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Elections with Three Candidates Four Candidates and Beyond: Counting Ties in the Borda Count with Permutahedra and Ehrhart Quasi-Polynomials

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Elections with Three Candidates, Four Candidates, and Beyond:
Counting Ties in the Borda Count with Permutahedra and Ehrhart Quasi-Polynomials

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Contents

Abstract: In voting theory, the Borda count's tendency to produce a tie in an election varies as a function of *n*, the number of voters, and *m*, the number of candidates. To better understand this tendency, we embed all possible rankings of candidates in a hyperplane sitting in m-dimensional space, to form an (m - 1)-dimensional polytope: the m-permutahedron. The number of possible ties may then be determined computationally using a special class of polynomials with modular coefficients. However, due to the growing complexity of the system, this method has not yet been extended past the case of $m = 3$. We examine the properties of certain voting situations for $m \geq 4$ to better understand an election's tendency to produce a Borda tie between all candidates.

Introduction

This paper covers a wide range of sub-topics in the field of social choice theory, with the overarching goal of chronicling and expanding research into the properties of a voting rule known as the Borda count. Specifically, our motivation is to count the number of different ways an election will produce a tie between all candidates according to this system. This problem is central to determining the Borda count's decisiveness, or proclivity to produce an individual winner from an election instead of a tie.[1] Over the last fifteen years, a great deal of research in this area has led to some useful results. However, due to the computational complexity of the problem, most of this research has been limited to the case of 3-candidate elections.

The paper is divided into 3 main sections. In Section I, we will provide general background on voting theory, introduce the Borda count, and present some simple results relevant to the problem of counting ties. In particular, we will discuss a geometric object called the permutahedron, and explain how it is used to count the number of "Borda ties" (i.e., ties using the Borda count as the voting rule) between all candidates, in a method first discovered by Union College Professor William Zwicker in 2008 [3].

In Section II, we will review research on the problem of counting ties in 3-candidate elections. The majority of this section focuses on two papers in particular, one by French

[1] The term *decisiveness* often refers to a black and white condition: a voting rule satisfies decisiveness if and only if it always produces a unique winner. A weaker version of decisiveness has been defined as *generic decisiveness*, where a voting rule satisfies generic decisiveness if and only if it nearly always produces a unique winner. Of course, this begs the question of what constitutes "nearly always". For a comparison of these definitions, we refer the reader to [1]. Our use of decisiveness, as a measure of how often a voting rule produces a tie, is more in the vein of Cervone et al [2].

economist Thierry Marchant, and the other by Steven Sam and Kevin Woods, professors at the Massachusetts Institute of Technology and Oberlin College, respectively [4 - 6]. Interestingly, both papers were themselves expansions of mathematical research done mainly in the 1960s for a purpose entirely unrelated to voting theory.[2] Marchant used the theory of random walks on special lattices, developed by Cyril Domb in 1960 for the study of crystallography, to count 3-way Borda ties in 3-candidate elections as a function of the number of voters in the electorate [7]. Sam and Woods, meanwhile, provided an alternative proof of a theorem first proved by French mathematician Eugene Ehrhart in 1962 [8]. We will conclude Section II by reviewing Union College thesis student Rhongua Dai's 2008 application of Ehrhart theory for counting Borda ties [9]. Dai's research came after a 2008 paper by Lepelley, Louichi, and Smaoui, in which they made the first connection between Ehrhart Theory and the problem of counting specific sets of ballots cast in an election [10]. Drawing on Lepelley et al, and using a different set of assumptions than Marchant (thus changing the problem), Dai used computer software to count the number of 3-ways Borda ties in 3-candidate elections as a function of the number of voters.

We will begin Section III by giving a brief overview of the other half of Dai's thesis, where he confirms his computer generated result with basic combinatorial methods. We will then present our own research into the problem of counting Borda ties. Since Dai's Ehrhart Theory approach for more than 3 candidates was impeded by the limitations of computing power, our research expands on his combinatorial methods. In essence, we have used the permutahedron to begin to classify and understand the relationships

[2] Sam and Woods do not work on voting theory either; however, we believe their new proof of Ehrhart's seminal theorem to be quite useful for anyone attempting to understand why functions that count ties in the Borda Count take the form that they do.

between different collections of ballots that produce an "all-way" tie when cast in an election. We conclude Section III by presenting the results from a computer program that we have written to aid us in our classification efforts.

While a great deal of work has been done in the pursuit of determining the decisiveness of the Borda count, there is still much that remains, particularly for elections with 4 or more candidates. As we will see, research related to this problem spans various topics in discrete mathematics, abstract and linear algebra, and affine geometry, to name just a few relevant fields. Therefore, just as recent efforts to count Borda ties have drawn on initially unrelated work, it is quite possible that future research in this direction will open doors outside of social choice theory in exciting and unforeseen ways.

