Communications in Applied Sciences
ISSN 2201-7372
Volume 2, Number 1, 2014, 65-84

On the Comparative Study of Some Numerical Methods for Vanilla Option Valuation

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Abstract. This paper presents some numerical methods for vanilla option valuation namely binomial tree model, Crank Nicolson method and Monte Carlo method. Binomial model is widely used in the finance community for numerical valuation of a wide variety of option models, due primarily to its ease of implementation and pedagogical appeal. Crank Nicolson approach seeks the discretization of the differential operators in the continuous Black Scholes model. Monte Carlo method simulates the random movement of the asset prices and provides a probabilistic solution to the option pricing models. We discuss the strengths, drawbacks and the performance of the methods under consideration. However, binomial model is the most accurate and converges faster than its two counterparts; Crank Nicolson method and Monte Carlo method.

2010 Mathematics Subject Classification: 41A25, 65C05, 65L12, 91G20, 91G60, 91G80.

Keywords: Binomial model, Crank Nicolson method, European option, Finite difference method, Monte Carlo method, Vanilla option

1 Introduction

In the last few years, financial derivatives, in particular options, become very popular financial contracts. A financial derivative is a financial asset whose value is derived in part from the value and characteristics of some other underlying assets. Options can be used for instance, to hedge assets and portfolios in order to control the risk due to movements in the share price. Options fall in two classes namely put and call. A vanilla option is a financial instrument that gives the right, but not obligation, to buy or sell an underlying asset at a predetermined price, within a given time frame. A vanilla option is a normal call and put option that has standardized terms and no special or unusual features. It is generally traded on an exchange such as the Chicago Board Options Exchange (CBOE). Examples are American and European options. The derivative of interest in this paper is an European option. European option contract is an agreement between two parties that gives one party the right to buy or sell the underlying asset at some specified date in the future for a fixed price. This right has a value and the party that provides the option will demand compensation at the inception of the option. Further, once initiated, options can be traded on an exchange much like the underlying asset.

A European call option on a stock gives the buyer the right but not the obligation to buy a number of shares of stocks (the underlying asset for stock options, S) for a specified price (the exercise or strike price, K) at a specified date, T in the future. In general, it seems clear that the higher the price of the stock, the greater the value of the option. When the stock price is much greater than the exercise price, the call option is almost sure to be exercised "in the money". If the price of the underlying asset is less than the exercise price, the call option is almost sure to expire without being exercised, so its value will be zero "out of the money" whereas a call option with an exercise price is equal to the price of underlying asset then the call option is "at the money" [9]. The value of the call option is thus a function of S and time and its payoff, which describes the option's value ate date T, is given by

$$C(S,T) = \max(S-K,0) \tag{1.1}$$

A European put option gives the buyer the right to sell a number of shares of stock for a specified time. The payoff which describes a put option's value at time T is given by

$$P(S,T) = \max(K-S,0) \tag{1.2}$$

If the expiration date of the option is very far in the future, then the price of a bond that pays the exercise price on the maturity date will be very low, and the value of the option will be approximately equal to the price of the stock [1]. On the other hand, if the expiration date is very near, the value of the option will be approximately equal to the stock price minus the exercise price or, zero, if the stock price is less than exercise price. Normally the value of an option declines as its maturity date approaches, if the value of the stock does not change [1].

European options are some of the simplest financial derivatives. They are interesting, because their valuation proved to be difficult until 1973. Before that there was no generally accepted model that could give option's trader the value of an option before expiry, the answer was provided by solving Black Scholes equation. In an idealized financial market the price of a European option can be obtained as the solution of the celebrated Black Scholes equation [1]. This equation provides a hedging portfolio that replicates the contingent claim.

One of the major contributors to the world of finance were Black and Scholes [1]. They ushered in the modern era of derivative securities with a seminar paper titled "Pricing and Hedging of European call and put options". In this paper, the famous Black-Scholes formula made its debut and the It$\hat{o}$ calculus was applied to finance. Later Merton (1976) proposed a jump-diffusion model. [6] derived the tree methods of pricing options based on risk-neutral valuation, the binomial option pricing European option prices under various alternatives, including the absolute diffusion, pre-jump and square root constant elasticity of variance methods [12]. J. Hull and A. White [13] used explicit finite difference method for the valuation of derivatives securities. P. Boyle [2] introduced a Monte Carlo approach for pricing options. Twenty years later, P. Boyle et al [3] described research advances that had improved efficiency and broadened the types of problem where simulation can be applied. M. Brennan and E. Schwartz [4] considered a finite difference methods for pricing American options for the Black-Scholes leading to one dimensional parabolic partial differential inequality. D. Tavella and C. Randall [18] considered pricing of options using finite differences for the space derivatives and a slightly stabilized Crank Nicolson method for the time derivative.

The comparative study of finite difference method and Monte Carlo method for pricing European option was considered by [8]. Some numerical methods for options valuation was considered by [16]. Later C. R. Nwozo and S. E. Fadugba [15] considered Monte Carlo method for pricing some path dependent options. The effect of volatility on binomial model for the valuation of American options was considered by [7].

A. Brick [5] had shown that geometric (exponential) Brownian motion could indeed be justified as the rational expectations equilibrium in a market with homogeneous agents. After F. Black and M. Scholes [1], a significant plateau has been reached by many authors in the modelling of stock price dynamics. J. Hull [12] and E. Stein et al [17] among others followed the traditional approach to pricing options on stocks with stochastic volatility, which start by specifying the joint process for the stock price and its volatility risk. Their models are typically calibrated to the prices of a few options or estimated from the time series of stock prices.

