Communications in Applied Sciences

ISSN 2201-7372

Volume 4, Number 2, 2016, 49-66

Numerical Solution of Fredholm Integral Equations of Second Kind using Haar Wavelets

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Abstract: Integral equations have been one of the most important tools in several areas of science and engineering. In this paper, we use Haar wavelet method for the numerical solution of one-dimensional and two-dimensional Fredholm integral equations of second kind. The basic idea of Haar wavelet collocation method is to convert the integral equation into a system of algebraic equations that involves a finite number of variables. The numerical results are compared with the exact solution to prove the accuracy of the Haar wavelet method.

Keywords: Integral equation, one-dimensional Fredholm integral equation of second kind, two-dimensional Fredholm integral equation of second kind, Haar wavelets, collocation points.

Mathematics Subject Classification: 65T60

1 Introduction

Fredholm integral equations are frequently encountered in many physical processes such as dynamic stiffness of rigid rectangular foundations, soil mechanics and rock mechanics, diffraction of waves by randomly rough surface in two dimensions, thermoelasticity, and scattering problem. Rahbar and Hashemizadeh [1] used modified quadrature method and Chen et. al. [2] applied discrete multi-projection method to solve Fredholm integral equations of second kind. Ray and Sahu [3] have discussed several numerical methods like B-spline wavelet method, method of moments, variational iteration method, quadrature method and so on, and Long and Nelakanti [4] used iteration methods like iterative Galerkin method, iterative projection method, iterative degenerate kernel method and so on for the solution of Fredholm integral equations of second kind. Maleknejad et. al. [5, 6] have determined the numerical solution of integral equations of the second kind by block-pulse functions and Taylor-series expansion method. Babolian and Fattahzadeh [7] have used Chebyshev wavelet operational matrix of integration to solve integral equations. Cattani and Kudreyko [8] used harmonic wavelet method for the solution of Fredholm type integral equations of the second kind. Yousefi and Banifatemi [9] have solved Fredholm integral equations by using CAS wavelets.

Two-dimensional Fredholm integral equations were solved using differential transform method by Ziyaee and Tari [10]. Lin [11] used wavelet based methods for numerical solutions of two dimensional integral equations. Alipanah and Esmaeili [12] applied Gaussian radial basis function for the numerical solution of the two-dimensional Fredholm integral equations. Fallahzadeh [13] solved two-dimensional Fredholm integral equation via RBF-triangular method. Tari and Shahmorad [14] determined a computational method for solving two dimensional linear Fredholm integral equations of the second kind. Mirzaee and Piroozfar [15] numerically solved linear two-dimensional Fredholm integral equations of the second kind via two-dimensional triangular orthogonal functions.

In the recent years, wavelets have been widely used to solve integral and differential equations. Sumana and Achala [17] have given a brief report on Haar wavelets. The main idea of Haar wavelet approach is to convert the integral equation into a system of algebraic equations that involves a finite number of variables which can be handled very conveniently. The paper is organized as follows. The Haar wavelet preliminaries and the function approximation are presented in Section 2. The method of solution of the one-dimensional and two-dimensional Fredholm integral equations using Haar wavelets is proposed in Section 3. The numerical examples and discussions are presented in Section 4. The conclusions drawn are presented in Section 5.

2 Haar Wavelets

The Haar wavelet family for $x \in [0,1]$ is defined [16] as follows

$$h_i(x) = \begin{cases} 1 & \text{for} x \in [\xi_1, \xi_2) \\ -1 & \text{for} x \in [\xi_2, \xi_3) \\ 0 & \text{elsewhere} \end{cases} \tag{1}$$

where

$$\xi_1 = \frac{k}{m}, \xi_2 = \frac{k+0.5}{m}, \xi_3 = \frac{k+1}{m}. \tag{2}$$

Here $m = 2^b, b = 0,1,\ldots,J$ indicates the level of the wavelet; $k = 0,1,\ldots,m-1$ is the translation parameter. J is the maximum level of resolution. The index i in equation

(1) is calculated by the formula $i = m + k + 1$. In the case of minimum values $m = 1, k = 0$ we have $i = 2$. The maximum value of i is $i = 2M = 2^{J+1}$. For $i = 1$, $h_1(x)$ is assumed to be the scaling function which is defined as follows.

$$h_1(x) = \begin{cases} 1 & \text{for} x \in [0,1) \\ 0 & \text{elsewhere} \end{cases} \tag{3}$$

Any function $f(x)$ defined on $[0,1]$ can be expressed in terms of Haar wavelets as follows.

$$f(x) = \sum_{i=1}^{2M} a_i h_i(x), \tag{4}$$

where the wavelet coefficients $a_i, i = 1,2,\ldots,2M$ are to be determined.

