6-2011

A General Look at Posets Rings and Lattices

Courtney A. Phillips
Union College - Schenectady, NY

A General Look at Posets,

Rings, and Lattices

By

Courtney Phillips

****

Submitted in partial fulfillment

of the requirements for

Honors in the Department of Mathematics.

UNION COLLEGE

June, 2011

ABSTRACT

PHILLIPS, COURTNEY Lattice Theory. Department of Mathematics, June 2011.

ADVISOR: Susan Niefield

A lattice is a type of structure that aims to organize certain relationships that exist between members of a set. This thesis seeks to define lattices, and demonstrate the different types. It will give examples of lattices, as well as various ways to describe and classify them.

Contents

1 Introduction

Mathematicians began studying what is now known as lattice theory in the early ninteenth century. English mathematician George Boole sought to formalize the concept of propositional logic, which led to the study of a type of lattice, known as a Boolean algebra, which will be discussed later. In the late ninteenth and early twentieth centuries, American mathematician Charles Sanders Peirce and German mathematician Ernst Schröder introduced the concept of lattices, while German mathematician Richard Dedekind introduced lattices in his research of algebraic numbers.

Lattice theory became securely rooted in the field of abstract algebra. The American mathematician Garrett Birkhoff, who studied abstract algebra and group theory, published a series of papers in the 1930s, as well as the book *Lattice Theory* in 1940 which converted lattice theory into a major branch of abstract algebra. Birkhoff used contributions both from Charles Sanders Peirce and Ernst Schröder, and showed that lattice theory could provide a unifying framework for various unrelated developments of mathematics.

Distributive lattices were among the first lattices to be considered, and are therefore the most extensive and well-researched subportion. Because of this, mathematicians find it easier to work with lattices after developing a strong grasp on distributive lattices. Distributivity has provided the motivation for many results in general lattice theory, and as well, weakened forms of distributivity have been used to prove conditions on lattices and on lattice elements. (For more information, see [6])

In sections two through four, we will introduce the notions of partially ordered sets, lattices, and rings, giving definitions, properties, and examples for each. In section five, we will describe and categorize functions that exist between different lattices. Section six will show an alternate method of describing a relationship between lattice elements. In section seven, we will consider different types of lattices, giving methods of distinguishing between them and examples of these types. Finally, section eight will introduce functions between posets (and between lattices) known as adjoints, and describe properties of these adjoints.

2 An Introduction to Posets

In this section, we will introduce the concept of a poset and give definitions, examples, and properties of partial orderings and posets. We will prove an important theorem that will give us the ability to establish properties of lattices.

Definition 2.1. A binary relation $\leq$ on a set P is called

(i) *reflexive* if $a \leq a$, for all $a \in P$

(ii) *antisymmetric* if $a \leq b$ and $b \leq a$ imply that $a = b$

(iii) *transitive* if $a \leq b$ and $b \leq c$ imply that $a \leq c$

(iv) a *partial ordering* if it is reflexive, antisymmetric, and transitive.

Note that for antisymmetry, we can consider more than just two elements of P. If $x_0 \leq x_1 \leq \ldots \leq x_{n-1} \leq x_0$, then $x_0 = x_1 = \ldots = x_{n-1}$.

Definition 2.2. A set P together with a partial ordering $\leq$ is called a *partially ordered set* or *poset*, and is denoted by $\langle P; \leq \rangle$, or merely P.

Like any mathematical structures, posets and partial orderings have certain properties that hold in all cases. For example, if $\langle P; \leq \rangle$ is a poset and $Q \subseteq P$ with $\leq_Q$ denoting the restriction of $\leq$ to Q, then Q is a poset with partial ordering $\leq_Q$.

Examples 2.3. The following are examples of posets:

(1) $\langle \mathcal{P}(X); \subseteq \rangle$, where $\mathcal{P}(X)$ is the set of subsets of a given set X. In this case, the relation $\leq$ is defined for $X_0, X_1 \in \mathcal{P}(X)$ by $X_0 \leq X_1$ if and only if $X_0 \subseteq X_1$.

(2) $\langle \mathcal{O}(\mathbb{R}^n); \subseteq \rangle$, where $\mathcal{O}(\mathbb{R}^n)$ is the set of open subsets of the real numbers.

(3) $\langle Idl(R); \subseteq \rangle$, where $Idl(R)$ is the set of ideals of a ring R.

(4) $\langle P; \leq \rangle$, where $P = \{0, a, b, 1\}$ and $\leq = \{(0, a), (0, b), (a, 1), (b, 1)\}$.

Since $\langle \mathcal{P}(X); \subseteq \rangle$ is clearly a poset, we know that Examples 2 and 3 are posets since $\mathcal{O}(\mathbb{R}^n) \subseteq \mathcal{P}(\mathbb{R}^n)$ and $Idl(R) \subseteq \mathcal{P}(R)$. We will go into detail in later sections concerning the definitions and properties of open subsets of $\mathbb{R}^n$ and ideals of a ring R. In Example 4, we defined the relation $\leq$ explicitly. We interpret this set as

$$(x, y) \in \leq \text{ if and only if } x \leq y$$

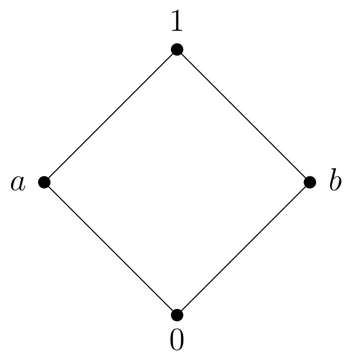
When referring to a general poset, we often refer to the set itself without explicit reference to the relation.

There are many ways to express a poset, particularly when the underlying set is finite. Our first option is to represent a poset P using a Hasse diagram, a visual representation of both the set, and the relationship between items, defined as follows.

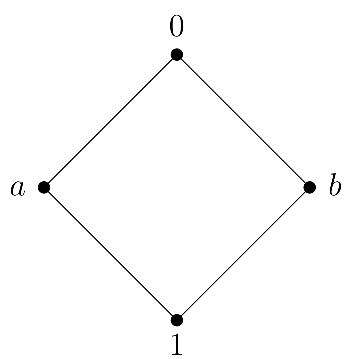
Definition 2.4. Let P be a poset, and let $a, b \in P$. Write $a < b$ if $a \leq b$ and $a \neq b$. Then a is *covered* by b, or b *covers* a, if $a < b$ and there is no c such that $a < c < b$.

Definition 2.5. A *Hasse diagram* for a poset P is the graph (as defined by [2]) such that its vertices are elements of P, its edges are sets $\{a, b\}$ such that b covers a, and it is drawn so that a is lower than b.

Definition 2.5 gives us the following Hasse diagram for Example 4 above.



With these beginning tools, we can now begin to demonstrate properties of posets. We will first look at the concept of duality, and how it relates to posets. The following is the Hasse diagram of “another” poset. We will see later that these posets are intrinsically the same.



This is the poset $\langle P^\circ; \leq^\circ \rangle$ obtained from Example 4 by the following general construction. The *dual* of a poset $\langle P; \leq \rangle$ is the poset denoted by $\langle P^\circ; \leq^\circ \rangle$ or just P°, and defined by $P^\circ = P$ and $x \leq^\circ y$ if and only if $y \leq x$. Note that for every poset P, $(P^\circ)^\circ = P$.

We can relate two elements of a poset with a partial ordering relation, deciding which element, if either, is greater than the other. We can also define relations to relate multiple elements, even entire subsets, of a poset.

Definition 2.6. Let $\langle P; \leq \rangle$ be a poset, $H \subseteq P$, and $a \in P$. Then a is an *upper bound* of H if $h \leq a$, for all $h \in H$. An upper bound a is a *least upper bound* if $a \leq b$, for all upper bounds b of H.

The least upper bound of H is also called the *supremum* or the *join* of H, and is denoted by $\sup H$ and $\bigvee H$, respectively. By replacing instances of $\leq$ with $\geq$, we get the definitions of *lower bound* and *greatest lower bound*. The greatest lower bound of H is called the *infimum* or the *meet* of H, denoted by $\inf H$ and $\bigwedge H$, respectively.

Proposition 2.7. *For every set P, if $x \in P$ and $S \subseteq P$ such that for all $y \in P$ ($s \leq y$, for all $s \in S$ if and only if $x \leq y$), then $x = \bigvee S$.*

Proof. Since $x \leq x$ by reflexivity, we know $s \leq x$, for all $s \in S$, and so x is an upper bound of S. Then, for every upper bound y, since $s \leq y$, for all $s \in S$, it must be that $x \leq y$. Thus, x is the least upper bound of S, and we conclude that $x = \bigvee S$. $\square$

Similar to posets, the concepts of upper bound and lower bound are dual to each other, because one can be obtained from the other by reversing the inequality. In particular, $a = \sup H$ in P if and only if $a = \inf H$ in P°. More generally, if Φ is a statement about posets, then the *dual* of Φ, denoted Φ°, is the statement obtained by replacing all occurances of $\leq$ with $\geq$.

Proposition 2.8. (Duality Principle) *If Φ is true for all posets, then Φ° is also true for all posets.*

Proof. Suppose Φ holds for all posets, and let P be an arbitrary poset. Since P° is a poset, it follows that Φ holds for P°. Then Φ° holds for P, and thus Φ° holds for all posets. $\square$

With the concepts of meets and joins, we can demonstrate another method of representing certain finite posets. For Example 4 above, we can create a meet table, a join table, and a join/meet table, pictured below.

In Table 1, the intersection between any two elements will give the meet of these two elements. Similarly, in Table 2, the intersection will give the join. In Table 3, we combined the previous two tables since both are symmetric about a forty-five degree

$\bigwedge$	0	a	b	1
0	0	0	0	0
a	0	a	0	a
b	0	0	b	b
1	0	a	b	1

Table 1: $\bigwedge$ Table

$\bigvee$	0	a	b	1
0	0	a	b	1
a	a	a	1	1
b	b	1	b	1
1	1	1	1	1

Table 2: $\bigvee$ Table

$\bigvee \backslash \bigwedge$	0	a	b	1
0	$\star$	0	0	0
a	a	$\star$	0	a
b	b	1	$\star$	b
1	1	1	1	$\star$

Table 3: $\bigvee / \bigwedge$ Table

line, marked with stars. To find the meet of two elements, we use the intersection that lies above the star line; to find the join, we use the intersection below the star line.

Since $\emptyset \subseteq P$, for every poset P, we can consider $\inf \emptyset$ and $\sup \emptyset$. We know that $\inf \emptyset$, if it exists, is the lower bound that is greater than every other lower bound. But since every element of P is a lower bound of $\emptyset$, $\inf \emptyset$ is the element that is greater than every element of P. Thus, $\inf \emptyset = \sup P$. This element is also called *top*, written $\top$ or 1. Dually, $\sup \emptyset = \inf P$ is called *bottom*, written $\perp$ or 0.

However, like with any subset, $\emptyset$ may not have a meet or a join in P. For example, the poset $\langle \mathbb{Z}; \leq \rangle$ is one in which $\inf \emptyset$ and $\sup \emptyset$ do not exist.

We can show a dual relationship between least upper bounds and greatest lower bounds.

Theorem 2.9. *Let P be a poset. Then $\bigwedge H$ exists, for all $H \subseteq P$, if and only if $\bigvee H$ exists, for all $H \subseteq P$.*

Proof. Let $\bigvee K$ exist for all $K \subseteq P$, and let $H \subseteq P$. Then H^l denotes the set of lower bounds of H. Now, since $H^l \subseteq P$ and all joins of P exist, $\bigvee H^l$ exists.

Since $m \leq h$, for all $m \in H^l$ and $h \in H$, it follows that $\bigvee H^l \leq h$, for all $h \in H$, and so $\bigvee H^l$ is a lower bound of H. To show that $\bigvee H^l = \bigwedge H = \inf H$, let a be a lower bound of H. Then $a \leq h$, for all $h \in H$. So $a \in H^l$, and thus $a \leq \bigvee H^l$, as desired.

By the Duality Principle, we get that if $\bigwedge H$ exists for all H, then $\bigvee H$ exists for all H. $\square$

Since we have defined least upper bound and greatest lower bound, we can see one type of a lattice:

Definition 2.10. A *complete lattice* is a poset P for which $\bigvee H$ (or equivalently $\bigwedge H$) exists, for all subsets $H \subseteq P$.

In the following section, we will see how this changes if $\bigvee$ and $\bigwedge$ exist only for finite subsets. We can also consider functions between posets.