In this paper we shall consider only the Matlab implementation and the comparative results analysis of binomial model, Crank Nicolson method and Monte Carlo method for European options.

1.1 The Black Scholes Equation

Speaking of the continuous-time model to price stock options, few can ignore the fundamental contribution that F. Black and M. Scholes [1] made in the early 1970s. They developed their European option pricing model under the assumption of the lognormal dynamics of derivatives. In its simplest form, the Black-Scholes (Merton) model equation is a linear partial differential equation with variable coefficient.

1.1.1 Assumptions

The assumptions used to derive the Black-Scholes partial differential equation are as follows:

- The stock price follows the geometric Brownian motion with μ and σ constants.
- The short selling of securities with full use of proceeds is permitted.
- There are no transaction costs or taxes; all securities are perfectly divisible.
- The underlying stock does not pay dividends.
- There are no arbitrage opportunities. That is, it is not possible to make risk free investments with a return greater than the risk free rate.
- The underlying asset trading is continuous and the change of its price is continuous.
- The risk-free rate of interest, r, is constant and the same for all maturities.
- Fractional shares of the underlying asset may be traded.

- The asset price follows a lognormal random walk.

In the sequel, we shall present the derivation of Black-Scholes model using a no-arbitrage approach.

1.1.2 Derivation of Black-Scholes Model

With these assumptions, one can begin to consider the value of a call or a put option. In this paper we will use $f(S,t)$ to denote the value of the option in order to emphasize that the analysis is independent of the form of the financial asset under consideration. As long as the value of the asset depends only on S and t. To understand the form that $f(S,t)$ will take, it is necessary to use a result from stochastic calculus known as It$\hat{o}$'s Lemma. We consider the equation of a stock price

$$dS_t = \mu S_t dt + \sigma S_t dW_t$$

The above equation can be written as

$$dS(t) = \mu S(t)dt + \sigma S(t)dW(t) \tag{1.3}$$

where μ is the rate of return, σ is the volatility and W follows a Wiener process on a filtered probability space $(\Omega, \mathbb{B}, \mu, \mathbb{F}(\mathbb{B}))$ in which filtration $\mathbb{F}(\mathbb{B}) = \{\mathbb{B}_t : t \geq 0\}$, where $\mathbb{B}_t$ is the sigma-algebra generated by $\{S_t : 0 \leq t \leq T\}$. Assuming that $f = f(S,t) \in C^{2,1}(\mathbb{R}, [0,T])$, then from It$\hat{o}$'s Lemma we have,

$$df = \left(\mu S\frac{\partial f}{\partial S} + \frac{\partial f}{\partial t} + \frac{1}{2}S^2\sigma^2\frac{\partial^2 f}{\partial S^2}\right)dt + \sigma S\frac{\partial f}{\partial S}dW \tag{1.4}$$

The Wiener process underlying f and S are the same and can be eliminated by choosing an appropriate portfolio of the stock and derivative. We choose a portfolio of

$$\begin{array}{rcl} -1 & : & \text{derivative} \\ +\dfrac{\partial f}{\partial S} & : & \text{shares} \end{array}$$

The holder is short of one derivative and long an amount $\frac{\partial f}{\partial S}$ of shares. Now we define θ as the value of the portfolio and we have

$$\theta = -f + S\frac{\partial f}{\partial S} \tag{1.5}$$

The change $d\theta$ in the value of the portfolio in the time interval dt is given by

$$d\theta = -df + \frac{\partial f}{\partial S}dS \tag{1.6}$$

Substituting (1.4) into (1.6), we get

$$d\theta = -\left[\left(\mu S\frac{\partial f}{\partial S} + \frac{\partial f}{\partial t} + \frac{1}{2}S^2\sigma^2\frac{\partial^2 f}{\partial S^2}\right)dt + \sigma S\frac{\partial f}{\partial S}dW\right] + \frac{\partial f}{\partial S}dS \tag{1.7}$$

Also substituting equation (1.3) into (1.7), yields

$$\begin{aligned} d\theta &= -\mu S\frac{\partial f}{\partial S}dt - \frac{\partial f}{\partial t}dt - \frac{1}{2}S^2\sigma^2\frac{\partial^2 f}{\partial S^2}dt - \sigma S\frac{\partial f}{\partial S}dW + \mu S\frac{\partial f}{\partial S}dt + \sigma S\frac{\partial f}{\partial S}dW \\ &= -\frac{\partial f}{\partial t}dt - \frac{1}{2}S^2\sigma^2\frac{\partial^2 f}{\partial S^2}dt \end{aligned}$$

$$d\theta = -\left[\frac{\partial f}{\partial t} + \frac{1}{2}S^2\sigma^2\frac{\partial^2 f}{\partial S^2}\right]dt \tag{1.8}$$

The portfolio is now risk-less due to elimination of the dW term. It must then earn a return similar to other short term risk-free securities such as bank account. Therefore

$$d\theta = r\theta dt \tag{1.9}$$

where r is the risk-free interest rate, substituting (1.5) and (1.8) into (1.9), we obtain

$$\begin{aligned}
-\left(\frac{\partial f}{\partial t} + \frac{1}{2}S^2\sigma^2\frac{\partial^2 f}{\partial S^2}\right)dt &= r\left(-f + S\frac{\partial f}{\partial S}\right)dt \\
\left(-\frac{\partial f}{\partial t} - \frac{1}{2}S^2\sigma^2\frac{\partial^2 f}{\partial S^2}\right)dt &= \left(-rf + rS\frac{\partial f}{\partial S}\right)dt \\
-\frac{\partial f}{\partial t} - \frac{1}{2}S^2\sigma^2\frac{\partial^2 f}{\partial S^2} &= -rf + rS\frac{\partial f}{\partial S} \\
\frac{\partial f}{\partial t} + rS\frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 f}{\partial S^2} - rf = 0 & \qquad (1.10)
\end{aligned}$$

(1.10) is called the Black-Scholes partial differential equation.