Any function $f(x, y)$ defined on $[0,1]\times[0,1]$ can be expressed in terms of Haar wavelets as follows.

$$f(x, y) = \sum_{i=1}^{2M} \sum_{j=1}^{2M^*} a_{ij} h_i(x) h_j(y), \tag{5}$$

where the wavelet coefficients $a_{ij}, i = 1,2,\ldots,2M, j = 1,2,\ldots,2M^*$ are to be determined. Here, $h_j(y)$ is defined as follows.

$$h_j(y) = \begin{cases} 1 & \text{for} y \in [\eta_1, \eta_2) \\ -1 & \text{for} y \in [\eta_2, \eta_3) \\ 0 & \text{elsewhere} \end{cases} \tag{6}$$

where

$$\eta_1 = \frac{k^*}{m^*}, \eta_2 = \frac{k^* + 0.5}{m^*}, \eta_3 = \frac{k^* + 1}{m^*}. \tag{7}$$

Here $m^* = 2^{b^*}, b^* = 0,1,\ldots, J^*$ indicates the level of the wavelet; $k^* = 0,1,\ldots, m^* - 1$ is the translation parameter. J^* is the maximum level of resolution. The index j in equation (6) is calculated by the formula $j = m^* + k^* + 1$. In the case of minimum values $m^* = 1, k^* = 0$ we have $j = 2$. The maximum value of j is $j = 2M^* = 2^{J^*+1}$.

3 Method of Solution

3.1 One-dimensional Fredholm Integral Equation

The one-dimensional Fredholm integral equation of the second kind is given by

$$u(x) = f(x) + \int_0^1 k(x,t)u(t)\,dt \tag{8}$$

where $f(x)$ is a known function, $k(x,t)$ is the kernel and $u(x)$ is the unknown function.

Let the Haar wavelet solution be in the form

$$u(x) = \sum_{i=1}^{2M} a_i h_i(x) \tag{9}$$

Substituting equation (9) in equation (8) gives

$$\sum_{i=1}^{2M} a_i \{h_i(x) - G_i(x)\} = f(x) \tag{10}$$

where

$$G_i(x) = \int_0^1 k(x,t)h_i(t)\,dt \tag{11}$$

The wavelet collocation points are defined as

$$x_l = \frac{l - 0.5}{2M}, l = 1,2,\ldots,2M. \tag{12}$$

Taking the collocation points $x \to x_l$ in equations (10) and (9), we obtain

$$\sum_{i=1}^{2M} a_i \{h_i(x_l) - G_i(x_l)\} = f(x_l) \tag{13}$$

$$u(x_l) = \sum_{i=1}^{2M} a_i h_i(x_l) \tag{14}$$

The wavelet coefficients $a_i, i = 1,2,\ldots,2M$ are obtained by solving the $2M$ system of equations in (13). These coefficients are then substituted in equation (14) to obtain the Haar wavelet solution at the collocation points $x_l, l = 1,2,\ldots,2M$.

3.2 Two-dimensional Fredholm Integral Equation

The two-dimensional Fredholm integral equation of the second kind is given by

$$u(x,y) = f(x,y) + \int_0^1\int_0^1 k(x,y,t,s)u(t,s)\,dt\,ds \tag{15}$$

where $f(x,y)$ is a known function, $k(x,y,t,s)$ is the kernel and $u(x,y)$ is the unknown function.

Let the Haar wavelet solution be in the form

$$u(x,y) = \sum_{i=1}^{2M}\sum_{j=1}^{2M^*} a_{ij} h_i(x) h_j(y) \tag{16}$$

Substituting equation (16) in equation (15) gives

$$\sum_{i=1}^{2M}\sum_{j=1}^{2M^*} a_{ij} \{h_i(x)h_j(y) - G_{ij}(x,y)\} = f(x,y) \tag{17}$$