Definition 2.11. Let P and Q be posets. A function $f\colon P \to Q$ is *order preserving*, or equivalently *monotone*, if $a \leq b$ implies that $fa \leq fb$.

We will give examples of order-preserving functions in later sections.

3 An Introduction to Lattices

Now we can begin our inquiry into lattices. In this section, we will start with two different definitions, one order theoretic and one algebraic, and then we will prove that these definitions are equivalent.

Definition 3.1. A poset $\langle P; \leq \rangle$ is a *lattice* if $\sup\{a, b\}$ and $\inf\{a, b\}$ exist for all $a, b \in P$, or equivalently, if $\sup H$ and $\inf H$ exist for every finite nonempty subset H of P.

There is also an algebraic approach to lattices, one that does not use the concept of posets.

Definition 3.2. A set together with binary operations $\vee$ and $\wedge$ is a *lattice* if

(i) $\vee$ and $\wedge$ are idempotent, i.e., $a \wedge a = a$ and $a \vee a = a$

(ii) $\vee$ and $\wedge$ are commutative, i.e., $a \wedge b = b \wedge a$ and $a \vee b = b \vee a$

(iii) $\vee$ and $\wedge$ are associative, i.e., $(a \wedge b) \wedge c = a \wedge (b \wedge c)$ and $(a \vee b) \vee c = a \vee (b \vee c)$

(iv) $\vee$ and $\wedge$ satisfy the absorption identities, i.e., $a \vee (a \wedge b) = a$ and $a \wedge (a \vee b) = a$

But since we have two definitions of lattices and only one concept, we are able to prove that these two definitions are equivalent and can thus be used interchangably.

First, we will relate the binary relations used in each definition.

Lemma 3.3. *If $L \times L \xrightarrow{*} L$ is commutative, associative, and idempotent, then $\leq^*$, defined by $a \leq^* b \Leftrightarrow a * b = b$, is a partial ordering on L and $a * b = \sup\{a, b\}$.*

Proof. We know $a \leq^* a \Leftrightarrow a*a = a$. But since $*$ is idempotent, we know by definition that $a * a = a$, and thus $\leq^*$ is reflexive.

Assume $a \leq^* b$ and $b \leq^* a$. Since $a \leq^* b$ and $b \leq^* a$, we know that $a * b = b$ and $b*a = a$. Since $*$ is commutative, it follows that $a*b = b*a$, and so $a = b*a = a*b = b$, as desired.

Assume $a \leq^* b$ and $b \leq^* c$. We know that $a * b = b$ and $b * c = c$. Then, by substituting $b * c$ for c and b for $a * b$, and by associativity of $*$, we have:

$$\begin{aligned} a * c &= a * (b * c) \\ &= (a * b) * c \\ &= b * c \\ &= c \end{aligned}$$

Thus, $a * c = c$, so $a \leq^* c$, and thus $\leq^*$ is transitive.

To show that $a * b = \sup\{a, b\}$, we know that $a \leq^* a * b \Leftrightarrow a * (a * b) = a * b$. We know already that $*$ is associative, idempotent, and commutative. Consider the following:

$$\begin{aligned} a * (a * b) &= (a * a) * b \\ &= a * b \end{aligned}$$

Similarly, for $b \leq^* a * b$, we first have

$$\begin{aligned} b * (a * b) &= b * (b * a) \\ &= (b * b) * a \\ &= b * a \\ &= a * b \end{aligned}$$

Then, $a * b$ is an upper bound of $\{a, b\}$.

Now, assume we have $c \in L$ such that $a \leq c$ and $b \leq c$. Then $a * c = c$ and $b * c = c$. Now, to show that $a * b \leq c$, we have the following:

$$\begin{aligned} (a * b) * c &= a * (b * c) \\ &= a * c \\ &= c \end{aligned}$$

Thus, $a * b = \sup\{a, b\}$, as desired. □

Next, we will show that our poset is an "algebraic semilattice". However, we must first define what this means.

Definition 3.4. Let $\langle A; \circ \rangle$ be a set with one binary operation $\circ$. Then $\langle A; \circ \rangle$ is called an *algebraic semilattice* if $\circ$ is idempotent, commutative, and associative. A poset $\langle P; \leq \rangle$ is a *join semilattice* if $\sup\{a, b\}$ exists, for all $a, b \in P$. A *meet semilattice* is defined dually.

Note that a poset is a lattice if and only if it is both a meet semilattice and a join semilattice.

Now we can continue on to our proof.

Lemma 3.5. *If $(P, \leq)$ is a join semilattice, then $(P, \vee)$ is an algebraic semilattice.*

Proof. Assume $(P, \leq)$ is a join semilattice, i.e., $a \vee b = \sup\{a, b\}$ exists, for all $a, b \in P$. To show that $\vee$ is commutative, consider $a \vee b$. Then:

$$\begin{aligned} a \vee b &= \sup\{a, b\} \\ &= \sup\{b, a\} \\ &= b \vee a. \end{aligned}$$

Thus, $\vee$ is commutative, as desired. Now, to show that $\vee$ is idempotent, consider $a \in P$. Then we have that $a \vee a = \sup\{a, a\} = \sup\{a\} = a$, so $\vee$ is idempotent.

Finally, to show that $\vee$ is associative, first, we will show that $a \leq b$ and $c \leq d \Rightarrow a \vee c \leq b \vee d$. We know that $a \leq b \leq b \vee d$, and $c \leq d \leq b \vee d$. So $b \vee d$ is an upper bound of $\{a, c\}$. But since $a \vee c = \sup\{a, c\}$, $a \vee c \leq b \vee d$, as desired. Now, consider $(a \vee b) \vee c \leq a \vee (b \vee c)$. To show this, since $a \leq a \vee (b \vee c)$ and $b \leq b \vee c \leq a \vee (b \vee c)$, we have that $a \vee b \leq a \vee (b \vee c)$. Also, we have that $c \leq b \vee c \leq a \vee (b \vee c)$. Then since we have $a \vee b \leq a \vee (b \vee c)$ and $c \leq a \vee (b \vee c)$, so we get that $(a \vee b) \vee c \leq a \vee (b \vee c)$. Similarly, consider $a \vee (b \vee c) \leq (a \vee b) \vee c$. For this, we have $a \leq (a \vee b) \leq (a \vee b) \vee c$. Also, since $b \leq (a \vee b) \leq (a \vee b) \vee c$ and $c \leq (a \vee b) \vee c$, we have that $b \vee c \leq (a \vee b) \vee c$. Thus, since we have $a \leq (a \vee b) \vee c$ and $(b \vee c) \leq (a \vee b) \vee c$, we have $a \vee (b \vee c) \leq (a \vee b) \vee c$, as desired. Now, since we have $a \vee (b \vee c) \leq (a \vee b) \vee c$ and $(a \vee b) \vee c \leq a \vee (b \vee c)$, we have $a \vee (b \vee c) = (a \vee b) \vee c$, and thus $\vee$ is associative. □

Finally, we are ready to prove the equivalence of our two definitions of a lattice.

Theorem 3.6. $(L, \leq)$ *is a lattice with* $a \wedge b = \inf\{a, b\}$ *and* $a \vee b = \sup\{a, b\}$ *if and only if* $\wedge, \vee$ *are idempotent, commutative, associative, and satisfy the two absorption identities.*

Proof. Assume $\wedge, \vee$ are idempotent, commutative, associative, and satisfy the two absorption identities. Define $\leq^{\vee}$ by

$$a \leq^{\vee} b \Leftrightarrow a \vee b = b$$

and define $\leq_{\wedge}$ by

$$a \leq_{\wedge} b \Leftrightarrow b \leq^{\wedge} a \Leftrightarrow b \wedge a = a$$

Then by Lemma 3.3, $\leq^{\vee}$ is a partial ordering on L with $a \vee b = \sup\{a, b\}$, and by the duel of Lemma 3.3, $\leq_{\wedge}$ is a partial ordering on L with $b \wedge a = \inf\{a, b\}$.

Now, we have assumed that $a \vee (a \wedge b) = a$ and $a \wedge (a \vee b) = a$. To show that $a \vee b = b \Rightarrow a \wedge b = a$, suppose $a \vee b = b$. Then

$$\begin{aligned} a \wedge b &= a \wedge (a \vee b) \\ &= a \end{aligned}$$

To show that $a \wedge b = a \Rightarrow a \vee b = b$, suppose $a \wedge b = a$. Then

$$\begin{aligned} a \vee b &= b \vee (b \wedge a) \\ &= b \end{aligned}$$

Thus, $a \vee b = b \Leftrightarrow a \wedge b = a$, so $(L, \leq)$ with $\leq$ defined by $a \leq b \Leftrightarrow (a \vee b = b$ and $a \wedge b = a)$ is a lattice.

Now, assume that $(L, \leq)$ is a lattice with $a \wedge b = \inf\{a, b\}$ and $a \vee b = \sup\{a, b\}$. Then $(L, \leq^{\vee})$ is a $\vee-$semilattice, so by Lemma 3.5, we have that $\vee$ is commutative, idempotent, and associative. Similarly, $\wedge$ is commutative, idempotent, and associative.

To show that $a \vee (a \wedge b) = a$, we know that $a \leq a \vee (a \wedge b)$. And, since $a \leq a$ and $a \wedge b \leq a$, we have that $a \vee (a \wedge b) \leq a$. Thus, we get that $a \vee (a \wedge b) = a$.

To show that $a \wedge (a \vee b) = a$, we first know that $a \wedge (a \vee b) \leq a$. Then, since $a \leq a$ and $a \leq a \vee b$, we know that $a \leq a \wedge (a \vee b)$. Thus, $a \wedge (a \vee b) = a$, as desired. $\square$

Consider the four examples in 2.3. Each of these examples is a complete lattice. In Example 1, $\bigwedge A_\alpha = \bigcap A_\alpha$ and $\bigvee A_\alpha = \bigcup A_\alpha$. For Example 2,

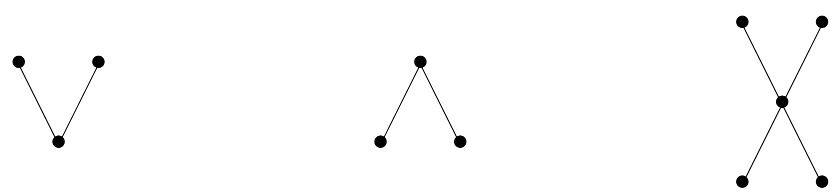
$$\bigvee U_\alpha = \bigcup U_\alpha$$

and since $\mathcal{O}(\mathbb{R}^n)$ is closed under unions, we have that

$$\bigwedge U_\alpha = \bigcup \{V | V \subseteq \bigcap U_\alpha\}$$

denoted $(\bigcap U_\alpha)^\circ$, by the proof of Theorem 2.9. In Example 3, $\bigwedge A_\alpha = \bigcap A_\alpha$, and $\bigvee A_\alpha = \Sigma A_\alpha$, which will be defined in a later section. For Example 4, $\bigwedge X = \inf X$ and $\bigvee X = \sup X$, for all $X \subseteq P$, as expected. So we must ask ourselves: Is every poset a lattice?

By definition, every lattice is a poset. The following Hasse diagrams are examples of posets:



The first poset is a meet semilattice, but not a join semilattice. Dually, the second poset is a join semilattice, but not a meet semilattice. Finally, the third poset is neither a meet semilattice nor a join semilattice. Thus, none of these three posets are lattices.

4 An Introduction to Ring Theory

This section will develop a basis for working with rings and ideals of a ring. We will begin by defining a ring and giving examples. Then, we will define different types of ideals. We will prove certain properties about ideals, and define residuals. In a later section, we will use this concept of rings and ideals to study lattices more in depth.

Definition 4.1. A *ring* is a set R with binary operations $+$ and $\cdot$ such that

(i) $(R, +)$ is an abelian group

(ii) $a(bc) = (ab)c$, for all $a, b, c \in R$

(iii) $a(b + c) = ab + ac$ and $(b + c)a = ba + ca$, for all $a, b, c \in R$

A ring is *commutative* if $ab = ba$, for all $a, b \in R$. A ring has *unity*, also called a ring with 1, if there exists an element 1 such that $1 \cdot a = a \cdot 1 = a$. Such an element is necessarily unique.

Examples 4.2. The following are rings. Unless otherwise stated, define $+$ and $\cdot$ as usual.