Solving the partial differential equation above gives an analytical formula for pricing the European style options. These options can only be exercised at the expiration date.

Lemma 1.1 *With the Black Scholes assumptions, where the expectation* E^* *is taken with respect to the so-called risk neutral probability* p^* *(equivalent to* p*)and under which* $dS_t = S_t(rdt + \sigma_t dW_t)$*,* W_t *being a wiener process under* p^* *and* $\mathbb{F}$ *which is called natural of* W_t*. Since* S_t *is a markov process, it can be shown that the option's price* f_t *is a function of* S_t *and* t*, i.e. there exists a two variable function called the pricing function such that* $f_t = f(S_t, t)$ *it is possible to prove that the option's price at time zero is given by*

$$f_t = \exp\left(\int_0^T r(s)ds\right)\mathrm{E}^*(f_0(S_T)|F_0) \tag{1.11}$$

1.1.3 Black-Scholes Pricing Formula

The major breakthrough in the pricing of options is that Black-Scholes obtained the closed form formula for European options. Calling $f(S,t)$ the price of an option with maturity T and payoff function f_0 and assuming that r, and $\sigma > 0$ are constants, then the Black Scholes formula

$$f(S,t) = e^{-rT}\boldsymbol{E}^*\left(f_0\left(Se^{rT}e^{W_T - W_0} - \frac{\sigma^2}{2}T\right)\right) \tag{1.12}$$

and since under p^*, $W_T - W_0$ is a centered Gaussian distribution with variance T

$$f(S,t) = \frac{1}{\sqrt{2\pi}}e^{-rT}\int_R f_0\left(Se^{(r-0.5\sigma^2)T + \sigma x\sqrt{T}}\right)e^{-\frac{x^2}{2}}dx \tag{1.13}$$

when the option is a vanilla European option where we shall denote the price of the call option by C_E and the price of the put by P_E, a more explicit formula can be deduced from (1.11) Take for example a call,

$$C_E(S,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{d_2}(Se^{\frac{-\sigma^2 T}{2} - \sigma x\sqrt{T}} - Ke^{-rT})e^{-\frac{x^2}{2}}dx \tag{1.14}$$

where,

$$d_1 = \frac{\ln(\frac{S}{K}) + (r + \frac{\sigma^2}{2})T}{\sigma\sqrt{T}} \tag{1.15}$$

$$d_2 = \frac{\ln(\frac{S}{K}) + (r - \frac{\sigma^2}{2})T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T} \tag{1.16}$$

and the Gaussian

$$N(d) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d} e^{\frac{-x^2}{2}} dx \tag{1.17}$$

Introducing the upper tail of (1.17), using (1.15), (1.16) and (1.17), we obtain the Black-Scholes formula for the prices at time zero of the European call option and the European put option on a non dividend paying stock respectively

$$C_E = SN(d_1) - Ke^{-rT}N(d_2) \tag{1.18}$$

and

$$P_E = Ke^{-rT}N(-d_2) - SN(-d_1) \tag{1.19}$$

We may extend the Black-Scholes analytic formula to price options on a dividend paying stock as follows,

Let λ denote the constant continuous dividend yield which is known, then the geometric Brownian motion model in (1.3) becomes

$$dS = S(\mu - \lambda)dt + S\sigma dW \tag{1.20}$$

and the modified partial differential equation is given by

$$\frac{\partial c}{\partial t} + (r - \lambda)S\frac{\partial c}{\partial S} + \frac{\sigma^2 S^2}{2}\frac{\partial^2 c}{\partial S^2} - rc = 0 \tag{1.21}$$

Solving (1.21) by applying the same method, then the European call option for a dividend paying stock is given by

$$C_E = Se^{-\lambda T}N(\hat{d}_1) - Ke^{-rT}N(\hat{d}_2) \tag{1.22}$$

and the European put option is

$$P_E = Ke^{-rT}N(-\hat{d}_2) - Se^{-\lambda T}N(-\hat{d}_1) \tag{1.23}$$

where

$$\hat{d}_1 = \frac{\ln(\frac{S}{K}) + (r - \lambda + \frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}}$$

$$\hat{d}_2 = \frac{\ln(\frac{S}{K}) + (r - \lambda - \frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}} = \hat{d}_1 - \sigma\sqrt{\tau}$$

Theorem 1.2 *When σ and r are constant, then the price of the call is given by*

$$C_E = Se^{-\lambda T}N(d_1) - Ke^{-rT}N(d_2)$$

and the price of the put is given by

$$P_E = Ke^{-rT}N(-d_2) - Se^{-\lambda T}N(-d_1),$$

where

$$d_1 = \frac{\ln(\frac{S}{K}) + (r + \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}$$

$$d_2 = \frac{\ln(\frac{S}{K}) + (r - \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}$$

and N is given by

$$N(d) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d} e^{\frac{-x^2}{2}} dx$$

Lemma 1.3 *If r is a function of time, then d_1 and d_2 become*

$$d_1 = \frac{\ln(\frac{S}{K}) + \int_t^T r(\tau)d\tau + \frac{\sigma^2}{2}(T-t)}{\sigma\sqrt{T-t}} \tag{1.24}$$

and

$$d_2 = \frac{\ln(\frac{S}{K}) + \int_t^T r(\tau)d\tau - \frac{\sigma^2}{2}(T-t)}{\sigma\sqrt{T-t}} \tag{1.25}$$

respectively.