where

$$G_{ij}(x,y)=\int_0^1\int_0^1 k(x,y,t,s)h_i(t)h_j(s)\,dt\,ds \tag{18}$$

The wavelet collocation points are defined as

$$x_l=\frac{l-0.5}{2M},\ l=1,2,\ldots,2M. \tag{19}$$

$$y_n=\frac{n-0.5}{2M^*},\ n=1,2,\ldots,2M^*. \tag{20}$$

Taking the collocation points $x\to x_l,\ \ y\to y_n$ in equations (17) and (16), we obtain

$$\sum_{i=1}^{2M}\sum_{j=1}^{2M^*}a_{ij}\{h_i(x_l)h_j(y_n)-G_{ij}(x_l,y_n)\}=f(x_l,y_n) \tag{21}$$

$$u(x_l,y_n)=\sum_{i=1}^{2M}\sum_{j=1}^{2M^*}a_{ij}h_i(x_l)h_j(y_n) \tag{22}$$

The wavelet coefficients $a_{ij}, i=1,2,\ldots,2M, j=1,2,\ldots,2M^*$ are obtained by solving the $2M*2M^*$ system of equations in (21). These coefficients are then substituted in equation (22) to obtain the Haar wavelet solution at the collocation points (x_l,y_n), $l=1,2,\ldots,2M$, $n=1,2,\ldots,2M^*$.

4 Numerical Examples and Discussion

In this section, examples are considered to check the efficiency and accuracy of the Haar wavelet collocation method (HWCM). Lagrange interpolation (in the case of one-dimensional Fredholm integral equation) and Lagrange bivariate interpolation (in the case two-dimensional of Fredholm integral equation) is used to find the solution at the specified points. The entire computational work has been done with the help of

MATLAB software.

Example 1:

$$u(x) = \frac{5}{6}x + \frac{1}{2}\int_0^1 xtu(t)\,dt \tag{23}$$

The exact solution is

$$u(x) = x \tag{24}$$

Solving equation (11) for $G_i(x)$, we obtain

$$G_i(x) = \begin{cases} \frac{1}{4}x & \text{for} i = 1 \\ \frac{1}{4}(2\xi_2^2 - \xi_1^2 - \xi_3^2)x & \text{for} i > 1 \end{cases} \tag{25}$$

The HWCM solution of the example with $J = 4$ in Table 1 and Figure 1. The results are compared with the exact solution. If $u_{ex}(x)$ is the exact solution (24), we define the error estimate as

$$\sigma = \frac{1}{2M}\left\| u(x) - u_{ex}(x) \right\| \tag{26}$$

We have obtained the following error estimates. [label=()]

1. $\sigma = 2.8172E-05$ in L_2 space and $\sigma = 1.1823E-05$ in L_∞ space for $J = 3$.
2. $\sigma = 4.9826E-06$ in L_2 space and $\sigma = 1.5020E-06$ in L_∞ space for $J = 4$.
3. $\sigma = 8.8093E-07$ in L_2 space and $\sigma = 1.8924E-07$ in L_∞ space for $J = 5$.

Example 2:

$$u(x) = \frac{3}{2}e^x - \frac{1}{2}xe^x - \frac{1}{2} + \frac{1}{2}\int_0^1 tu(t)\,dt \tag{27}$$

The exact solution is

$$u(x)=\frac{3}{2}e^{x}-\frac{1}{2}xe^{x}-\frac{1}{3}e+1 \tag{28}$$

Solving equation (11) for $G_i(x)$, we have

$$G_i(x)=\begin{cases}\dfrac{1}{4} & \text{for} i=1\\ \dfrac{1}{4}(2\xi_2^2-\xi_1^2-\xi_3^2) & \text{for} i>1\end{cases} \tag{29}$$

The HWCM solution of the example with $J=5$ in Table 2 and Figure 2. The results are compared with the exact solution. We have obtained the following error estimates. [label=()]

1. $\sigma=6.9926E-05$ in L_2 space and $\sigma=1.7481E-05$ in L_∞ space for $J=3$.
2. $\sigma=1.2360E-05$ in L_2 space and $\sigma=2.1850E-06$ in L_∞ space for $J=4$.
3. $\sigma=2.1849E-06$ in L_2 space and $\sigma=2.7312E-07$ in L_∞ space for $J=5$.