(1) $(\mathbb{Z}, +, \cdot)$

(2) $(2\mathbb{Z}, +, \cdot)$, the even integers

(3) $(\mathbb{Z}_n, +_n, \cdot_n)$, with $\mathbb{Z}_n = \{x \in \mathbb{Z} | 0 \leq x \leq n - 1\}$, $+_n$ defined as $(x + y) \mod n$, and $\cdot_n$ defined as $(x \cdot y) \mod n$

(4) $(\{0\}, +, \cdot)$, known as the trivial ring

Note that the only ring listed above without unity is Example 2. Unless otherwise stated, all rings we consider will be commutative rings with 1.

Definition 4.3. An ideal of a ring is a nonempty subset $I \subseteq R$ such that $a + b \in I$ and $ra \in I$, for all $a, b \in I$ and $r \in R$.

One example of an ideal is a generated ideal. Let R be a ring, and let $S \subseteq R$. The *ideal generated by* S, denoted $\langle S \rangle$, is the set

$$\langle S \rangle = \left\{ \sum_{i=1}^{n} r_i s_i \middle| r_i \in R \text{ and } s_i \in S \right\}$$

Since $\bigcap I_\alpha$ is an ideal for all ideals I_α, this set is the smallest ideal that contains S, and equivalently is

$$\langle S \rangle = \bigcap \{ I \in Idl(R) | S \subseteq I \}$$

Note that if $S = \{a\}$ is a set with a single element, then $\langle S \rangle$ is called a *principal ideal*, and is also written $\langle a \rangle$ or Ra.

Proposition 4.4. *Let I be an ideal of a ring R. Then $S \subseteq I \Leftrightarrow \langle S \rangle \subseteq I$.*

Proof. Consider the following:

$$\begin{aligned} \langle S \rangle \subseteq I \quad & \Leftrightarrow \quad r_1 s_1 + \ldots + r_n s_n \in I, \text{ for all } r_1, \ldots, r_n \in R \text{ and } s_1, \ldots, s_n \in S \\ & \Leftrightarrow \quad s \in I, \text{ for all } s \in S, \text{ since } I \text{ is an ideal} \\ & \Leftrightarrow \quad S \subseteq I \end{aligned}$$

So we have $\langle S \rangle \subseteq I \Leftrightarrow S \subseteq I$, as desired. □

With this result, we can look at some properties of ideals.

Definition 4.5. For ideals I, J of a ring R, the *product* of I and J, written $I \cdot J$ or equivalently IJ, and the *sum*, written $I + J$, are defined as

$$IJ = \langle \{ ij | i \in I, \ j \in J \} \rangle \qquad\qquad I + J = \{ i + j | i \in I, j \in J \}$$

Clearly IJ is an ideal by definition, and we note also that $I + J$ is an ideal.

Proposition 4.6. *Let I, J, K be ideals of a ring R. Then* *(a)* $I(JK) = (IJ)K$

(b) $IJ = JI$

(c) $IJ \subseteq I \cap J$

(d) $I(J + K) = IJ + IK$

(e) $I \subseteq J \Rightarrow IK \subseteq JK$

(f) $I(J \cap K) \subseteq IJ \cap IK$

(g) $I \subseteq K \Rightarrow I + (J \cap K) = (I + J) \cap K$

Proof.

(a) Since I, J, K are ideals, by Proposition 4.4, we know that

$$I(JK) = \langle i(jk) | i \in I, j \in J, k \in K \rangle$$

To show that $I(JK) \subseteq (IJ)K$, we need only show that $i(jk) \in (IJ)K$, for all $i \in I$, $j \in J$, $k \in K$. Given i, j, k, we know $i(jk) = (ij)k$, since multiplication is associative, and so $i(jk) \in (IJ)K$. A similar proof is used to show that $(IJ)K \subseteq I(JK)$. Thus, $I(JK) = (IJ)K$.

(b) To show that $IJ \subseteq JI$, we need only show that $ij \in JI$, for all $i \in I, j \in J$. But $ij \in JI$ since $ij = ji$. So $IJ = JI$, since the other subset inclusion is proved similarly.

(c) Take $ij \in IJ$ with $i \in I$ and $j \in J$. Since I and J are both ideals of R, we know $i, j \in R$. Then, since $i \in I$ and $j \in R$, it follows that $ij \in I$. Similarly, $ij \in J$. Thus, $ij \in I \cap J$. So $IJ \subseteq I \cap J$.

(d) Suppose $i \in I$, $j \in J$, $k \in K$. Then $i(j + k) = ij + ik$, and it follows that $I(J + K) = IJ + IK$.

(e) Let $ik \in IK$, for $i \in I$ and $k \in K$. Then since $I \subseteq J$ implies that $i \in J$, and so $ik \in JK$. Thus, $I \subseteq J \Rightarrow IK \subseteq JK$.

(f) Let $ir \in I(J \cap K)$ with $i \in I$ and $r \in J \cap K$. Then $r \in J$ and $r \in K$. So $ir \in IJ$ and $ir \in IK$. Thus, $ir \in IJ \cap IK$, and so $I(J \cap K) \subseteq IJ \cap IK$.

(g) Let $I, J, K \in Idl(R)$ with $I \subseteq K$, and consider $I + (J \cap K)$. We know that $I + (J \cap K) \subseteq I + J$. Since $I \subseteq K$, we have that $I + (J \cap K) \subseteq K$, and so $I + (J \cap K) \subseteq (I + J) \cap K$. Let $r \in (I + J) \cap K$. Then $r \in K$ and $r = i + j$, for some $i \in I$ and $j \in J$. So $i \in K$ since $I \subseteq K$. Then, since $i \in K$ and $r \in K$, and since K is an ideal, $r - i \in K$. Thus, since $j \in J$ and $j = r - i$, it follows that $j \in J \cap K$, and so $r \in I + (J \cap K)$. Then we have that $(I + J) \cap K = I + (J \cap K)$.

□

Definition 4.7. Let I and J be ideals of a ring R. Then the residuation of I by J, written $I \colon J$, is the set

$$I \colon J = \{r \in R | rJ \subseteq I\}$$

5 Lattice Homomorphisms

Like with any structure, we can study not only lattices, but also functions between lattices. In this section, let L and M be lattices.

Definition 5.1. A function $f \colon L \to M$ is a *homomorphism* if

(i) $f(a \vee b) = fa \vee fb$, for all $a, b \in L$

(ii) $f(a \wedge b) = fa \wedge fb$, for all $a, b \in L$

Proposition 5.2. *If $f \colon L \to M$ is a homomorphism, then f is order preserving.*

Proof. If $a, b \in L$ with $a \leq b$, then $a \vee b = b$ and so $f(a \vee b) = fb$. Since f is a homomorphism, $f(a \vee b) = fa \vee fb$. Then we have that $fa \vee fb = fb$, so that $fa \leq fb$. Thus, f is order preserving. $\square$

Definition 5.3. A function $f \colon L \to M$ is an *isomorphism* if f is a 1-1 and onto homomorphism. We say two lattices are *isomorphic* if there exists an isomorphism between them.

Consider Examples 2.3. We saw the Hasse diagram of Example 4, and we saw that this poset is a lattice. It is easy to see that this lattice P is isomorphic to its dual P°. As well, both P and P° are isomorphic to $\mathcal{P}(\{1,2\})$. In fact, two finite lattices are isomorphic if they have the same unlabeled Hasse diagram.

Proposition 5.4. *A homomorphism $f \colon L \to M$ is an isomorphism if and only if there exists a homomorphism $g \colon M \to L$ such that $g \circ f = id_L$ and $f \circ g = id_M$.*

Proof. We know that $f \colon L \to M$ is invertible if and only if f is a bijection. It suffices to show that if f is an isomorphism, then f^{-1} is a homomorphism. Assume f is a bijective homomorphism. Then we have

$$\begin{aligned} f(f^{-1}(a) \wedge f^{-1}(b)) &= f(f^{-1}(a)) \wedge f(f^{-1}(b)) \\ &= a \wedge b \\ &= f(f^{-1}(a \wedge b)) \end{aligned}$$

Then, since f is 1-1, $f^{-1}(a) \wedge f^{-1}(b) = f^{-1}(a \wedge b)$. A similar equation holds for $\vee$, so f^{-1} is a homomorphism. $\square$

6 Properties of Lattice Elements

This section will give more detail about lattices. We will give relationships between lattice elements, and categorize different lattice examples.

First, we will look at lattice elements and how we can relate them to each other.

Definition 6.1. Let L be a complete lattice, and let $a, b \in L$. Then a is *way below* b, denoted $a << b$, if $b \leq \bigvee S$ implies that $a \leq \bigvee F$, for some finite $F \subseteq S$. If $a << a$, we say that a is *compact*.

If S is nonempty, we can rewrite this as $b \leq \bigvee S$ implies that $a \leq s_1 \vee s_2 \vee \ldots \vee s_n$, for some $s_1, \ldots, s_n \in S$.

Definition 6.2. A nonempty subset S of a lattice L is:

(i) *directed* if $x, y \in S$ implies that $x \vee y \in S$

(ii) an *ideal* if S is directed, and if $x \in S$ and $y \leq x$ implies that $y \in S$

Note that if S is a directed set and $a << b$, then $b \leq \bigvee S$ implies that $a \leq s$ for some $s \in S$. If S is an ideal, then $b \leq \bigvee S$ implies that $a \in S$.

The following are properties of complete lattices.

Proposition 6.3. *Suppose $a << b$ in L. Then*

(a) $a \leq b$

(b) $b \leq c \Rightarrow a << c$

(c) $c \leq a \Rightarrow c << b$

Proof. For (a), assume that $a << b$ and let $S = \{b\}$. Then clearly $b \leq \bigvee S$. Since $a << b$, there exists $s_1, \ldots, s_n \in S$ with $a \leq s_1 \vee \ldots \vee s_n$. But since $S = \{b\}$, by letting $n = 1$ and $s_1 = b$, we get that $a \leq b$.

Now, for (b), suppose $b \leq c$, and let $c \leq \bigvee S$, for $S \subseteq L$. Then since $b \leq c \leq \bigvee S$, we know $b \leq \bigvee S$. So, since $a << b$, it follows that $a \leq s_1 \vee s_2 \vee \ldots \vee s_n$, for some $s_1, s_2, \ldots, s_n \in S$. Thus, $a << c$.

For (c), let $b \leq \bigvee S$, for $S \subseteq L$, and assume $c \leq a$. Since $a << b$, we know $a \leq s_1 \vee s_2 \vee \ldots \vee s_n$, for some $s_1, s_2, \ldots, s_n \in S$. But, then we have

$$c \leq a \leq s_1 \vee s_2 \vee \ldots \vee s_n$$

and it follows that $c << b$. □

Proposition 6.4. *If $a << c$ and $b << c$ in L, then $a \vee b << c$.*

Proof. Assume that $a << c$ and $b << c$, and assume that $c \leq \bigvee S$, for some $S \subseteq L$. Then $a \leq s_1 \vee s_2 \vee \ldots \vee s_n$, for some $s_1, \ldots, s_n \in S$, and $b \leq t_1 \vee t_2 \vee \ldots \vee t_m$, for some $t_1, \ldots, t_m \in S$, and so $a \vee b \leq s_1 \vee s_2 \vee \ldots \vee s_n \vee t_1 \vee t_2 \vee \ldots \vee t_m$, for these $s_1, \ldots, s_n, t_1, \ldots, t_m \in S$. Thus, $a \vee b << c$. □

Corollary 6.5. *The set $\Downarrow b = \{a | a << b\}$ is an ideal of L.*

Proof. This follows directly from Propositions 6.3c and 6.4 □

There is a stronger relationship that elements of a lattice can have.

Definition 6.6. Let L be a complete lattice, and let $a, b \in L$. Then a is *completely below* b, denoted $a <<< b$, if, for every set $S \neq \emptyset$, $b \leq \bigvee S$ implies that $a \leq s$, for some $s \in S$.

Remark 6.7. The concept of $<<<$ has properties similar to those of $<<$ as described in 6.3.

Proposition 6.8. *Let L be a lattice with $a, b \in L$. If $a <<< b$, then $a << b$.*

Proof. Let $S \subseteq L$ be a set such that $b \leq \bigvee S$. Then, since $a <<< b$, we know that $a \leq s$, for some $s \in S$. So we have that $a << b$. □

We can categorize which elements are way below and completely below other elements in certain lattices.

Example 6.9. Consider $\mathcal{P}(X)$, where $X = \mathbb{R}$. Let $A_1 = 3\mathbb{N} = \{3, 6, 9, \ldots\}$ and $B = \mathbb{N}$. We know that $B \subseteq \bigcup S$, for $S = \{\{1\}, \{2\}, \{3\}, \ldots\}$. But since A_1 has multiple elements, and each $C \in S$ has only one, there does not exist any $C \in S$ such that $A_1 \subseteq C$, so $A_1 <\not<< B$.