1.2 Boundary Conditions for European Call and Put Options

The boundary conditions for European call and put options are given below.

1.2.1 European Call Option

The boundary conditions for a European call option are given by

$$C(S,T) = \max(S-K,0),\ S > 0 \tag{1.26}$$

$$C(0,t) = 0,\ t > 0 \tag{1.27}$$

$$C(S,t) \approx S - Ke^{-r(T-t)},\ \text{as } S \to \infty,\ t > 0 \tag{1.28}$$

1.2.2 European Put Option

The boundary conditions for a European call option are given by

$$P(S,T) = \max(K-S,0),\ S > 0 \tag{1.29}$$

$$P(0,t) = Ke^{-r(T-t)},\ t > 0 \tag{1.30}$$

$$P(S,t) \to 0,\ \text{as } S \to \infty,\ t > 0 \tag{1.31}$$

1.3 Factors Affecting Option Value

The fundamental direct determinants of option value are the current stock price S_0, the interest rate r, the strike price or exercise price K, the expiration date T, the stock price volatility σ and the dividend λ expected during the life of the option. It is also important to consider whether the option is an American or European style option. These factors affecting an option are summarized in the Table 1.1 for both the call and put options.

Table 1.1 A Summary of the General Effect of the Six Variables

Factors	**Call Option**	**Put Option**
Strike price, K	Decrease	Increase
Current stock price, S_0	Increase	Decrease
Interest rate, r	Increase	Decrease
Expiration date, T	Increase	Increase
Volatility, σ	Increase	Increase
Dividend, λ	Decrease	Increase

2 Some Numerical Methods for European Options Valuation

This section presents the procedures for the implementation of binomial model, Crank Nicolson method and Monte Carlo method for the valuation of European options as follows;

2.1 Binomial Model

The binomial model breaks down the time to expiration into potentially a very large number of time intervals, or steps. A tree of stock prices is initially produced working forward from the present to expiration.

2.1.1 Procedures for the Implementation of Binomial Model

When stock price movements are governed by a multi-step binomial tree, we can treat each binomial step separately. The multi-step binomial tree can be used for the American and European style options.

Like the Black-Scholes model, the CRR formula in [9] can be used in the pricing of European style options and can easily be implemented in Matlab. To overcome this problem, we use a different multi-period binomial model for the American style options on both the dividend and non dividend paying stocks.

The no-arbitrage arguments are used and no assumptions are required about the probabilities of up and down movements in the stock price at each node. We now explain the procedures for the implementation of the multi-period binomial model.

At time zero, the stock price S is known, at time δt, there are two possible stock prices Su and Sd, at time $2\delta t$, there are three possible stock prices Su^2, Sud and Sd^2 and so on. In general, at time $i\delta t$ where $0 \leq i \leq N$, $(i+1)$ stock price are considered, given by

$$Su^j d^{N-j}, \quad \text{for } j = 0, 1, 2, ..., N \tag{2.1}$$

where N is the total number of movements and j is the total number of up movements. The multi-period binomial model can reflect numerous stock price outcomes if there are numerous periods. Fortunately. the binomial option pricing model is based on recombining trees, otherwise the computational burden quickly become overwhelming as the number of moves in the tree is increased.

Options are evaluated by starting at the end of the tree at time T and working backward. We know the worth of a call and put at time T is

$$\left.\begin{array}{c} \max(S_T - K, 0) \\ \\ \max(K - S_T, 0) \end{array}\right\} \tag{2.2}$$

respectively. Because we are assuming the risk neutral world, the value at each node at time $(T - \delta t)$ can be calculated as the expected value at time T discounted at rate r for a time period δt. similarly, the value at each node at time $(2T - \delta t)$ can be calculated as the expected value at time $(T - \delta t)$ discounted for a time period δt at rate r, and so on. By working back through all the nodes, we are able to obtain the value of the option at time zero.

Suppose that the life of an European option on a non-dividend paying stock is divided into N subintervals of length δt. Denote the j^{th} node at time $i\delta t$ as the (i, j) node, where $0 \leq i \leq N$ and $0 \leq j \leq i$. Define $f_{i,j}$ as the value of the option at the (i, j) node. The stock price at the (i, j) node is $Su^j d^{N-j}$. Then, the respective European call and put can be expressed as

$$f_{N,j} = \max(Su^j d^{N-j} - K, 0) \tag{2.3}$$

$$f_{N,j} = \max(K - Su^j d^{N-j}, 0), \quad \text{for } j = 0, 1, 2, ..., N \tag{2.4}$$

There is a probability p of moving from the (i, j) node at time $i\delta t$ to the $(i+1, j+1)$ node at time $(i+1)\delta t$ and a probability $(1-p)$ of moving from the (i, j) node at the $i\delta t$ to the $(i+1, j)$ node at time $(i+1)\delta t$. Then the neutral valuation is

$$f_{i,j} = e^{-r\delta t}\left[p f_{i+1,j+1} + (1-p) f_{i+1,j}\right] \tag{2.5}$$

and $0 \leq i \leq N-1$, $0 \leq j \leq i$

2.2 Crank Nicolson Method

The Crank Nicolson finite difference method is the average of the implicit and explicit methods.