Example 3:

$$u(x,y)=xy-\frac{1}{4}(x+y)-\frac{1}{3}+\int_0^1\int_0^1(x+y+t+s)u(t,s)\,dt\,ds \tag{30}$$

The exact solution is

$$u(x,y)=xy \tag{31}$$

Solving equation (18) for $G_{ij}(xy)$, we obtain

$$G_{ij}(x,y)=\begin{cases} x+y+1 & \text{for} i=1,\ j=1 \\ \frac{1}{2}(2\eta_2^2-\eta_1^2-\eta_3^2) & \text{for} i=1,\ j>1 \\ \frac{1}{2}(2\xi_2^2-\xi_1^2-\xi_3^2) & \text{for} i>1,\ j=1 \\ 0 & \text{for} i>1,\ j>1 \end{cases} \tag{32}$$

The HWCM solution of the example with $J = J^* = 4$ are given in Table 3 and Figure 3. The results are compared with the exact solution. Figure 4 shows the physical behavior of the HWCM solution in contour and 3D. If $u_{ex}(x,y)$ is the exact solution (31), we define the error estimate as

$$\mu=\frac{1}{2M_1 2M_2}\left\|u(x,y)-u_{ex}(x,y)\right\| \tag{33}$$

We have obtained the following error estimates. [label=()]

1. $\mu = 1.5048E-04$ in L_2 space and $\mu = 2.0099E-04$ in L_∞ space for $J = J^* = 2$.
2. $\mu = 1.8793E-05$ in L_2 space and $\mu = 2.5627E-05$ in L_∞ space for $J = J^* = 3$.
3. $\mu = 2.3486E-06$ in L_2 space and $\mu = 3.2361E-06$ in L_∞ space for $J = J^* = 4$.

Example 4:

$$u(x,y)=\frac{1}{1+x+y}-\frac{x}{1+y}+\int_0^1\int_0^1\frac{x}{1+y}(1+t+s)u(t,s)\,dt\,ds \tag{34}$$

The exact solution is

$$u(x,y)=\frac{1}{1+x+y} \tag{35}$$

Solving equation (18) for $G_{ij}(x,y)$, we have

$$G_{ij}(x,y)=\begin{cases}\dfrac{2x}{1+y} & \text{for} i=1,\, j=1\\ \dfrac{(2\eta_2^2-\eta_1^2-\eta_3^2)x}{2(1+y)} & \text{for} i=1,\, j>1\\ \dfrac{(2\xi_2^2-\xi_1^2-\xi_3^2)x}{2(1+y)} & \text{for} i>1,\, j=1\\ 0 & \text{for} i>1,\, j>1\end{cases} \tag{36}$$

The HWCM solution of the example with $J=J^*=2$ are given in Table 4 and Figure 5. The results are compared with the exact solution. Figure 6 shows the physical behavior of the HWCM solution in contour and 3D. We have obtained the following error estimates. [label=()]

1. $\mu=4.1297E-17$ in L_2 space and $\mu=5.5511E-17$ in L_∞ space for $J=J^*=1$.
2. $\mu=3.9492E-17$ in L_2 space and $\mu=5.3776E-17$ in L_∞ space for $J=J^*=2$.
3. $\mu=3.2655E-17$ in L_2 space and $\mu=6.2450E-17$ in L_∞ space for $J=J^*=3$.

5 Conclusion

In this paper, an efficient numerical scheme based on uniform Haar wavelets is used to solve one-dimensional and two-dimensional Fredholm integral equations. The numerical scheme is tested for four examples. The obtained numerical results are compared with the exact solutions. We observe that the error values are negligibly small which indicate that the HWCM solution is very close to the exact solution. Thus the Haar wavelet method guarantees the necessary accuracy with a small number of grid points and a wide class of integral equations can be solved using this approach.

