\infty \frac{e^{-y}s^{n-1}}{y/(\epsilon c)}(\sin\theta)^{p-1}\cos\theta\,\frac{s}{ky}\,dyd\theta$

$$= \frac{\Omega_p\Omega_q}{(2\pi)^n}\int_0^{\frac{\pi}{2}}\int_0^\infty \frac{e^{-y}\epsilon}{ky^2}\left(\frac{y}{c\epsilon(\cos\theta)^k}\right)^{n/k}(\sin\theta)^{p-1}\cos\theta\, dyd\theta$$

$$= \frac{\Omega_p\Omega_q}{(2\pi)^n}\int_0^{\frac{\pi}{2}}\int_0^\infty \frac{e^{-y}y^{n/k-2}}{c^{n/k}k\epsilon^{\frac{n}{k}-1}}(\sin\theta)^{p-1}(\cos\theta)^{1-n}dyd\theta$$

$$= \frac{\Omega_p\Omega_q}{(2\pi)^n}\frac{\Gamma(\frac{n}{k}-1)}{c^{n/k}k\epsilon^{\frac{n}{k}-1}}\int_0^{\frac{\pi}{2}}(\sin\theta)^{p-1}(\cos\theta)^{1-n}d\theta$$

$$= \frac{\Omega_p\Omega_q}{2c^{n/k}(2\pi)^n k\epsilon^{\frac{n}{k}-1}}\Gamma\left(\frac{n}{k}-1\right)B\left(\frac{p}{2},\frac{2-n}{2}\right).$$

$$|\phi_t^\epsilon(x)| \le \frac{\Omega_p\Omega_q}{2c^{n/k}(2\pi)^n k\epsilon^{\frac{n}{k}-1}}\frac{\Gamma\left(\frac{n}{k}-1\right)\Gamma(\frac{p}{2})\Gamma(\frac{2-n}{2})}{\Gamma(\frac{2-q}{2})}$$

Set $\quad E^\epsilon(x,t) = \phi_t^\epsilon(x) \qquad (3.10)$

Which is $\epsilon-$approximation of $E(x,t)$ in (3.10) and for $\epsilon \to 0$, $E^\epsilon(x,t) \to E(x,t)$ uniformly. Now$|E^\epsilon(x,t)| \le \frac{\Omega_p\Omega_q}{2c^{n/k}(2\pi)^n k\epsilon^{\frac{n}{k}-1}}\frac{\Gamma\left(\frac{n}{k}-1\right)\Gamma(\frac{p}{2})\Gamma(\frac{2-n}{2})}{\Gamma(\frac{2-q}{2})}$

$$\epsilon^{\frac{n}{k}-1}|\,E^{\epsilon}(x,t)| \leq \frac{\Omega_p\Omega_q}{2kc^{n/k}(2\pi)^n}\frac{\Gamma\left(\frac{n}{k}-1\right)\Gamma\left(\frac{p}{2}\right)\Gamma\left(\frac{2-n}{2}\right)}{\Gamma\left(\frac{2-q}{2}\right)}$$

$$\lim_{\epsilon\to 0}\epsilon^{\frac{n}{k}-1}|E^{\epsilon}(x,t)| \leq \frac{\Omega_p\Omega_q}{2(2\pi)^{\frac{n}{2}}kc^{n/k}}\frac{\Gamma\left(\frac{n}{k}\right)\Gamma\left(\frac{p}{2}\right)\Gamma\left(\frac{2-n}{2}\right)}{\Gamma\left(\frac{2-q}{2}\right)} = K.$$

$$E^{\epsilon}(x,t) = O(\epsilon^{-n/k+1})$$

It follows that $E(x,t) = O(\epsilon^{-n/k+1})$ as $\epsilon \to 0$.where $E(x,t)$ is an elementary solution.