Now let $A_2 = \{5, 10, 15, \ldots, 100\}$ and let

$$S' = \{\{1, 2, \ldots, 10\}, \{11, 12, \ldots, 20\}, \ldots, \{10n-9, 10n-8, \ldots, 10n\}, \ldots\}$$

But since $5 \in \{1, 2, \ldots, 10\}$ and $15 \in \{11, 12, \ldots, 20\}$, again, we find that there does not exist any $C \in S$ such that $A_2 \subseteq C$, so $A_2 <\not<< B$.

Finally, let $A_3 = \{26\}$, and consider S from the first example. Then $26 \in \{26\}$, so $A_3 \subseteq \{26\}$, and thus $A_3 <<< B$. Now consider S' from the second example. Then $26 \in \{21, 22, \ldots, 30\}$, so $A_3 \subseteq \{21, 22, \ldots, 30\}$, so again $A_3 <<< B$. Finally let $S'' = \{2\mathbb{Z}, \{\ldots, -5, -3, -1, 1, 3, 5, \ldots\}\}$. Then since $26 \in 2\mathbb{Z}$, $A_3 \subseteq 2\mathbb{Z}$, and so $A_3 <<< B$.

Proposition 6.10. *In $\mathcal{P}(X)$, $A <<< B$ if and only if $A = \emptyset$ or $A = \{x\}$, for some $x \in B$.*

Proof. Assume $A <<< B$. We know, then, that $B \subseteq \bigcup S \Rightarrow A \subseteq C$, for some $C \in S$. Let $S = \{\{b\} | b \in B\}$. Then $B \subseteq \bigcup S$. Then we have that $A \subseteq C$, for some $C \in S$. But then since each $C \in S$ is some set $\{x\} \subseteq B$, it must be that $A = \emptyset$ or $A = \{x\}$, for some $x \in B$.

Note that if $A = \emptyset$, $A <<< B$ since the empty set is completely below everything. Assume that $A \neq \emptyset$ and $A = \{x\}$, for some $x \in B$. To show that $A <<< B$, let $B \subseteq \bigcup S$, for some set S. Then $x \in \bigcup S$, so $x \in C$, for some $C \in S$, and it follows that $\{x\} \subseteq C$, for some $C \in S$. Then, since $A = \{x\}$, we have that $A \subseteq C$, for some $C \in S$. $\square$

Example 6.11. Again, consider $\mathcal{P}(X)$ where $X = \mathbb{R}$. Let $A_1 = 2\mathbb{N} = \{2, 4, 6, \ldots\}$

and $B = \mathbb{N}$. For $S = \{\{1\}, \{2\}, \{3\}, \ldots\}$, we know that $B \subseteq \bigcup S$. But since A_1 is infinite, there does not exist any finite set $\{C_1, C_2, \ldots, C_n\} \subseteq S$ such that $A_1 \subseteq C_1 \cup C_2 \cup \ldots \cup C_n$. Thus, $A_1 \notin \Downarrow B$.

Let $A_2 = \{1, 2, 3, \ldots, 10\}$. We will show for multiple S that

$$B \subseteq \bigcup S \Rightarrow A_2 \subseteq C_1 \cup C_2 \cup \ldots \cup C_n$$

for some $C_1, \ldots, C_n \in S$.

(1) Consider the same S as in the previous example. Then, letting $C_i = \{i\}$, for $i = 1, 2, \ldots, 10$, we have that $A_2 \subseteq C_1 \cup C_2 \cup \ldots \cup C_{10}$.

(2) Now let $S = \{2\mathbb{Z}, \{\ldots, -3, -1, 1, 3, \ldots\}\}$. So we have that $B \subseteq \bigcup S$. Then, letting $C_1 = 2\mathbb{Z}$ and $C_2 = \{\ldots, -3, -1, 1, 3, \ldots\}$, $A_2 \subseteq C_1 \cup C_2$.

(3) Finally, let $S = \{\{1\}, \{1, 2\}, \{1, 2, 3\}, \ldots, \{1, 2, \ldots, n\}, \ldots\}$. Then clearly $B \subseteq \bigcup S$. Let $C_1 = \{1, 2, \ldots, 10\}$. Then $A_2 \subseteq C_1$.

Proposition 6.12. *In $\mathcal{P}(X)$, $A << B$ if and only if A is a finite subset of B.*

Proof. Assume $A << B$ and let $S = \{\{b\} | b \in B\}$. Then clearly $B \subseteq \bigcup S$. So $A \subseteq \{b_1\} \cup \{b_2\} \cup \ldots \cup \{b_n\}$, for some $b_1 \ldots, b_n \in B$. Then A is finite since

$$\{b_1\} \cup \{b_2\} \cup \ldots \cup \{b_n\} = \{b_1, b_2, \ldots, b_n\}$$

Now assume $A \neq \emptyset$ and $A \subseteq B$ with A finite, and let $B \subseteq \bigcup S$. Then $A \subseteq \bigcup S$. Since A is finite, we can write $A = \{a_1, a_2, \ldots, a_n\}$, for $a_1, \ldots, a_n \in A$. Since $A \subseteq \bigcup S$, we know that for each $a_i \in A$, there exists some $C_i \in S$ such that $a_i \in C_i$. Thus, $A \subseteq C_1 \cup C_2 \cup \ldots \cup C_n$, for $C_1, \ldots, C_n \in S$, so $A << B$. $\square$

We can classify the elements of the lattice $\langle Idl(\mathbb{Z}); \subseteq \rangle$ that are way below and completely below each other. First recall that $I \in Idl(\mathbb{Z})$ if and only if $I = n\mathbb{Z}$, for

$n \in \mathbb{Z}$, and that $n\mathbb{Z} + m\mathbb{Z} = \gcd(m, n)\mathbb{Z}$ (see [7]). First, we look at the way below relationship.

Proposition 6.13. *$m\mathbb{Z} << n\mathbb{Z}$ if and only if $n|m$.*

Proof. If $m\mathbb{Z} << n\mathbb{Z}$, then $m\mathbb{Z} \subseteq n\mathbb{Z}$, and so $n|m$.

If $n|m$, then $m\mathbb{Z} \subseteq n\mathbb{Z}$, so it suffices to show that $n\mathbb{Z} << n\mathbb{Z}$. Suppose

$$n\mathbb{Z} \subseteq \sum_{\alpha \in A} n_\alpha \mathbb{Z}$$

Then $n = n_{\alpha_1}k_1 + \ldots + n_{\alpha_N}k_N$, where $n_{\alpha_i} \in \mathbb{Z}$ and $k_i \in \mathbb{Z}$. So $n \in n_{\alpha_1}\mathbb{Z} + \ldots + n_{\alpha_N}\mathbb{Z}$, and thus $n\mathbb{Z} << n\mathbb{Z}$. So by Proposition 6.3c, we conclude that $m\mathbb{Z} << n\mathbb{Z}$. $\square$

Now, for completely below, we begin with two examples, which we will generalize and prove later.

Examples 6.14.

(1) Let $m = 0$ and $n = 2$. Then $m\mathbb{Z} = \{0\}$ and $n\mathbb{Z} = \{\ldots, -4, -2, 0, 2, 4, \ldots\}$. Then $n\mathbb{Z} \subseteq \bigvee S$, for $S = \{3\mathbb{Z}, 5\mathbb{Z}\}$. Note that $\bigvee S = 3\mathbb{Z} + 5\mathbb{Z} = \mathbb{Z}$. Then since $\{0\} \subseteq 3\mathbb{Z}$, we know $m\mathbb{Z} \subseteq I$, for $I \in S$. Similarly, for any nonempty set S of ideals, since $0 \in I$ for every ideal I, this result can be generalized.

(2) Let $m = 6$ and $n = 2$. Then $m\mathbb{Z} = \{\ldots, -18, -12, -6, 0, 6, 12, 18, \ldots\}$ and $n\mathbb{Z} = \{\ldots, -6, -4, -2, 0, 2, 4, 6, \ldots\}$. Now consider $n\mathbb{Z} \subseteq \bigvee S$, for $S = \{12\mathbb{Z}, 14\mathbb{Z}\}$. Then $\bigvee S = \gcd(12, 14)\mathbb{Z} = 2\mathbb{Z}$. But $6\mathbb{Z} \not\subseteq I$, for any $I \in S$, and so $6\mathbb{Z}$ is not completely below $2\mathbb{Z}$

Proposition 6.15. *$m\mathbb{Z} <<< n\mathbb{Z}$ if and only if $m = 0$.*

Proof. Let $m = 0$ and suppose $n\mathbb{Z} \subseteq \bigvee S$, where $S = \{n_\alpha\mathbb{Z} | \alpha \in A\}$. Then equivalently, $n\mathbb{Z} \subseteq \Sigma_{\alpha \in A} n_\alpha \mathbb{Z}$. But since $0 \in n_\alpha\mathbb{Z}$ for all α, it follows that $m\mathbb{Z} \subseteq n_\alpha\mathbb{Z}$, for all α. Thus, $m\mathbb{Z} <<< n\mathbb{Z}$.

Now, suppose $m\mathbb{Z} <<< n\mathbb{Z}$. Then $n|m$ since $m\mathbb{Z} \subseteq n\mathbb{Z}$. Now, let p, q be prime numbers such that $p \neq q$ and $p, q \nmid m$. Consider the set $pn\mathbb{Z} + qn\mathbb{Z}$. Recall that $pn\mathbb{Z} + qn\mathbb{Z} = \gcd(pn, qn)\mathbb{Z}$. But since p and q are prime and thus relatively prime, $\gcd(pn, qn) = n$. So $pn\mathbb{Z} + qn\mathbb{Z} = n\mathbb{Z}$. Then $m\mathbb{Z} \subseteq pn\mathbb{Z}$ or $m\mathbb{Z} \subseteq qn\mathbb{Z}$. This means that either $pn|m$ or $qn|m$. But by our choice of p and q, each of these are an impossibility if $m \neq 0$. Thus, it must be that $m = 0$. □

Recall for the remaining propositions that all rings are assumed to be commutative rings with 1. First, we can categorize ideals of a ring.

Lemma 6.16. *For all $I \in Idl(R)$,*

$$I = \sum_{a \in I} Ra$$

Proof. Let $I \in Idl(R)$, and let $x \in I$. Since $1 \in R$, $1x \in Rx$, so $x \in \Sigma_{a \in I} Ra$.

Now let $x \in \Sigma_{a \in I} Ra$. Then $x = r_1 a_1 + r_2 a_2 + \ldots + r_n a_n$, for $r_1, \ldots, r_n \in R$ and $a_1, \ldots, a_n \in I$. But since $a_1, a_2, \ldots, a_n \in I$, and since $I \in Idl(R)$, we have that $x \in I$. Thus, we can conclude that $I = \Sigma_{a \in I} Ra$, for all $I \in Idl(R)$. □

Lemma 6.17. *Let R be a ring with $a \in R$. Then Ra is compact.*

Proof. Let $Ra \subseteq \Sigma_\alpha J_\alpha$. Since $a \in \Sigma_\alpha J_\alpha$, we know that $a = r_1 + \ldots + r_n$ where $r_i \in J_{\alpha_i}$ for some $a_1, \ldots, a_n$. Thus,

$$a \in \sum_{i=1}^{n} J_{\alpha_i} \text{ and so } Ra \subseteq \sum_{i=1}^{n} J_{\alpha_i}$$

by Lemma 4.4. Therefore, $Ra << Ra$. □

Consequently, for $a \in I$, $Ra \subseteq I$, and so $Ra << I$.

We can categorize and relate $<<$ for ideals of a ring just as we did before with $\mathcal{P}(X)$

Proposition 6.18. *In $Idl(R)$, $I << J$ if and only if $I \subseteq Ra_1 + \ldots + Ra_n$, for some $a_1, \ldots, a_n \in J$.*

Proof. Suppose $I << J$. Since $J \subseteq \Sigma_{a \in J} Ra$, by Lemma 6.16, we know

$$I \subseteq Ra_1 + Ra_2 + \ldots + Ra_n, \text{ for some } a_1, \ldots, a_n \in J$$

Suppose $I \subseteq Ra_1 + \ldots + Ra_n$, for $a_1, \ldots, a_n \in J$. Since $Ra << Ra$, for all $a \in R$ by Lemma 6.17, we know $Ra_1 + \ldots + Ra_n << Ra_1 + \ldots + Ra_n$, by Propositions 6.3b and 6.4. Again by Proposition 6.3b, $Ra_1 + \ldots + Ra_n << J$. Finally, by 6.16 and 6.3c, $I << J$. □

7 Types of Lattices

There are several different types of lattices, many of which we will be considering later.