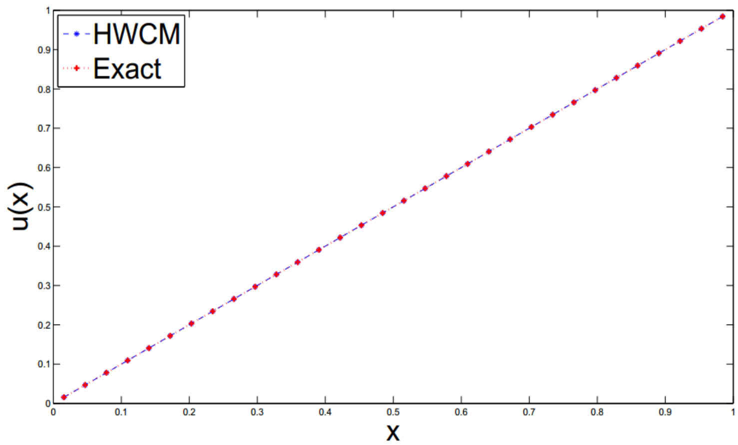

Figure 1: Comparison of the HWCM solution and exact solution of Example 1

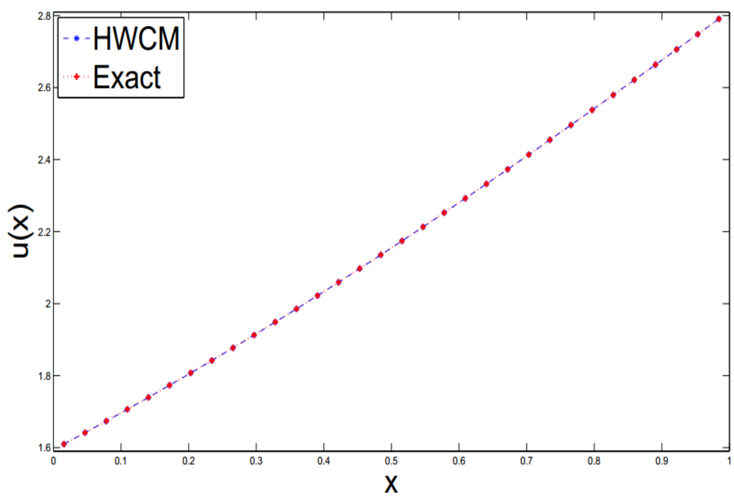

Figure 2: Comparison of the HWCM solution and exact solution of Example 2

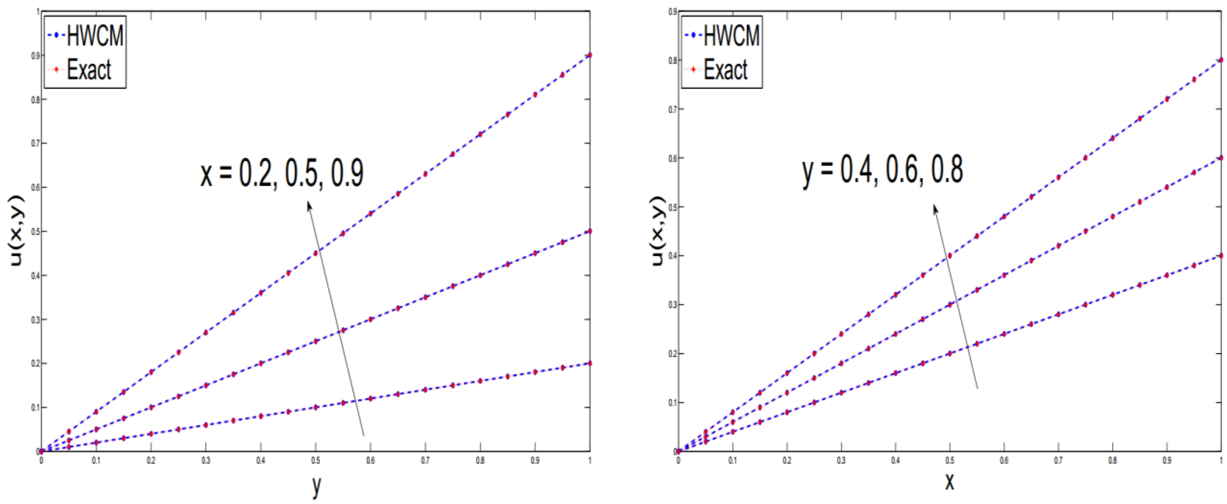

Figure 3: Comparison of the HWCM solution and exact solution of Example 3

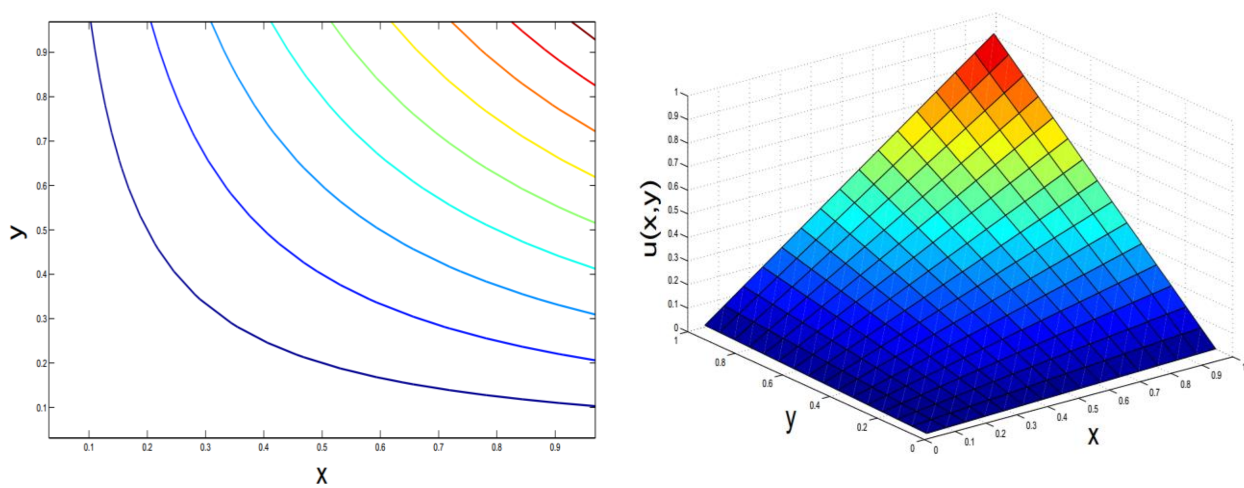

Figure 4: Physical behaviour of the HWCM solution of Example 3

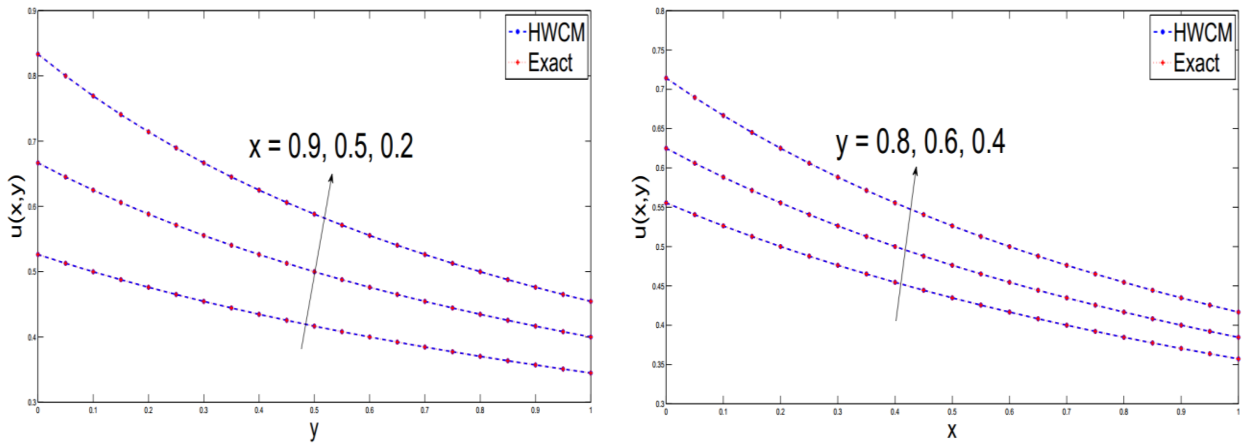

Figure 5: Comparison of the HWCM solution and exact solution of Example 4

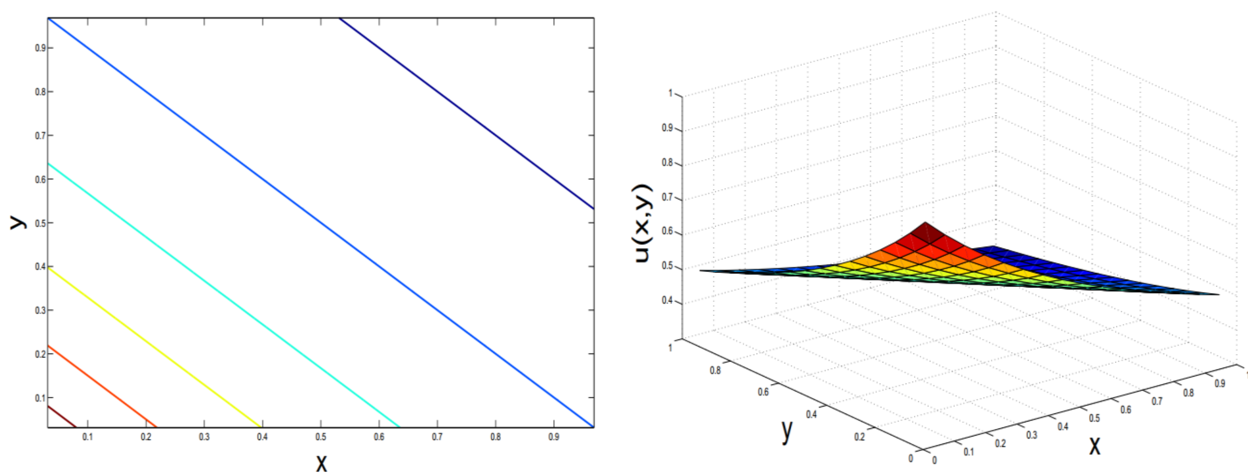

Figure 6: Physical behaviour of the HWCM solution of Example 4

Table 1: Comparison of the HWCM solution and exact solution of Example 1

x	$u(x)$	
	HWCM	Exact
0.1	0.09999512	0.10000000
0.2	0.19999023	0.20000000
0.3	0.29998535	0.30000000
0.4	0.39998047	0.40000000
0.5	0.49997559	0.50000000
0.6	0.59997070	0.60000000
0.7	0.69996582	0.70000000
0.8	0.79996094	0.80000000