We take the average of the two methods to get

$$\frac{f_{n+1,m}-f_{n,m}}{\delta t}+\frac{rm\delta S}{4\delta S}\left[f_{n+1,m+1}-f_{n+1,m-1}+f_{n,m+1}-f_{n,m-1}\right]+$$
$$\frac{\sigma^2 m^2 \delta S^2}{4\delta S}\left[f_{n,m-1}-2f_{n,m}+f_{n+1,m-1}-2f_{n+1,m}f_{n+1,m+1}\right]$$
$$=\frac{1}{2}\left[rf_{n,m}+rf_{n+1,m}\right] \tag{2.6}$$

Re-arranging (2.6), we get

$$\left[\frac{rm\delta t}{4}-\frac{\sigma^2 m^2\delta t}{4}\right]f_{n,m-1}+\left[1+\frac{r\delta t}{2}+\frac{\sigma^2 m^2\delta t}{2}\right]f_{n,m}$$
$$+\left[-\frac{\sigma^2 m^2\delta t}{4}-\frac{rm\delta t}{4}\right]f_{n,m+1}=\left[\frac{\sigma^2 m^2\delta t}{4}-\frac{rm\delta t}{4}\right]f_{n+1,m-1}$$
$$+\left[1-\frac{r\delta t}{2}-\frac{\sigma^2 m^2\delta t}{2}\right]f_{n+1,m}+\left[\frac{rm\delta t}{4}-\frac{\sigma^2 m^2\delta t}{4}\right]f_{n+1,m+1} \tag{2.7}$$

and we simplify to get

$$\gamma_{1m}f_{n,m-1}+\gamma_{2m}f_{n,m}+\gamma_{3m}f_{n,m+1}$$
$$=\rho_{1m}f_{n+1,m+1}+\rho_{2m}f_{n+1,m}+\rho_{3m}f_{n+1,m+1} \tag{2.8}$$

for $n=0,1,...,N-1$ and $m=1,2,...,M-1$ [10]. Then the parameters γ_{km} and ρ_{km} for $k=1,2,3$ are given by

$$\gamma_{1m}=\frac{rm\delta t}{4}-\frac{\sigma^2 m^2\delta t}{4},$$
$$\gamma_{2m}=1+\frac{r\delta t}{2}+\frac{\sigma^2 m^2\delta t}{2},$$
$$\gamma_{3m}=-\frac{\sigma^2 m^2\delta t}{4}-\frac{rm\delta t}{4},$$
$$\rho_{1m}=\frac{\sigma^2 m^2\delta t}{4}-\frac{rm\delta t}{4},$$
$$\rho_{2m}=1-\frac{r\delta t}{2}-\frac{\sigma^2 m^2\delta t}{2},$$
$$\rho_{3m}=\frac{rm\delta t}{4}+\frac{\sigma^2 m^2\delta t}{4} \tag{2.9}$$

We express the system of equations in (2.8) as

$$Cf_n=Df_{n+1}$$

This results into a tridiagonal system given by

$$\begin{bmatrix}\gamma_{20} & \gamma_{30} & 0 & \dots & 0 & 0 & 0\\ \gamma_{11} & \gamma_{21} & \gamma_{31} & \dots & 0 & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots\\ 0 & 0 & 0 & \dots & \gamma_{1M-1} & \gamma_{2M-1} & \gamma_{3M-1}\\ 0 & 0 & 0 & \dots & 0 & \gamma_{1M} & \gamma_{2M}\end{bmatrix}\begin{bmatrix}f_{n,0}\\ f_{n,1}\\ \vdots\\ f_{n,M-1}\\ f_{n,M}\end{bmatrix}=$$

$$\begin{bmatrix} \rho_{20} & \rho_{30} & 0 & \dots & 0 & 0 & 0 \\ \rho_{11} & \rho_{21} & \rho_{31} & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \rho_{1M-1} & \rho_{2M-1} & \rho_{3M-1} \\ 0 & 0 & 0 & \dots & 0 & \rho_{1M} & \rho_{2M} \end{bmatrix} \begin{bmatrix} f_{n+1,0} \\ f_{n+1,1} \\ \vdots \\ f_{n+1,M-1} \\ f_{n+1,M} \end{bmatrix} \tag{2.10}$$

The elements of vector f_{n+1} are known at maturity time T and we express the system as $f_n = C^{-1}Df_{n+1}$. By repeatedly iterating from time T to zero, we obtain the value of f as the price of the option. The diagonal entries of matrix C is

$$\gamma_{2m} = 1 + \frac{r\delta t}{2} + \frac{\sigma^2 m^2 \delta t}{2} \tag{2.11}$$

are always positive and thus the diagonal elements are non zero. Therefore the matrix is non singular as the diagonal entries are non zero.

The boundary conditions and (2.8) result in some entry changes in the tridiagonal matrices C and D. For the matrix C, $\gamma_{20}, \gamma_{2M} = 1$ and $\gamma_{30}, \gamma_{1M} = 0$. For the matrix D, $\rho_{20}, \rho_{2M} = 1$ and $\rho_{30}, \rho_{1M} = 0$. Crank Nicolson method has a leading error of order $(\delta t^2, \delta S^2)$ [14].