Proposition 7.1. *Let L be a lattice. The following are equivalent:*

(a) $(x \wedge y) \vee (x \wedge z) = x \wedge (y \vee z)$, *for all* $x, y, z \in L$

(b) $(x \vee y) \wedge (x \vee z) = x \vee (y \wedge z)$, *for all* $x, y, z \in L$

Proof. Assume (a) holds. Then we have:

$$\begin{aligned}
(x \vee y) \wedge (x \vee z) &= ((x \vee y) \wedge x) \vee ((x \vee y) \wedge z) \\
&= x \vee (z \wedge (x \vee y)) \\
&= x \vee ((z \wedge x) \vee (z \wedge y)) \\
&= (x \vee (z \wedge x)) \vee (z \wedge y) \\
&= x \vee (z \wedge y) \\
&= x \vee (y \wedge z)
\end{aligned}$$

Thus, (a) implies (b). Dually, (b) implies (a). $\square$

Definition 7.2. A lattice L is *distributive* if $(x \wedge y) \vee (x \wedge z) = x \wedge (y \vee z)$, or equivalently, if $(x \vee y) \wedge (x \vee z) = x \vee (y \wedge z)$, for all $x, y, z \in L$. L is *modular* if $x \geq z$ implies that $(x \wedge y) \vee z = x \wedge (y \vee z)$.

Note that for a distributive lattice L with $x, y, z \in L$ such that $x \geq z$, we have

$$\begin{aligned}(x \wedge y) \vee z &= (x \vee z) \wedge (y \vee z) \\ &= x \wedge (y \vee z)\end{aligned}$$

Thus, distributivity implies modularity.

Remark 7.3. In the modular law, we can replace z with $x \wedge z$ to get a more general form: $(x \wedge y) \vee (x \wedge z) = x \wedge (y \wedge (x \wedge z))$.

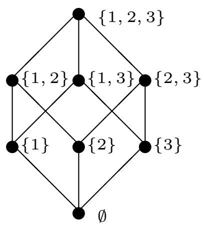
The lattices $\mathcal{P}(X)$ and $\mathcal{O}(\mathbb{R}^n)$ are distributive lattices, since it is easy to see that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$, for all sets A, B, C. By Proposition 4.6g, $Idl(R)$ is modular for all commutative rings with 1, but is not, in general, distributive. In fact, by Theorem 6.20 of [8], a Noetherian integral domain is distributive if and only if it is a Dedekind domain.

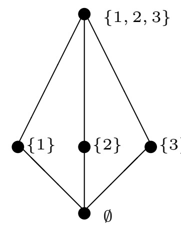
We can categorize exactly the lattices that are distributive and modular. But first, we need a definition.

Definition 7.4. Let L be a lattice with $A \subseteq L$. Then A is a *sublattice* if $x, y \in A$ implies that $x \vee y \in A$ and $x \wedge y \in A$, i.e., A is closed under the operations $\vee$ and $\wedge$.

Note that $A \subseteq L$ is a sublattice if and only if the inclusion function $i \colon A \to L$ is a homomorphism.

For example, consider the lattice $\mathcal{P}(\{1, 2, 3\})$ and the subset A, pictured below.



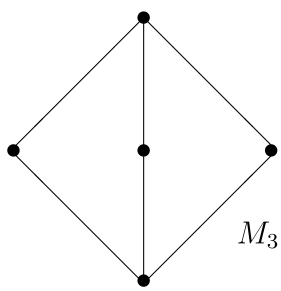


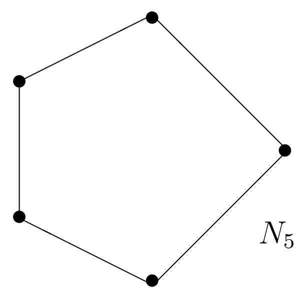
Then clearly A is a subset of $\mathcal{P}(\{1,2,3\})$, but $\{1\} \in A$ and $\{3\} \in A$, so $\{1\} \cup \{3\}$ should also be in A. However, $\{1\} \cup \{3\} = \{1,3\} \notin A$, so A is not a sublattice of $\mathcal{P}(\{1,2,3\})$.

Theorem 7.5. (M_3N_5 Theorem) *Let L be a lattice.*

(a) L is non-modular iff L has a sublattice isomorphic to N_5

(b) L is non-distributive iff L has a sublattice isomorphic to M_3 or N_5.

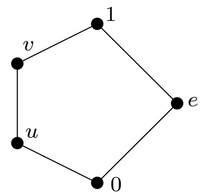




Proof. (a) Note that if L has a sublattice isomorphic to N_5, then since N_5 is non-modular, L is non-modular. Assume L is not modular. Then we have elements $d, e, f \in L$ such that $d \geq f$ and $(d \wedge e) \vee f \neq d \wedge (e \vee f)$. If $d = f$, then we have $(d \wedge e) \vee f = (f \wedge e) \vee f = f$ and $d \wedge (e \vee f) = f \wedge (e \vee f) = f$, so it must be that $d > f$.

Now let $u = (d \wedge e) \vee f$ and $v = d \wedge (e \vee f)$, and note that $d \wedge e \leq d$ and $d \wedge e \leq e \leq e \vee f$. So we have that $d \vee e \leq d \wedge (e \vee f)$. Similarly, since $f \leq d$ and $f \leq (e \vee f)$, we have that $f \leq d \wedge (e \vee f)$. Thus, $u = (d \wedge e) \vee f \leq d \wedge (e \vee f) = v$. But we already have that $u \neq v$, so it must be that $u < v$.

We will show that the following is a sublattice of L.



We have that

$$\begin{aligned} e \wedge v &= e \wedge ((e \vee f) \wedge d) \\ &= (e \wedge (e \vee f)) \wedge d \\ &= e \wedge d \end{aligned}$$

and

$$\begin{aligned} e \vee u &= e \vee ((e \wedge d) \vee f) \\ &= (e \vee (e \wedge d)) \vee f \\ &= e \vee f \end{aligned}$$

Now, consider

$$d \wedge e = (d \wedge e) \wedge e \leq u \wedge e \leq v \wedge e = d \wedge e$$

and

$$e \vee f = e \vee u \leq e \vee v \leq e \vee (e \vee f) = e \vee f$$

Thus, since $d \wedge e \leq u \wedge e \leq d \wedge e$, we have that $u \wedge e = d \wedge e = v \wedge e$. Similarly, we have $v \vee e = e \vee f = u \vee e$. Thus, since $u \vee e = v \vee e$ and $u \wedge e = v \wedge e$, we can conclude that L has a sublattice isomorphic to N_5.

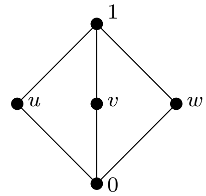
(b) Note that if L has a sublattice isomorphic to M_3 or N_5, then L is not distributive since neither of these lattices are distributive. Also, if L is non-distributive and non-modular, by (i) L has a sublattice isomorphic to N_5. So it suffices to show that if L is a modular, non-distributive lattice, then L has a sublattice isomorphic to M_3.

Let L be modular and not distributive. Then, for all $a, b, c \in L$, we know that $a \geq c$ implies that $(a \wedge b) \vee c = a \wedge (b \vee c)$, and there exist $d, e, f \in L$ such that $d \wedge (e \vee f) \neq (d \wedge e) \vee (d \wedge f)$. Note that since $d \wedge e \leq d$ and $d \wedge e \leq e \leq e \vee f$, then $d \wedge e \leq d \wedge (e \vee f)$. Also, since $d \wedge f \leq d$ and $d \wedge f \leq f \leq e \vee f$, we have that $d \wedge f \leq d \wedge (e \vee f)$ and so $(d \wedge e) \vee (d \wedge f) \leq d \wedge (e \vee f)$. Thus, since we know $d \wedge (e \vee f) \neq (d \wedge e) \vee (d \wedge f)$, we can conclude that $(d \wedge e) \vee (d \wedge f) < d \wedge (e \vee f)$.

Now, we define

$$\begin{aligned}
p &:= (d \wedge e) \vee (e \wedge f) \vee (f \wedge d) \\
q &:= (d \vee e) \wedge (e \vee f) \wedge (f \vee d) \\
u &:= (d \wedge q) \vee p \\
v &:= (e \wedge q) \vee p \\
w &:= (f \wedge q) \vee p
\end{aligned}$$

We will show that the following is a sublattice of L:



Clearly, $p \leq u$, $p \leq v$, and $p \leq w$. We can also prove that $p \leq q$. So,

$u \leq (d \wedge q) \vee q = q$, and similarly $v \leq q$ and $w \leq q$. Then since L is modular, we have that $d \wedge q = d \wedge (e \vee f)$. We also have that

$$\begin{aligned} d \wedge p &= d \wedge ((e \wedge f) \vee ((d \wedge e) \vee (d \wedge f))) \\ &= (d \wedge (e \wedge f)) \vee ((d \wedge e) \vee (d \wedge f)) \\ &= (d \wedge e) \vee (d \wedge f). \end{aligned}$$

The second equality holds since L is modular. So, since $d \wedge q \neq d \wedge p$, $p \neq q$. Thus, $p < q$. To show that $u \wedge v = p$, we have

$$\begin{aligned} u \wedge v &= (((d \wedge q) \vee p) \wedge (e \wedge q) \vee p) \\ &= (((e \wedge q) \vee p) \wedge (d \wedge q)) \vee p \\ &= ((q \wedge (e \vee p)) \wedge (d \wedge q)) \vee p \\ &= ((e \vee p) \wedge (d \wedge q)) \vee p \\ &= ((d \wedge (e \vee f)) \wedge (e \vee (f \wedge d))) \vee p \\ &= (d \wedge ((e \vee f) \wedge (e \vee (f \wedge d)))) \vee p \\ &= (d \wedge (((e \vee f) \wedge (f \wedge d)) \vee e)) \vee p \\ &= (d \wedge ((f \wedge d) \vee e)) \vee p \\ &= ((d \wedge e) \vee (f \wedge d)) \vee p \\ &= p \end{aligned}$$

The second, third, seventh, and ninth equalities hold since L is modular. Similarly, $v \wedge w = w \wedge u = p$. By similar calculations, $u \vee v = v \vee w = w \vee u = q$. Note then that if any pair of u, v, w, p, q are equal, then $p = q$, an impossibility. $\square$

Since a distributive lattice has no sublattice isomorphic to N_5, if a lattice is distributive, it is modular.

We have seen complete lattices in Section 2, and now we have seen distributive and modular lattices. There are several other types of lattices as well.

Definition 7.6. A lattice L is

(i) *bounded* if $\bigvee L$ and $\bigwedge L$ exist, i.e., if $\top$ and $\bot$ exist in L.

(ii) *complemented* if L is bounded, and if, for every $a \in L$, there exists $a' \in L$ such that $a \vee a' = \top$ and $a \wedge a' = \bot$. We say that a' is a *complement* of a.

(iii) a *Boolean algebra* if L is a complemented, distributive lattice.

Note that $\mathcal{P}(X)$ is a Boolean algebra with $\top = X$, $\bot = \emptyset$, and $A' = X \backslash A$, for every set $A \subseteq X$. The lattices $Idl(R)$ and $\mathcal{O}(\mathbb{R}^n)$ are clearly bounded, but are not generally complemented, and hence not Boolean algebras.