0.9	0.89995606	0.90000000

Table 2: Comparison of the HWCM solution and exact solution of Example 2

x	$u(x)$	
	HWCM	Exact
0.1	1.69638676	1.69640389
0.2	1.80385244	1.80386992
0.3	1.91619797	1.91621545
0.4	2.03326068	2.03327816
0.5	2.15479017	2.15480765
0.6	2.28043114	2.28044862
0.7	2.40970419	2.40972167
0.8	2.54198360	2.54200108
0.9	2.67647185	2.67648932

Table 3: Comparison of the HWCM solution and exact solution of Example 3

(x, y)	$u(x, y)$		(x, y)	$u(x, y)$	
	HWCM	Exact		HWCM	Exact
(0.1,0.2)	0.02002093	0.02000000	(0.5,0.6)	0.30007674	0.30000000
(0.1,0.4)	0.04003488	0.04000000	(0.5,0.8)	0.40009069	0.40000000
(0.1,0.6)	0.06004883	0.06000000	(0.7,0.2)	0.14006279	0.14000000
(0.1,0.8)	0.08006279	0.08000000	(0.7,0.4)	0.28007674	0.28000000
(0.3,0.2)	0.06003488	0.06000000	(0.7,0.6)	0.42009069	0.42000000
(0.3,0.4)	0.12004883	0.12000000	(0.7,0.8)	0.56010465	0.56000000
(0.3,0.6)	0.18006279	0.18000000	(0.9,0.2)	0.18007674	0.18000000
(0.3,0.8)	0.24007674	0.24000000	(0.9,0.4)	0.36009069	0.36000000