Corollary 3.2.

$$|E| \leq \frac{2^{2-n}M(t)}{\pi^{\frac{n}{2}}\Gamma(\frac{p}{2})\Gamma(\frac{q}{2})}, \tag{3.11}$$

$$M(t) = \int_0^\infty \int_0^s \frac{\operatorname{sinc}\left(\sqrt{s^2-r^2}\right)^k t}{c\left(\sqrt{s^2-r^2}\right)^k} r^{p-1}s^{q-1}drds \tag{3.12}$$

Proof: By applying the inverse of Fourier transform to (3.6), we obtain the solution $E(x,t)$ in the form

$$E(x,t) = \frac{1}{(2\pi)^n}\int_{\mathcal{R}^n} e^{i(\xi,x)}\widehat{E}(\xi,t)d\xi$$

$$|E(x,t)| \leq \frac{1}{(2\pi)^n}\int_{\mathcal{R}^n} \frac{\operatorname{sinc}\left(\sqrt{s^2-r^2}\right)^k t}{c\left(\sqrt{s^2-r^2}\right)^k}d\xi$$

By changing to bipolar coordinates, we get

$$|E| \leq \frac{1}{(2\pi)^n}\int_{\mathcal{R}^n} \frac{\operatorname{sinc}\left(\sqrt{s^2-r^2}\right)^k t}{c\left(\sqrt{s^2-r^2}\right)^k} r^{p-1}s^{q-1}drdsd\Omega_p d\,\Omega_q,$$

Where $d\xi = r^{p-1}s^{q-1}drdsd\Omega_p d\Omega_q$, $d\Omega_p$and $d\Omega_q$are the elements of surface area on the unit sphere in $\mathcal{R}^p$ and $\mathcal{R}^q$ respectively, we have

$$|E(x,t)| \leq \frac{\Omega_p\Omega_q}{(2\pi)^n}\int_0^\infty \int_0^s \frac{\operatorname{sinc}\left(\sqrt{s^2-r^2}\right)^k t}{c\left(\sqrt{s^2-r^2}\right)^k} r^{p-1}s^{q-1}drds = \frac{\Omega_p\Omega_q}{(2\pi)^n}M(t)$$

where$\Omega_p = \frac{2\pi^{\frac{p}{2}}}{\Gamma(\frac{p}{2})}$ and $\Omega_q = \frac{2\pi^{\frac{q}{2}}}{\Gamma(\frac{q}{2})}$,

Thus

$$|E(x,t)| \leq \frac{2^{2-n}M(t)}{\pi^{\frac{n}{2}}\Gamma\left(\frac{p}{2}\right)\Gamma\left(\frac{q}{2}\right)}$$

Theorem 3.3. *given the nonlinear equation*

$$\frac{\partial^2}{\partial t^2}u(x,t) + C^2(\Box)^k u(x,t) = f(x,t,u(x,t)) \tag{3.13}$$

$$u(x,0) = 0 \text{ and } \frac{\partial u(x,0)}{\partial t} = 0$$

for $(x,t)\epsilon\mathcal{R}^n X(0,\infty)$, k is a positive number and with the following conditions on u and f as follows:

(1) $u(x,t)$is the space of function on $\mathcal{R}^n X(0,\infty)$.

(2) f satisfies the Lipchitz condition,

$$|f(x,t,u) - f(x,t,w)| \le A|u - w| \quad ,$$

for some constant $A > 0$.

(3) $\int_0^\infty \int_{\mathcal{R}^n} |f(x,t,u(x,t))|\, dxdt < \infty$ for $x = (x_1, x_2, \dots, x_n)\epsilon\mathcal{R}^n$,

$0 < t < \infty$ and $u(x,t)$ is a function on $\mathcal{R}^n X(0,\infty)$.

Then we obtain

$$u(x,t) = E(x,t) * f\big(x,t,u(x,t)\big). \tag{3.14}$$

as a unique solution of (3.13) for $x \,\epsilon\, \Omega_0$ is a compact subset of $\mathcal{R}^n$ and $0 \le t \le T$ with T is constant and $E(x,t)$ is an elementary solution defined by (3.3) and also $u(x,t)$ is bounded for any fixed $t > 0$.