Proposition 7.7. *Let L be a Boolean algebra, and let $a \in L$. If b is a complement of a, and c is a complement of a, then $b = c$. In other words, complements are unique in Boolean algebras.*

Proof. If b and c are complements of a, then $b \vee a = \top = c \vee a$ and $b \wedge a = \bot = c \wedge a$. Then we have that:

$$\begin{aligned}
b &= b \wedge \top \\
&= b \wedge (a \vee c) \\
&= (b \wedge a) \vee (b \wedge c) \\
&= \bot \vee (b \wedge c) \\
&= (c \wedge a) \vee (c \wedge b) \\
&= c \wedge (a \vee b) \\
&= c \wedge \top \\
&= c
\end{aligned}$$

Thus $b = c$ and so complements are unique. $\square$

Proposition 7.8. De Morgan's Laws *Let L be a complemented, distributive lattice. Then $(a \wedge b)' = a' \vee b'$ and dually, $(a \vee b)' = a' \wedge b'$, for all $a, b \in L$.*

Proof. Consider the following.

$$\begin{aligned}
(a \wedge b) \vee (a' \vee b') &= (a \vee (a' \vee b')) \wedge (b \vee (a' \vee b')) \\
&= ((a \vee a') \vee b') \wedge (a' \vee (b \vee b')) \\
&= (\top \vee b') \wedge (a' \vee \top) \\
&= \top \wedge \top \\
&= \top
\end{aligned}$$

$$\begin{aligned}
(a \wedge b) \wedge (a' \vee b') &= ((a \wedge b) \wedge a') \vee ((a \wedge b) \wedge b') \\
&= (b \wedge (a \wedge a')) \vee (a \wedge (b \wedge b')) \\
&= (b \wedge \bot) \vee (a \wedge \bot) \\
&= \bot \vee \bot \\
&= \bot
\end{aligned}$$

Thus, $(a \wedge b) \vee (a' \vee b') = \top$ and $(a \wedge b) \wedge (a' \vee b') = \bot$, so $(a \wedge b)' = a' \vee b'$. $\square$

We can categorize exactly the finite lattices that are Boolean algebras. To do this, recall the definition of cover from Definition 2.4, and consider the following.

Definition 7.9. Let L be a lattice with $\bot \in L$. Then $a \in L$ is an *atom* if a covers $\bot$. The set of atoms of L is denoted $\mathcal{A}(L)$.

Lemma 7.10. *If L is a finite Boolean algebra, then $a = \bigvee\{x \in \mathcal{A}(L) | x \leq a\}$, for all $a \in L$.*

Proof. If $a = \bot$, then $\{x|x \leq a\} = \emptyset$, and so $\bot = \bigvee \emptyset = \bigvee\{x|x \leq a\}$. If a is an atom, then $\{x|x \leq a\} = \{a\}$ since a is an atom, and so $a = \bigvee\{x|x \leq a\}$.

Suppose $a \neq \bot$, and a is not an atom. Then there exists some $b \in L$ such that $\bot < b < a$. Now we have:

$$\begin{aligned} a &= a \wedge \top \\ &= a \wedge (b \vee b') \\ &= (a \wedge b) \vee (a \wedge b') \\ &= b \vee (a \wedge b') \end{aligned}$$

So $a = b \vee c$, where $c = a \wedge b'$. By assumption, $\bot < b < a$. To see that $\bot < c < a$, assume $c = \bot$. Then we have that $a = b \vee c = b \vee \bot = b$, contradicting that $b < a$, so $c \neq \bot$. Now, assume $c = a$, so we have $a \wedge b' = a$ and so $a' \vee b = a'$ by 7.8. Then $b < a$ and $b \leq a'$, so $b \leq a \wedge a' = \bot$ and thus $b = \bot$, a contradiction, so $c \neq a$.

Then $\bot < b, c < a$. If b and c are atoms, then $a \leq \bigvee\{b, c\} \leq \bigvee\{x|x \leq a\} \leq a$, as desired. If b and c are not atoms, repeat this process substituting a for b and c until they can be written as the joins of atoms. We know we will eventually reach this point since L is finite. Then, a is the join of atoms, so $a = \bigvee\{x|x \leq a\}$. □

Proposition 7.11. *A lattice L is a finite Boolean algebra if and only if L is isomorphic to $\mathcal{P}(X)$, for some finite set X.*

Proof. First we show if L is isomorphic to $\mathcal{P}(X)$, for some set X, then L must be a Boolean algebra. Note first that $\top_{\mathcal{P}(X)} = X$ and $\bot_{\mathcal{P}(X)} = \emptyset$. Suppose $f\colon \mathcal{P}(X) \to L$ is an isomorphism. Then $fX \leq \top_L = f(A)$, for some A since f is onto. Then $f^{-1}fX \leq f^{-1}f(A)$, so $X \subseteq A$ and thus $A = X$. So $\top_L = f(X)$. The proof that $f(\emptyset) = \bot_L$ is similar. Finally, for L to be a Boolean algebra, it must be distributive and complemented. To see that it is distributive, consider the following:

$$\begin{aligned}
a \wedge (b \vee c) &= ff^{-1}a \wedge (ff^{-1}b \vee ff^{-1}c) \\
&= f(f^{-1}a \cap (f^{-1}b \cup f^{-1}c)) \\
&= f((f^{-1}a \cap f^{-1}b) \cup (f^{-1}a \cap f^{-1}c)) \\
&= (ff^{-1}a \wedge ff^{-1}b) \vee (ff^{-1}a \wedge ff^{-1}c) \\
&= (a \wedge b) \vee (a \wedge c)
\end{aligned}$$

To see that L is complemented, let $a \in L$. Then since f is onto, there exists some $A \in \mathcal{P}(X)$ such that $a = fA$. So $a \wedge f(A') = fA \wedge f(A') = f(A \cap A') = f(\emptyset) = \perp_L$, and $a \vee f(A') = fA \vee f(A') = f(A \cup A') = f(X) = \top_L$. Thus, $f(A') = a'$ and so L is complemented.

Now let L be a finite Boolean algebra with $X = \mathcal{A}(L)$, and define $f \colon L \to \mathcal{P}(X)$ by $fa = \{x \in X | x \leq a\}$. To see that f is 1-1, let $fa = fb$, for some $a, b \in L$. Then $\{x \in X | x \leq a\} = \{x \in X | x \leq b\}$. So we have

$$\bigvee \{x \in X | x \leq a\} = \bigvee \{x \in X | x \leq b\}$$

and so $a = b$ by Lemma 7.10. To see that f is onto, let $S = \{x_1, \ldots, x_n\} \subseteq X$, and let $a = x_1 \vee \ldots \vee x_n$. Clearly, $S \subseteq fa$ since $x_i \leq a$, for $i = 1, \ldots, n$, and so $x_i \in fa$. Let $x \in fa$. Then x is an atom such that $x \leq a$. So

$$x = x \wedge a = x \wedge (x_1 \vee \ldots \vee x_n) = (x \wedge x_1) \vee \ldots \vee (x \wedge x_n)$$

Since $x \neq \perp$, we know that there exists some i such that $x \wedge x_i \neq \perp$. Then for this i, $\perp < x \wedge x_i \leq x$, so it follows that $x \wedge x_i = x$ since x is an atom. But then $x \leq x_i$, so it must be that $x = x_i$ since x_i is an atom, and thus $x \in S$. Then $S = fa$, and so we have that f is onto.

To show that f is a homomorphism, we must prove the following:

(a) $f(a \wedge b) = fa \cap fb$

(b) $f(a \vee b) = fa \cup fb$

First we show that f is order preserving. Assume $a \leq b$ and consider $fa = \{x | x \leq a\}$ and $fb = \{x | x \leq b\}$. Since $x \leq a$ for each $x \in fa$, by transitivity, we have $x \leq b$, so $x \in fb$. Thus, $fa \subseteq fb$, and so f is order preserving.

For (a), since f is order preserving, we need only show that $fa \cap fb \subseteq f(a \wedge b)$. Consider the following:

$$
\begin{aligned}
x \in fa \cap fb \ &\Rightarrow\ x \leq a, x \leq b, \text{ and } x \text{ is an atom} \\
&\Rightarrow\ x \leq a \wedge b \text{ and } x \text{ is an atom} \\
&\Rightarrow\ x \in f(a \wedge b)
\end{aligned}
$$

Similarly in (b), since f is order preserving, we need only show that $f(a \vee b) \subseteq fa \cup fb$. Consider the following:

$$
\begin{aligned}
x \in f(a \vee b) \ &\Rightarrow\ x \leq a \vee b \text{ and } x \text{ is an atom} \\
&\Rightarrow\ x = x \wedge (a \vee b) \\
&\Rightarrow\ x = (x \wedge a) \vee (x \wedge b)
\end{aligned}
$$

Then since $x \neq \perp$, $x \wedge a \neq \perp$ or $x \wedge b \neq \perp$. If $x \wedge a \neq \perp$, then $\perp < x \wedge a \leq x$, so $x \wedge a = x$ and thus $x \leq a$. So we have that $x \in fa$ and thus $x \in fa \cup fb$. Similarly, if $x \wedge b \neq \perp$, then $x \in fb$ and so $x \in fa \cup fb$.

So, we have that f is a 1-1 and onto homomorphism, and thus L is isomorphic to $\mathcal{P}(X)$, as desired. $\square$

Definition 7.12. A lattice L is

(i) *algebraic* if it is complete and if every element is the join of compact elements.

(ii) *continuous* if it is complete and $x = \bigvee\{y|y << x\}$, for all $x \in L$.

(iii) *completely continuous* if it is complete and $x = \bigvee\{y|y <<< x\}$, for all $x \in L$.

Note that there are certain relationships between lattices. We have seen already that a distributive lattice is a modular lattice. By definition, a Boolean algebra is a complemented lattice and a distributive lattice, and similarly a complemented lattice is a bounded lattice. To see that an algebraic lattice is a continuous lattice, assume L is an algebraic lattice and let $x \in L$. We have that $\bigvee\{y|y << x\} \leq x$, since $y << x$ implies that $y \leq x$. Now, since L is algebraic, $x = \bigvee\{a|a << a, a \leq x\}$. But since, for each a in this set, $a << a$ and $a \leq x$, $a << x$. Thus,

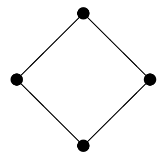
$$\{a|a << a, a \leq x\} \subseteq \{y|y << x\}, \text{ and so } \bigvee\{a|a << a, a \leq x\} \leq \bigvee\{y|y << x\}$$

So $x = \bigvee\{y|y << x\}$ and L is continuous. To see that a completely continuous lattice is a continuous lattice, let L be a completely continuous lattice, let $x \in L$ be arbitrary, and consider the following:

$$\begin{aligned} x &= \bigvee\{a|a <<< x\} \\ &\leq \bigvee\{a|a << x\} \\ &\leq x \end{aligned}$$

Thus, since $x \leq \bigvee\{a|a << x\} \leq x$, we have that $x = \bigvee\{a|a << x\}$, and so L is continuous.

The following are examples of continuous lattices from the posets we introduced in Example 2.3. Clearly,



is continuous. We will see that the other three examples, introduced in 2.3, are as well.

Definition 7.13. Let $r \in \mathbb{R}^+$, $c \in \mathbb{R}^n$ and $U, F \subseteq \mathbb{R}^n$. Then

(i) $D_r(c) = \{x \in \mathbb{R}^n \mid \|x - c\| < r\}$ is called an *open disk.*

(ii) $\overline{D}_r(c) = \{a \in \mathbb{R}^n \mid \|a - c\| \leq r\}$ is called a *closed disk.*

(iii) c is an *interior point* of U if $D_r(c) \subseteq U$, for some $r \in \mathbb{R}^+$.

(iv) U is *open* if every element of U is an interior point of U.

(v) F is *closed* if its complement $\mathbb{R}^n \backslash F$ is open.

Note that $D_r(x)$ is open, and $\overline{D}_r(x)$ is closed. One can also show that $\overline{D}_r(x)$ is a compact subset of $\mathbb{R}^n$ since it is closed and bounded (see [11]).