(0.5,0.2)	0.10004883	0.10000000	(0.9,0.6)	0.54010465	0.54000000
(0.5,0.4)	0.20006279	0.20000000	(0.9,0.8)	0.72011860	0.72000000

Table 4: Comparison of the HWCM solution and exact solution of Example 4

(x, y)	$u(x, y)$		(x, y)	$u(x, y)$	
	HWCM	Exact		HWCM	Exact
(0.1,0.2)	0.76923117	0.76923077	(0.5,0.6)	0.47619048	0.47619048
(0.1,0.4)	0.66666687	0.66666667	(0.5,0.8)	0.43478260	0.43478261
(0.1,0.6)	0.58823540	0.58823529	(0.7,0.2)	0.52631577	0.52631579
(0.1,0.8)	0.52631577	0.52631579	(0.7,0.4)	0.47619047	0.47619048
(0.3,0.2)	0.66666663	0.66666667	(0.7,0.6)	0.43478261	0.43478261
(0.3,0.4)	0.58823530	0.58823529	(0.7,0.8)	0.40000000	0.40000000
(0.3,0.6)	0.52631580	0.52631579	(0.9,0.2)	0.47619077	0.47619048
(0.3,0.8)	0.47619046	0.47619048	(0.9,0.4)	0.43478272	0.43478261
(0.5,0.2)	0.58823526	0.58823529	(0.9,0.6)	0.40000004	0.40000000
(0.5,0.4)	0.52631579	0.52631579	(0.9,0.8)	0.37037039	0.37037037

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