2.2.1 Procedures for the Implementation of Crank Nicolson Method

The reason that Crank Nicolson finite difference method is a popular choice for pricing options is that all options will satisfy the Black-Scholes partial differential equation (1.10) or appropriate variants of it. The difference between each option contract is in determining the boundary conditions that it satisfies. Crank Nicolson method can be applied to American options. To illustrate the finite difference method in practice, we will now price a European put option using the Crank Nicolson method. We first define the domain discretization in the underlying price of the asset S and the time t as follows:

- Asset value Discretization: $0, \delta S, 2\delta S, 3\delta S, ..., M\delta S$, where $M\delta S = S_{max}$
- Time Discretization: $0, \delta t, 2\delta t, 3\delta t..., N\delta t$, where $N\delta t = T$
- The Price of the Option: $f_{n,m} = f(n\delta t, m\delta S), n = 0, 1, 2, 3, ..., N$and $m = 0, 1, 2, 3, ..., M$

S_{max} is a maximum value for the underlying asset price, S that we must choose sufficiently large. The boundary conditions for European put are:

- $f_{t,S} = \max(K - S)$
- $f_{t,0} = Ke{-}r(T - t)$
- $f_{t,S_{max}} = 0$

Changing these into mesh notation, we have

- $f_{N,m} = \max(K - m\delta S, 0), m = 0, 1, 2, 3, ..., M$
- $f_{n,0} = Ke^{-r(N-n)\delta t}, n = 0, 1, 2, 3, ..., N$
- $f_{n,M} = 0, n = 0, 1, 2, 3, ..., N$

from (2.8), Crank Nicolson method can be expressed as

$$Cf_n = Df_{n+1} \tag{2.12}$$

The above equation (2.12) resulting into tridiagonal matrices. These matrices occupy huge amounts of memory and its processing can consume a lot of computer time. For example, a system of n simultaneous linear equations requires n^2 matrix entries and the computing time to solve them is proportional to n^3. In our case, the matrices have very few non-zero entries. Such matrices are called sparse as opposed to full. Matlab has facilities for exploiting the sparsity of matrices and the potential of saving huge amounts of memory and processing time. It also has an inbuilt function to cater for the inverse of a matrix. It is accurate and efficient as it uses the Gauss elimination method [11]. This inbuilt function will ease our implementation of the explicit and implicit methods in Matlab.

2.3 Monte Carlo Method

The basis of Monte Carlo simulation is the strong law of large numbers, stating that the arithmetic mean of independent, identically distributed random variables, converges towards their mean almost surely. Monte Carlo simulation method uses the risk valuation result. The expected payoff in a risk neutral world is calculated using a sampling procedure

2.3.1 Principles of Theory for Monte Carlo Method

Monte Carlo methods are based on the analogy between probability and volume. The mathematics of measure formalizes the intuitive notion of probability, associating an event with a set of outcomes and defining the probability of the event to be its volume or measure relative to that of a universe of possible outcomes. Monte Carlo uses this identity in reverse, calculating the volume of a set by interpreting the volume as a probability. In the simplest case, this means sampling randomly from a universe of possible outcomes and taking the fraction of random draws that fall in a given set as an estimate of the set's volume. The law of large numbers ensures that this estimate converges to the correct value as the number of draws increases. The central limit theorem provides information about the likely magnitude of the error in the estimate after a finite number of draws.

The principles of theory for Monte Carlo method are as follows [9]:

- If a derivative security can be perfectly replicated through trading in other assets, then the price of the derivative security is the cost of the replicating trading strategy.

- Discounted asset prices are martingales under a probability measure associated with the choice of discount factor. Prices are expectation of discounted payoffs under such martingale measure.
- In a complete market, any payoff can be realized through a trading strategy and the martingale measure associated with the discount rate is unique.

2.3.2 Procedures for the Implementation of Monte Carlo Method

The main procedures are followed when using Monte Carlo simulation.

- Simulate a path of the underlying asset under the risk neutral condition within the desired time horizon
- Discount the payoff corresponding to the path at the risk-free interest rate. The structure of the security in question should be adhered to
- Repeat the procedure for a high number of simulated sample path
- Average the discounted cash flows over sample paths to obtain the option's value.

Now we consider an European option which is an example of an vanilla options that has path dependent payoff and this makes it ideally suited for pricing using Monte Carlo approach.

Computing an European option price means computing the discounted expectation of the payoff. This suggests the following algorithm to determine the European option price through Monte Carlo method.

We simulate M independent realization X^j of the final payoffs X given by

$$X^j_{call} = \max(S^j_T - \bar{S}_t, 0)$$

and

$$X^j_{put} = \max(\bar{S}_t - S^j_T, 0),$$

where X^j_{call} and X^j_{put} are called the payoff for the European call and put options respectively. The discretely monitored European call(put) option has the estimated value in the jth path given by

$$C^j_{call} = e^{-rT} \max(S^j_T - \bar{S}_t, 0) \tag{2.13}$$

and

$$C^j_{put} = e^{-rT} \max(\bar{S}_t - S^j_T, 0) \tag{2.14}$$

respectively, where S^j_T and $\bar{S}_t$ are called stock price at maturity time T and strike price respectively and this strike price is either given by arithmetic average or geometric average [15]. This is repeated for $j = 1, 2, ..., M$, where M denotes the number of trials. These M simulations are the possible paths that a stock price can have at maturity time T. The final estimated call option value is

$$C_{call} = \frac{1}{M}\sum_{j=1}^{M} C_{call}^{j} = \frac{1}{M}\sum_{j=1}^{M} e^{-rT}\max(S_T^j - \bar{S}_t, 0) \tag{2.15}$$

and corresponding put option value is given by

$$C_{put} = \frac{1}{M}\sum_{j=1}^{M} C_{put}^{j} = \frac{1}{M}\sum_{j=1}^{M} e^{-rT}\max(\bar{S}_t - S_T^j, 0) \tag{2.16}$$

The variance of the estimate is computed for European call and put option respectively by

$$\hat{S}^2 = \frac{1}{M-1}\sum_{j=1}^{M}(C_{call}^{j} - C_{call})^2 \tag{2.17}$$

$$\hat{S}^2 = \frac{1}{M-1}\sum_{j=1}^{M}(C_{put}^{j} - C_{put})^2 \tag{2.18}$$

For a sufficiently large value of M, the distribution for European call

$$\frac{(C_{call} - C)}{\sqrt{\frac{\hat{S}^2}{M}}}$$

and for put option

$$\frac{(C_{put} - C)}{\sqrt{\frac{\hat{S}^2}{M}}}$$

where C is the time call value, tends to the standard normal distribution. Note that the standard deviation of C_{call} and C_{put} is equal to $\frac{\hat{S}}{\sqrt{M}}$ and so the confidence limits of estimation can be reduced by increasing the number of simulation runs M. The appearance of M as the factor $\frac{1}{\sqrt{M}}$ implies that the reduction of the standard deviation by a factor of 10 will require an increase of the number of simulation runs by 100 times.