Proposition 7.14. *The set $\mathcal{O}(\mathbb{R}^n)$ of open subsets of $\mathbb{R}^n$ is a continuous lattice.*

Proof. First, note that, for all $U \in \mathcal{O}(\mathbb{R}^n)$, $\bigvee U_\alpha = \bigcup U_\alpha$ since $\bigcup U_\alpha$ is clearly open, and so $\mathcal{O}(\mathbb{R}^n)$ is complete. By Theorem 2.9,

$$\bigwedge U_\alpha = \left(\bigcap U_\alpha\right)^\circ = \bigcup \left\{ W \in \mathcal{O}(\mathbb{R}^n) \middle| W \subseteq \bigcap U_\alpha \right\}$$

It is easy to show that each of these are open sets, and so $\mathcal{O}(\mathbb{R}^n)$ is complete since all meets and joins exist in the lattice. Let $V \in \mathcal{O}(\mathbb{R}^n)$ be arbitrary, and consider the set $\bigcup\{U | U << V\}$. By Proposition 6.3a, we have that $U << V \Rightarrow U \leq V$, thus, for all $V \in \mathcal{O}(\mathbb{R}^n)$, $V \supseteq \bigcup\{U | U << V\}$. Let $x \in V$. Since V is open, we can find an $r \in \mathbb{N}$

such that $D_r(x) \subseteq V$. Let $U = D_{\frac{r}{2}}(x)$. Then $U \subseteq \overline{D}_{\frac{r}{2}}(x) \subseteq D_r(x) \subseteq V$. Since $\overline{D}_{\frac{r}{2}}(x)$ is compact, we have that $U << V$ by Proposition 6.3, and so $V \subseteq \bigcup\{U | U << V\}$. Then, $\mathcal{O}(\mathbb{R}^n)$ is continuous. □

Proposition 7.15. *$\mathcal{P}(X)$ is completely continuous, and hence continuous.*

Proof. First note that $\mathcal{P}(X)$ is complete with $\bigvee A_\alpha = \bigcup A_\alpha$ and $\bigwedge A_\alpha = \bigcap A_\alpha$, for all $A_\alpha \subseteq X$. Let $B \in \mathcal{P}(X)$. By Proposition 6.10, we found that $A <<< B$ if and only if $A = \{x\}$, for $x \in B$. Since $B = \bigcup\{\{b\} | b \in B\} = \bigcup\{A | A <<< B\}$, it follows that $B = \bigcup\{A | A <<< B\}$, and so $\mathcal{P}(X)$ is completely continuous. □

Proposition 7.16. *$Idl(R)$ is algebraic, and hence completely continuous and continuous.*

Proof. We first note that $Idl(R)$ is complete with

$$\bigwedge I_\alpha = \bigcap I_\alpha \text{ and } \bigvee I_\alpha = \sum_{\alpha \in A} I_\alpha$$

By Lemma 6.16, we have that $I = \Sigma\{Ra | a \in I\}$, for all $I \in Idl(R)$, and by Lemma 6.17, we have that Ra is compact. Thus, every $I \in Idl(R)$ is the join of compact elements, and so $Idl(R)$ is an algebraic lattice. □

Let $r < s$. We showed in Proposition 7.14 that $D_r(x) << D_s(x)$, and similarly one can show that there exists a set W such that $D_r(x) << W << D_s(x)$, namely $W = D_{\frac{r+s}{2}}(c)$. In fact, for all open sets $U << V$, we can find an open set between them. We will see that every continuous lattice satisfies this property.

Lemma 7.17. *Suppose I and J are ideals of a complete lattice L. If $I \subseteq J$, then $\bigvee I \leq \bigvee J$.*

Proof. Let $i \in I$. Then, since $I \subseteq J$, we have that $i \in J$. So $i \leq \bigvee J$. Then, since $i \leq \bigvee J$, for all $i \in I$, we conclude that $\bigvee I \leq \bigvee J$. □

Lemma 7.18. *Suppose $\{I_s\}_{s\in S}$ is a family of ideals of a complete lattice L, and $I = \bigcup_{s\in S} I_s$ is an ideal of L. Then $\bigvee I = \bigvee\{\bigvee I_s | s \in S\}$.*

Proof. Note that for every set X, we have that $\bigvee X \leq b$ iff $x \leq b$, for all $x \in X$. Let $i \in I$. Then there exists $s \in S$ such that $i \in I_s$. So $i \leq \bigvee I_s \leq \bigvee\{\bigvee I_s|s \in S\}$. Thus, $\bigvee I \leq \bigvee\{\bigvee I_s|s \in S\}$.

Now let $x \in \{\bigvee I_s|s \in S\}$ be arbitrary. Then there exists $s \in S$ such that $x = \bigvee I_s$. Note that $I_s \subseteq I$, so by Lemma 7.17, we can see that $x = \bigvee I_s \leq \bigvee I$. Then, since $x \leq \bigvee I$, for all $x \in \{\bigvee I_s|s \in A\}$, we have that $\bigvee\{\bigvee I_s|s \in A\} \leq \bigvee I$, as desired.

Then, we conclude that $\bigvee I = \bigvee\{\bigvee I_s|s \in A\}$, as desired. □

Proposition 7.19. *Let L be a continuous lattice, and let $a, b \in L$ with $a << b$. Then there exists $c \in L$ such that $a << c << b$.*

Proof. First, fix $a, b \in L$ with $a << b$, and define

$$I = \{x|x << c << b, \text{ for some } c \in L\}.$$

We claim that I is an ideal of L. Note that $\perp \in I$ since $\perp << a << b$, and thus $I \neq \emptyset$.

Suppose $x \in I$ with $y \leq x$. Then there exists $c \in L$ such that $x << c << b$. So, by Proposition 6.3c, we have that $y << c << b$, so $y \in I$. Now, let $x \in I$ and $y \in I$. Then there exists $c \in L$ and $d \in L$ such that $x << c << b$ and $y << d << b$. Then by Proposition 6.3c, since $c \leq c \vee d$ and $d \leq c \vee d$, we have that $x << c \vee d$ and $y << c \vee d$. Also, by Proposition 6.4, since $c << b$ and $d << b$, we have that $c \vee d << b$. Again by Proposition 6.4, since $x << c \vee d$ and $y << c \vee d$, we have that $x \vee y << c \vee d$. Thus, since $x \vee y << c \vee d << b$, we have that $x \vee y \in I$. So clearly, I is an ideal.

Now, since L is continuous and by Lemmas 7.17 and 7.18, we have the following system:

$$\begin{aligned} b &\leq \bigvee \{c | c << b\} \\ &= \bigvee \left\{ \bigvee \{x | x << c\} | c << b \right\} \\ &= \bigvee \left\{ \bigvee \{x | x << c << b, \text{ for some } c \in L\} \right\} \\ &= \bigvee I \end{aligned}$$

Thus, we have that $b \leq \bigvee I$. Then, since I is an ideal, and $a << b$, $a \in I$. So, we have that there exists $c \in L$ such that $a << c << b$, as desired. □

8 Adjoints Between Posets

In addition to lattices and lattice elements, we can also consider functions between lattices. First, we consider functions between posets.

Definition 8.1. Let P and Q be posets, and let $P \underset{g}{\overset{f}{\rightleftarrows}} Q$ be order-preserving maps. Then f is *left adjoint* to g, denoted $f \dashv g$, if $fx \leq y \Leftrightarrow x \leq gy$, for all $x \in P$, $y \in Q$.

Equivalently, we say that g is right adjoint to f. We will prove an equivalent definition.

Proposition 8.2. *Let* $P \underset{g}{\overset{f}{\rightleftarrows}} Q$ *be order-preserving maps. Then* $f \dashv g$ *if and only if*

(a) $fgy \leq y, \ \forall y \in Q$

(b) $x \leq gfx, \ \forall x \in P$

Proof. Assume $f \dashv g$, and let $y \in Q$ be arbitrary. Then, since $gy \in P$ and $gy \leq gy$ by reflexivity of $\leq$, we have that $fgy \leq y$. Now, let $x \in P$ be arbitrary. Then since $fx \in Q$ and $fx \leq fx$ by reflexivity of $\leq$, we have that $x \leq gfx$.

Now assume that $fgy \leq y$, for all $y \in Q$, and $x \leq gfx$, for all $x \in P$. Now, if $fx \leq y$, we have $x \leq gfx \leq gy$ since g and f are order preserving. Then, if

$x \leq gy$, we have that $fx \leq fgy \leq y$ again since f and g are order preserving. So, by transitivity, we have that $fx \leq y \Leftrightarrow x \leq gy$, and so $f \dashv g$. $\square$

For example, $\langle - \rangle \dashv i$, for $\mathcal{P}(R) \underset{i}{\overset{\langle - \rangle}{\rightleftarrows}} Idl(R)$, by Proposition 4.4.

Next, we show that right adjoints (dually, left adjoints) are unique, and preserve all greatest lower bounds (dually, least upper bounds).

Proposition 8.3. *Let* $P \overset{g_1}{\underset{g_2}{\leftleftarrows}} \!\!\xrightarrow{f}\! Q$. *If* $f \dashv g_1$ *and* $f \dashv g_2$*, then* $g_1 = g_2$.

Proof. Let f, g_1 and g_2 be as described, and let $f \dashv g_1$ and $f \dashv g_2$. Then we have:

(1) $fx \leq y \Leftrightarrow x \leq g_1 y$.

(2) $fx \leq y \Leftrightarrow x \leq g_2 y$.

Now, by reflexivity of $\leq$, we have that $g_1 y \leq g_1 y$, so by (1), we know $fg_1 y \leq y$. Thus, by (2), we have that $g_1 y \leq g_2 y$. Similarly, we can conclude that $g_2 y \leq g_1 y$. Now, since $g_1 y \leq g_2 y$ and $g_2 y \leq g_1 y$, for all $y \in Q$, we have $g_1 = g_2$, as desired. $\square$

Proposition 8.4. *Let* $P \underset{g}{\overset{f}{\rightleftarrows}} Q$ *with* $f \dashv g$. *If* $S \subseteq P$ *and* $\bigvee S$ *exists in* P*, then* $f(\bigvee S) = \bigvee fS$. *Dually, if* $S \subseteq Q$ *and* $\bigwedge S$ *exists in* Q*, then* $g(\bigwedge S) = \bigwedge gS$.

Proof. Given $y \in Q$,

$$\begin{aligned} fs \leq y, \text{ for all } s \in S \quad &\Leftrightarrow \quad s \leq gy, \text{ for all } s \in S && (1) \\ &\Leftrightarrow \quad \bigvee S \leq gy && (2) \\ &\Leftrightarrow \quad f(\bigvee S) \leq y && (3) \end{aligned}$$

We get (1) and (3) by definition of $f \dashv g$, and (2) by definition of $\bigvee$. Thus, by Proposition 2.7, $f(\bigvee S) = \bigvee fS$, and so f preserves $\bigvee$. $\square$

We can expand this to a stronger proposition.

Proposition 8.5. *Let $f\colon P \to Q$ with P and Q complete. Then f has a right adjoint if and only if f preserves $\bigvee$. Moreover, the right adjoint is given by*

$$gy = \bigvee\{x|fx \leq y\}$$

Proof. Assume that f has a right adjoint. Then by Proposition 8.4, f preserves $\bigvee$.

Now assime f preserves $\bigvee$. Define $g\colon Q \to P$ by $gy = \bigvee\{x|fx \leq y\}$. Then $fgy = f\bigvee\{x|fx \leq y\} = \bigvee\{fx|fx \leq y\}$, since f preserves $\bigvee$. Since for each $a \in \{fx|fx \leq y\}$, we have $a \leq y$, we conclude that $\bigvee\{fx|fx \leq y\} \leq y$ and so $fgy \leq y$.

Consider $gfx = \bigvee\{z|fz \leq fx\}$. Then, $x \in \{z|fz \leq fx\}$ since $\leq$ is reflexive and so $x \leq \bigvee\{z|fz \leq fx\} = gfx$.

Thus, since $fgy \leq y$, for all $y \in Q$ and $x \leq gfx$, for all $x \in P$, by Proposition 8.2, we have that $f \dashv g$ and so f has a right adjoint. □

We conclude the following by duality:

Corollary 8.6. *Let $g\colon Q \to P$ with P and Q complete. Then g has a left adjoint if and only if g preserves $\bigwedge$. Moreover, the left adjoint is given by*

$$fx = \bigwedge\{y|x \leq gy\}$$

Proposition 8.7. *If X and Y are sets, and $f\colon X \to Y$ is a function, then*

(a) $f\colon \mathcal{P}(X) \to \mathcal{P}(Y)$ *preserves* $\bigcup$

(b) $f^{-1}\colon \mathcal{P}(Y) \to \mathcal{P}(X)$ *preserves* $\bigcup$ *and* $\bigcap$

(c) $f \dashv f^{-1}$

Proof. First, we show that $f \dashv f^{-1}$. Let $B \in \mathcal{P}(Y)$ and $y \in f(f^{-1}(B))$. Then $y = fx$, for some $x \in f^{-1}(B)$. But, since $x \in f^{-1}(B)$, we know that $fx \in B$, and so $y \in B$. Thus, $f(f^{-1}(B)) \subseteq B$. Now let $A \in \mathcal{P}(X)$, and $x \in A$. Then $fx \in fA$, and so $x \in f^{-1}(f(A))$. Thus, $A \subseteq f^{-1}(f(A))$. So by Proposition 8.2, we have $f \dashv f^{-1}$. Then by Proposition 8.4, f preserves $\bigcup$, and by the dual of Proposition 8.4, f^{-1} preserves $\bigcap$.