Proof: convolving both sides of (1.4) with $\mathrm{E(x,t)}$, that is

$$\mathrm{E(x,t)} * [\frac{\partial^2}{\partial \mathrm{t}^2}\mathrm{u(x,t)} + \mathrm{C}^2(\Box)^\mathrm{k}\mathrm{u(x,t)}] = \mathrm{E(x,t)} * \mathrm{f(x,t,u(x,t))}$$

Or

$$[\frac{\partial^2}{\partial \mathrm{t}^2}\mathrm{E(x,t)} + \mathrm{C}^2(\Box)^\mathrm{k}\mathrm{E(x,t)}] * \mathrm{u(x,t)} = \mathrm{E(x,t)} * \mathrm{f(x,t,u(x,t))}$$

So

$$\delta(\mathrm{x,t}) * \mathrm{u(x,t)} = \mathrm{E(x,t)} * \mathrm{f}\big(\mathrm{x,t,u(x,t)}\big).$$

Thus

$$\mathrm{u(x,t)} = \mathrm{E(x,t)} * \mathrm{f}\big(\mathrm{x,t,u(x,t)}\big) = \int_{-\infty}^{\infty}\int_{\mathcal{R}^\mathrm{n}} \mathrm{E(r,s)f}\big(\mathrm{x-r,t-s,u(x-r,t-s)}\big)\mathrm{drds}$$

We next show that $u(x,t)$ is bounded on $\mathcal{R}^n X(0,\infty)$. Using (3.11), we have,

$$|u(x,t)| \leq \int_{-\infty}^{\infty}\int_{\mathcal{R}^n}|E(r,s)|\left|f\big(x-r,t-s,u(x-r,t-s)\big)\right|drds$$

$$\leq \frac{2^{2-n}M(t)N}{\pi^{\frac{n}{2}}\Gamma(\frac{p}{2})\Gamma(\frac{q}{2})}$$

Where $M(t)$ was defined in (3.12) and

$$N = \int_{-\infty}^{\infty}\int_{\mathcal{R}^n}\left|f\big(x-r,t-s,u(x-r,t-s)\big)\right| drds,$$

Thus $u(x,t)$ is bounded on $\mathcal{R}^n X(0,\infty)$.

We next show that $u(x,t)$ is unique. Let $w(x,t)$ be another solution of (1.4), then

$$w(x,t) = E(x,t) * f\big(x,t,w(x,t)\big)$$

for $(x,t)\epsilon\Omega_0 X(0,T]$ the compact subset of $\mathcal{R}^n X[0,\infty)$

and $E(x,t)$ is defined in (3.3). Now define

$$\|u(x,t)\| = \sup_{\substack{x\epsilon\Omega_0 \\ 0<t\leq T}} |u(x,t)|.$$

So ,

$$|u(x,t)-w(x,t)| = \left|E(x,t) * f\big(x,t,u(x,t)\big) - E(x,t) * f\big(x,t,w(x,t)\big)\right|$$

$$\leq \int_{-\infty}^{\infty}\int_{\mathcal{R}^n}|E(r,s)|.\left|f\big(x-r,t-s,u(x-r,t-s)\big) - f\big(x-r,t-s,w(x-r,t-s)\big)\right|drds$$

$$\leq A|E(r,s)|\int_{-\infty}^{\infty}\int_{\mathcal{R}^n}|u(x-r,t-s) - w(x-r,t-s)|drds$$

By the condition (2) on f, and for $(x,t)\epsilon\Omega_0 X(0,T]$

we have

$$|u-w| \leq A|E(r,s)|\|u-w\|\int_0^T ds\int_{\Omega_0} dr$$

$$= A|E(r,s)|T\,V(\Omega_0)\|u-w\| \qquad (3.15)$$

where $V(\Omega_0)$ is the volume of the surface on Ω_0. Choose

$A < \frac{1}{|E(r,s)|TV(\Omega_0)}$. Thus from (3.15)

$\|u - w\| \leq \alpha\|u - w\|$ where $\alpha = A|E(r,s)|TV(\Omega_0) < 1$.

It follows that $\|u - w\| = 0$, thus $u = w$.

That is the solution u of (3.13) is unique for $(x,t) \in \Omega_0 X(0,T]$ where $u(x,t)$ is defined by (3.14).

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