The algorithm above can easily be implemented in Matlab to estimate the price of European call(put) options.

3 Numerical Examples and Results

This section presents some numerical examples and results generated as follows:

Example 1

We consider the accuracy and the convergence of Binomial model, Crank Nicolson method and Monte Carlo method with relation to the Black-Scholes value of the option. We price a European put option on non-dividend paying stock with the following parameters:

$$K = 60, r = 5\%, \sigma = 25\%, T = 3$$

The results generated from the three methods are presented in the Tables 1 and 2 below using MATLAB.

Example 2

We consider the performance of the three numerical methods against the 'true' Black-Scholes price for a vanilla option with the following parameters:

$$S = 60, 70, 80, 90, 100, K = 90, r = 4\%, \sigma = 0.2, 0.4, 0.6, 0.8, T = 2$$

The results obtained are shown in the Table 3, 4,5 and 6 below.

3.1 Table of Results

Table 1: The Accuracy of Binomial Model, Crank Nicolson Method and Monte Carlo Method with relation to the 'True' Black-Scholes Values for European Call Option

S	Black-Scholes Values	Binomial Model	Crank Nicolson Method	Monte Carlo Method
10	0.1372	0.1372	0.1372	0.1382
20	1.3580	1.3580	1.3470	1.3579
30	4.1503	4.1538	4.0855	4.1430
40	8.3687	8.3722	8.1614	8.3305
50	13.7407	13.7439	13.2598	13.7503
60	20.2665	20.0220	19.0979	20.0072
70	26.9975	26.9969	25.4517	27.0079
80	34.5287	34.5384	32.1512	34.5043
90	42.4935	42.5021	39.0694	42.4557
100	50.8025	50.8121	46.1115	50.8121

Table 2: The Accuracy of Binomial Model, Crank Nicolson Method and Monte Carlo Method with relation to the 'True' Black-Scholes Values for European Put Option

S	**Black-Scholes Values**	**Binomial Model**	**Crank Nicolson Method**	**Monte Carlo Method**
10	50.3868	50.3868	48.1807	50.3887
20	41.6076	41.6075	41.2856	41.6062
30	34.3998	34.4033	34.3108	34.4078
40	28.6128	28.6217	28.5147	28.6170
50	23.9903	23.9935	23.9449	23.9816
60	20.2665	20.2715	20.1941	20.2615
70	17.2470	17.2465	17.1252	17.2496
80	14.7783	14.7879	14.5832	14.7759
90	12.7430	12.7526	12.4490	12.7469
100	11.0520	11.0617	10.4313	11.0510

Table 3: The Performance of Binomial Model, Crank Nicolson Method and Monte Carlo Method against the 'True' Black-Scholes Values for European Put Option with $\sigma = 0.20$

S	**Black-Scholes Values**	**Binomial Model**	**Crank Nicolson Method**	**Monte Carlo Method**
60	24.3107	24.3119	24.3100	24.3162
70	16.6658	16.6690	16.6611	16.6759
80	10.7916	10.7949	10.7843	10.7881
90	6.6567	6.6591	6.6491	6.6559
100	3.9483	3.9504	3.9420	3.9512

Table 4: The Performance of Binomial Model, Crank Nicolson Method and Monte Carlo Method against the 'True' Black-Scholes Values for European Put Option with $\sigma = 0.40$

S	**Black-Scholes Values**	**Binomial Model**	**Crank Nicolson Method**	**Monte Carlo Method**
60	29.9441	29.9420	29.9392	29.9379
70	24.3389	24.3331	24.3297	24.3289
80	19.7389	19.7442	19.7210	19.7466
90	16.0002	15.9926	15.9663	16.0105
100	12.9789	12.9641	12.9181	12.9786

Table 5: The Performance of Binomial Model, Crank Nicolson Method and Monte Carlo Method against the 'True' Black-Scholes Values for European Put Option with $\sigma = 0.60$

S	**Black-Scholes Values**	**Binomial Model**	**Crank Nicolson Method**	**Monte Carlo Method**
60	36.6412	36.6387	36.4648	36.6560
70	32.1729	32.1697	31.8663	32.1804
80	28.3615	28.3396	27.8683	28.3693
90	25.0985	25.0982	24.3573	25.1079
100	22.2935	22.2713	21.2386	22.3020

Table 6: The Performance of Binomial Model, Crank Nicolson Method and Monte Carlo Method against the 'True' Black-Scholes Values for European Put Option with $\sigma = 0.80$

S	**Black-Scholes Values**	**Binomial Model**	**Crank Nicolson Method**	**Monte Carlo Method**
60	43.3059	43.3029	41.9470	43.3063
70	39.6653	39.6362	37.8850	39.6572
80	36.4889	36.4347	34.1409	36.5031
90	33.6965	33.6842	30.6548	33.6971
100	31.2252	31.2103	27.3765	31.2259

3.2 Discussion of Results

Tables 1 and 2 show the accuracy of binomial model, Crank Nicolson method and Monte Carlo method with relation to the 'true' Black-Scholes values for European call and put options respectively. The Tables show the variation of the option price with the stock price, S. The results demonstrate that binomial model and Monte Carlo method perform well, are mutually consistent, more accurate than Crank Nicolson method and agree with the Black-Scholes values. From Tables 3, 4, 5 and 6, we can see that binomial model is more accurate than its counterparts. Also the higher the volatility, the higher the values of the three methods under consideration. Hence binomial model and Monte Carlo method work very well for the valuation of European options.