To show that f^{-1} preserves $\bigcup$, consider the following:

$$\begin{aligned} x \in f^{-1}\left(\bigcup B_\alpha\right) &\Leftrightarrow fx \in \bigcup B_\alpha \\ &\Leftrightarrow fx \in B_\alpha, \text{ for some } \alpha \\ &\Leftrightarrow x \in f^{-1}B_\alpha \\ &\Leftrightarrow x \in \bigcup f^{-1}B_\alpha \end{aligned}$$

Thus, $f^{-1}(\bigcup B_\alpha) = \bigcup f^{-1}(B_\alpha)$, and so f^{-1} preserves $\bigcup$, as desired. ☐

We can use adjoints to categorize complete lattices. Suppose P is a poset and $x \in P$. Let $\downarrow x = \{y \in P | y \leq x\}$. Then $\downarrow$ defines an order-preserving function from P to $\mathcal{P}(P)$, since $\leq$ is transitive.

Proposition 8.8. *The function* $\downarrow : P \to \mathcal{P}(P)$ *is order preserving.*

Proof. Let $a, b \in P$ with $a \leq b$. Then $\downarrow a = \{x | x \leq a\}$ and $\downarrow b = \{x | x \leq b\}$. But then $x \leq a \leq b$ for $x \in \downarrow a$, so by transitivity, $x \leq b$. Thus, $x \in \downarrow b$, and so $\downarrow a \subseteq \downarrow b$. So, we conclude that $\downarrow$ is order preserving. ☐

Proposition 8.9. $\downarrow : P \to \mathcal{P}(P)$ *has a left adjoint if and only if* P *is complete. In this case,* $\bigvee \dashv \downarrow$.

Proof. Assume $\downarrow$ has a left adjoint. Then there exists some $f : \mathcal{P}(P) \to P$ such that

$f \dashv \downarrow$. So, for every set $S \in \mathcal{P}(P)$, we have:

$$\begin{aligned} s \leq x, \text{ for all } s \in S &\Leftrightarrow S \subseteq \downarrow x \\ &\Leftrightarrow fS \leq x \end{aligned}$$

Then, since $s \leq x$, for all $s \in S \Leftrightarrow fS \leq x$, we have that $\bigvee S$ exists, and $fS = \bigvee S$, and thus P is complete.

Now, assume P is complete. Then we know that for every $S \in \mathcal{P}(P)$, we have:

$$\begin{aligned} \bigvee S \leq y &\Leftrightarrow s \leq y, \text{ for all } s \in S \\ &\Leftrightarrow S \subseteq \downarrow y \end{aligned}$$

So, since $\bigvee S \leq y \Leftrightarrow S \subseteq \downarrow y$, we conclude that $\bigvee \dashv \downarrow$. $\square$

Proposition 8.10. *Suppose P is a join-semilattice. Then*

(a) $\downarrow x$ is an ideal of P, for all $x \in P$.

(b) $\downarrow: P \to Idl(P)$ has a left adjoint if and only if $\bigvee I$ exists for all ideals I. In this case, $\bigvee \dashv \downarrow$.

Proof. Consider $y \in \downarrow x$ and let $z \leq y$. Since $y \in \downarrow x$, $y \leq x$. Then, by transitivity, $z \leq x$ so $z \in \downarrow x$. Now, let $y, z \in \downarrow x$. Then $y \leq x$ and $z \leq x$. So $y \vee z \in \downarrow x$ by definition of least upper bound. Then, since $\downarrow x$ is downward closed and closed under joins, $\downarrow x$ is an ideal.

Suppose there exists some $f: Idl(P) \to P$ such that $f \dashv \downarrow$. So, for every set $I \in Idl(P)$, we have:

$$\begin{aligned} a \leq x, \text{ for all } a \in I &\Leftrightarrow I \subseteq \downarrow x \\ &\Leftrightarrow fI \leq x \end{aligned}$$

Then, since $a \leq x$, for all $a \in I \Leftrightarrow fI \leq x$, we have that $fI = \bigvee I$, and thus $\bigvee I$ exists for all ideals I of P.

Now assume $\bigvee I$ exists for all ideals. Then we know that for every $J \subseteq Idl(P)$, we have:

$$\begin{aligned} \bigvee J \leq x &\Leftrightarrow a \leq x, \text{ for all } a \in J \\ &\Leftrightarrow J \subseteq \downarrow x \end{aligned}$$

Thus, since $\bigvee J \leq x \Leftrightarrow J \subseteq \downarrow x$, we conclude that $\bigvee \dashv \downarrow$. $\square$

In a lattice L, we recall that $\Downarrow x = \{y \in L | y << x\}$. Note that $\Downarrow x$ is an ideal by Proposition 6.5.

Proposition 8.11. *Assume $\bigvee I$ exists for all $I \in Idl(P)$. Then $\bigvee \colon Idl(P) \to P$ has a left adjoint if and only if P is continuous. In this case, $\Downarrow \dashv \bigvee$.*

Proof. Assume P is continuous. Consider $\Downarrow \colon P \to Idl(P)$, and let $I \in Idl(P)$. Then $\Downarrow \bigvee I = \{y | y << \bigvee I\}$. Given $a \in \{y | y << \bigvee I\}$, since I is an ideal, $a << \bigvee I$ implies that $a \in I$. So we have that $\Downarrow \bigvee I \subseteq I$.

Now let $x \in P$. Then $\bigvee \Downarrow x = \bigvee \{y | y << x\} = x$ since P is continuous. Thus, by Proposition 8.2, $\Downarrow \dashv \bigvee$, so $\bigvee$ has a left adjoint.

Assume there exists $f \colon P \to Idl(P)$ such that $f \dashv \bigvee$. First, we show $fx \subseteq \Downarrow x$, for all $x \in P$. Let $x \in P$, and assume $a \in fx$. To show $a << x$, let $x \leq \bigvee I$, for some $I \in Idl(P)$. Then since f is order preserving, $a \in fx \Rightarrow a \in f \bigvee I$. But since $f \dashv \bigvee$, we know $f \bigvee I \subseteq I$, and so $a \in I$. Thus, $a << x$, and we can conclude that $fx \subseteq \Downarrow x$.

Now, since $f \dashv \bigvee$, we know that $x \leq \bigvee fx$. Then, since $fx \subseteq \Downarrow x$, we know $\bigvee fx \leq \bigvee \Downarrow x$ by Lemma 7.17. Since $x \leq \bigvee fx \leq \bigvee \Downarrow x \leq x$, it follows that $x = \bigvee \Downarrow x$, for all $x \in P$. Thus P is continuous. $\square$

Propositions 8.10 and 8.11 give us that if P is a continuous lattice, then $\Downarrow \dashv \bigvee \dashv \downarrow$, for

$$\begin{tikzcd} P \arrow[r, "\vee" description] & Idl(P) \arrow[l, bend right, "\Downarrow"'] \arrow[l, bend left, "\downarrow"] \end{tikzcd}$$

We can prove properties of functions between ideals of a ring. Recall that for $I, J \in Idl(R)$,

$$IJ = \left\{ \sum_{i=1}^{n} a_i b_i \middle| a_i \in I, b_i \in J \right\} \text{ and } I \colon J = \{r \in R | rJ \subseteq I\}$$

Proposition 8.12. *Suppose $J \in Idl(R)$. Then $-\cdot J \dashv - \colon J$, where $Idl(R) \underset{-\colon J}{\overset{-\cdot J}{\rightleftarrows}} Idl(R)$.*

Proof. By Proposition 4.4,

$$\begin{aligned} IJ \subseteq K &\Leftrightarrow \langle \{ab | a \in I, b \in J\} \rangle \subseteq K \\ &\Leftrightarrow ab \in K, \text{ for all } a \in I, b \in J \\ &\Leftrightarrow aJ \subseteq K \\ &\Leftrightarrow a \in K \colon J, \text{ for all } a \in I \\ &\Leftrightarrow I \subseteq K \colon J \end{aligned}$$

Then, since $IJ \subseteq K \Leftrightarrow I \subseteq K \colon J$, by definition, we have that $- \cdot J \dashv - \colon J$. □

Proposition 8.13. *Consider $\mathcal{P}(X)$ and $\mathcal{O}(\mathbb{R}^n)$. Then $- \cap B \colon \mathcal{P}(X) \to \mathcal{P}(X)$ and $- \cap V \colon \mathcal{O}(\mathbb{R}^n) \to \mathcal{O}(\mathbb{R}^n)$ have right adjoints.*

Proof. Let $A, B, C \in \mathcal{P}(X)$, and define the function $B \Rightarrow -$ by $B \Rightarrow C = (X \backslash B) \cup C$. To show that $- \cap B \dashv B \Rightarrow -$, assume that $A \cap B \subseteq C$ and let $x \in A$. Then, if $x \in B$, we know that $x \in A \cap B$, so $x \in C$ and thus $x \in (X \backslash B) \cup C$. Alternatively, if $x \notin B$, then $x \in X \backslash B$, and so $x \in (X \backslash B) \cup C$. Thus, $A \cap B \subseteq C \Rightarrow A \subseteq (X \backslash B) \cup C$. Now assume $A \subseteq (X \backslash B) \cup C$, and let $x \in A \cap B$. Then $x \in A$ and $x \in B$. But since $x \in A$, we know that $x \in (X \backslash B) \cup C$, so $x \in X \backslash B$ or $x \in C$. Since $x \in B$, we know $x \notin X \backslash B$, so it must be that $x \in C$. Thus, these give us that $A \cap B \subseteq C \Leftrightarrow A \subseteq (X \backslash B) \cup C$, so by definition, $- \cap B \dashv B \Rightarrow -$.

By 8.5 we have that $- \cap B$ preserves joins in $\mathcal{P}(X)$, and so

$$\left(\bigcup_{\alpha} A_{\alpha}\right) \cap B = \bigcup_{\alpha}(A_{\alpha} \cap B)$$

In particular, we have that $- \cap V$ preserves joins in $\mathcal{O}(\mathbb{R}^n)$ from the proof in $\mathcal{P}(\mathbb{R}^n)$. So also by 8.5, $- \cap V$ has a right adjoint, and if $f(U) = U \cap V$, we get that $g(W) = \bigcup\{U | f(U) \subseteq W\}$. Then we can find the right adjoint, $V \Rightarrow -$. So we have:

$$\begin{aligned} V \Rightarrow W &= \bigcup\{U | U \cap V \subseteq W\} \\ &= \bigcup\{U | U \subseteq (\mathbb{R}^n \backslash V) \cup W\} \\ &= [(\mathbb{R}^n \backslash V) \cup W]^{\circ} \end{aligned}$$

Then we have that $- \cap B \dashv B \Rightarrow -$ in $\mathcal{P}(X)$, and $- \cap V \dashv V \Rightarrow -$ in $\mathcal{O}(\mathbb{R}^n)$. $\square$

We have shown in 8.9 and 8.10 that $Idl(R)$, $\mathcal{P}(X)$, and $\mathcal{O}(\mathbb{R}^n)$ are each what is known as a commutative quantale, i.e. a complete lattice Q together with a commutative, associative operation $\cdot$ such that $a \cdot (\bigvee b_{\alpha}) = \bigvee(a \cdot b_{\alpha})$, for all $a \in Q$ and $\{b_{\alpha}\} \in Q$. For more on quantales, see [10].

References

[1] B. A. Davey and H. A. Priestly. *Introduction to Lattices and Order*. Cambridge Mathematical Textbooks, New York, 1990.

[2] Reinhard Diestel. *Graph Theory*. Springer-Verlag, New York, 2006.

[3] G. Gierz, K. H. Hofmann, K. Keimel, J. D. Lawson, M. Mislove, and D. S. Scott. *A Compendium of Continuous Lattices*. Springer-Verlag, 1980.

[4] G. Gierz, K. H. Hofmann, K. Keimel, J. D. Lawson, M. Mislove, and D. S. Scott. *Continuous Lattices and Domains*. Cambridge University Press, New York, 2003.

[5] George Gratzer. *Lattice Theory: First Concepts and Distributive Lattices*. Dover Publications, Inc, Mineola, N.Y., 1971.

[6] George Gratzer. *General Lattice Theory*. Birkhauser Verlag, Boston, 2003.

[7] I. N. Herstein. *Abstract Algebra*. John Wiley & Sons, Inc., Hoboken, NJ, 1999.

[8] Max D. Larson and Paul J. McCarthy. *Multiplicative Theory of Ideals*. Academic Press, New York, 1971.

[9] J. J. O'Connor and E. F. Robertson. Garrett Birkhoff. Website, May 2000. `http://www-history.mcs.st-andrews.ac.uk/Biographies/Birkhoff_Garrett.html`.

[10] Kimmo I. Rosenthal. *Quantales and their Applications*. Longman Scientific & Technical, New York, 1990.

[11] Walter Rudin. *Principles of Mathematical Analysis*. McGraw-Hill, New York, 1986.