4 Conclusion

In this paper we consider three numerical methods namely: binomial model, Crank Nicolson method and Monte Carlo method. Each of the numerical methods for vanilla option valuation has its own advantages and disadvantages of use as outlined below.

Advantages of Binomial Model

- It is very flexible in pricing options.
- This method is both computationally efficient and accurate.
- It is very simple but powerful technique that can be used to solve many complex option pricing problem.
- It can be used to accurately price American style options than the Black-Scholes model as it takes into consideration the possibilities of early exercise and other factors like dividends.

Disadvantages of Binomial Model

- It is very slow.
- This method is not adequate to deal with path dependent options.

Advantages of Crank Nicolson Finite Difference Methods

- They are fairly robust.
- Finite difference methods are a good choice for solving partial differential equations over a complex domains.
- They are very flexible in handling different processes for the underlying state variables.
- They can be used to accurately price American style options where there is possibilities of early exercise.

Disadvantages of Finite Difference Methods

- These methods can not be used in high dimensions.
- Finite difference methods are somewhat problematic for path dependent options.
- Finite difference methods are a little harder to code than Monte Carlo method and the binomial model, and can be prone to instability (in the case of an explicit method).

Advantages of Monte Carlo Method

- Easy computation.
- It accommodates complex payoffs and stochastic processes.
- It is flexible in handling varying and even high dimensional financial problems.
- It provides standard error for the estimates that it makes.
- It is often used as the benchmark valuation technique for many complex options.

- It is quite easy to implement and can be used without too much difficulty to value a large range of European style exotics.

Disadvantages of Monte Carlo Method

- It is very computationally expensive in terms of time and computing resources.
- Early exercise is problematic for simulation methods.

From the above Tables, we can see the effect of volatility on the valuation of European put options that increase in volatility will lead to increase in the price of the option and among the methods considered in this paper, we conclude that binomial model and Monte Carlo method perform better and more accurate than Crank Nicolson method when pricing European options.

References

[1] F. Black and M. Scholes, *The pricing of options and corporate liabilities*, Journal of Political Economy, **81**(1973), 637–654.

[2] P. Boyle, *Options: a Monte Carlo approach*, Journal of Financial Economics. **4**(1977), 323–338.

[3] P. Boyle , M. Broadie, and P. Glasserman, *Monte Carlo methods for security pricing*, Journal of Economic Dynamics and Control, **21**(1997), 1267–1321.

[4] M. Brennan and E. Schwartz, *Finite difference methods and jump processes arising in the pricing of contingent claims*, Journal of Financial and Quantitative Analysis, **5**(1978), 461–474.

[5] A. Brick, *On the consistence of the Black-Scholes model with a general equilibrium frame work*, Journal of Financial Quantitative Analysis, **22**(1987), 259–275.

[6] J. Cox, S. Ross and M. Rubinstein, *Option pricing: a simplified approach*, Journal of Financial Economics, **7**(1979), 229–263.

[7] S. E. Fadugba, A. O. Ajayi and C. R. Nwozo, *Effect of volatility on binomial model for the valuation of American options*, International Journal of Pure and Applied Sciences and Technology, **18**(2013), 43–53.

[8] S. Fadugba, C. Nwozo, and T. Babalola, *The comparative study of finite difference method and Monte Carlo method for pricing European option*, Mathematical Theory and Modeling, **2**(2012), 60–66.

[9] S. E. Fadugba, C. R. Nwozo, J. T. Okunlola, O. A. Adeyemo and A. O. Ajayi, *On the accuracy of binomial model and Monte Carlo method for pricing European options*, International Journal of Mathematics and Statistics Studies, European Centre for Research Training and Development, **1**(2013), 38–54.

[10] S. E. Fadugba and C. R. Nwozo, *Crank Nicolson finite difference method for the valuation of options*, Pacific Journal of Science and Technology, **14**(2013), 136–146.

[11] B. Hahn, *Essential Matlab for Scientists and Engineers*, Pearson Education Publishers, Third Edition, South Africa, 2002.

[12] J. Hull, *Options, futures and other derivatives*, Pearson Education Inc, Fifth Edition, Prentice Hall, New Jersey, 2003.

[13] J. Hull and A. White *Valuing derivatives securities using the explicit Finite difference method*, Journal of Financial and Quantitative Analysis, **25**(1990), 88–99.

[14] J. Kerman, *Numerical methods for option Pricing: binomial and finite difference approximations*, www.stat.columbia.edu/kerman, 2002.

[15] C. R. Nwozo and S. E. Fadugba, *Monte Carlo method for pricing some path dependent options*, International Journal of Applied Mathematics, **25**(2012), 763–778.

[16] C. R. Nwozo and S. E. Fadugba, *Some numerical methods for options valuation*, Communications in Mathematical Finance, **1**(2012), 57–74.

[17] E. Stein and Stein, *stock price distribution with stochastic volatility: an analysis approach*, Review of Financial Studies, **4**(1997): 727-752.

[18] D. Tavella and C. Randall, *Pricing financial instruments: the finite difference method*, John Wiley and Sons, New York, 2000.