Section I:

The Borda Count, Permutahedra, and Central Voting Situations

1.0 Introduction to the Borda count

Historically, methods for choosing the winner of an election have varied from society to society. The best-known method for a two-candidate election is *Majority Rule* voting, wherein each member of an electorate casts one vote for a preferred candidate. Of course, the candidate receiving the greatest number of votes wins the election. In 1952, American mathematician Kenneth May demonstrated that Majority Rule is the only method for determining the winner of a two-candidate election that meets a specific set of desirable criteria[3] [11]. Examples of Majority Rule voting systems abound. For instance, most U.S. states use Majority Rule to determine which presidential candidate will be awarded that state's total number of electoral votes.

For elections consisting of three or more candidates, the choice of voting rule (formally defined in Section 1.3) is less clear. Different rules obey different sets of desirable and undesirable properties, and in 1951, American economist Kenneth Arrow showed that there is no one 'perfect' voting rule for an election between three or more candidates, i.e., no such voting rule could possess every desirable property[4] [12]. Our

[3] While we do not delve into detail here, the conditions that hold for Majority Rule in two-candidate elections are anonymity, neutrality, and positive responsiveness. Formal definitions and May's original proof of his theorem may be found in [11].
[4] Again, we do not define these properties here, but they include Pareto efficiency, non-dictatorship, and independence of irrelevant alternatives (IIA), in addition to some other conditions [12].

paper is concerned with certain properties of the Borda count, a rule proposed by French political scientist and mathematician Jean-Charles De Borda in 1770. The Borda count is currently used by various political and private organizations.[5]

We will now define the Borda count. Let $A = \{a_1, a_2, \ldots, a_m\}$ be a finite set of *alternatives* (or *candidates*); let $V = \{v_1, v_2, \ldots, v_n\}$ be a finite set of *voters*; let a *ranking* σ be a strict linear ordering of the alternatives; and let L(A) be the set of all rankings. Since |A| = m, then there are m! possible rankings, so |L(A)| = m!. In the Borda count, each voter casts a ballot that corresponds to one of the m! rankings, rather than voting for a single alternative. The first-place alternative in a single voter's ballot is assigned m - 1 points, the second-place alternative is assigned m - 2 points, and so on, until the last-place alternative receives 0 points.

For instance, consider the three-candidate case where A = {p, q, r}. Then there are 3! (so 6) possible rankings in L(A). For each ballot cast in the election, 2 points are awarded to the first-choice candidate on the ballot's ranking, 1 point to the second-choice candidate, and no points to the candidate lowest on the ranking.

The *winner(s)* or *social choice(s)* of the election is/are the alternative(s) with a greatest sum of points over all voters at the end of voting. In some instances, the Borda count may result in a tie in the election.

[5] The National Assembly of Slovenia uses the Borda Count to elect two ethnic minority (Italian and Hungarian) members and the Associated Press uses the Borda Count to rank American college (NCAA) athletes, to name just a few current uses [13-14]. Interestingly, the Borda Count dates back far before Jean-Charles De Borda described it in the 18th century: the Roman Senate used the Borda Count as a voting method as early as 105 AD [15].

1.1 Independent Culture and Independent Anonymous Culture

A *profile* reflects all ballots cast in some election, and it is convention to represent a profile by listing each ballot below a corresponding voter in column form.[6] For example, consider the election with three candidates (A = {p, q, r}) and four voters (V = {v_1, v_2, v_3, v_4}), where v_1 and v_2 both cast ballots for the ranking with p first, q second, and r third (denoted $p > q > r$) , v_3 casts a ballot for $q > r > p$, and v_4 casts a ballot for $r > p > q$. This profile may then be written as

v_1	v_2	v_3	v_4
p	p	q	r
q	q	r	p
r	r	p	q

According to the Borda count, in the above profile, p receives 5 points (2 each from v_1 and v_2, 0 from v_3, and 1 from v_4); similarly, q receives 4 points and r receives 3 points. Thus p is the winner.

An *anonymous profile* or *voting situation* is an m!-tuple

$$\boldsymbol{\Pi} = (n_1, n_2, \dots, n_{m!})$$

of non-negative integers satisfying $n = \sum_{j=1}^{m!} n_j$, where n is the number of voters in the electorate. For the case of three alternatives, we interpret each n_j as the number of voters who cast the j^{th} ballot in the following list:

[6] More formally, a profile may be defined as a function P: N→L(A), i.e., as a function that assigns a voter to a ranking in L(A). We say $P \in L(A)^N$.

1. p > q > r
2. p > r > q
3. q > p > r
4. q > r > p $\circledast$
5. r > p > q
6. r > q > p

Note that for our purposes, the choice of how to index the rankings is largely arbitrary. In $\circledast$, we listed them by *lexicographic* order, where rankings are positioned according to their linear ordering in a dictionary. By way of example, the voting situation corresponding to the previous profile would be

$$\boldsymbol{\Pi} = (2, 0, 0, 1, 1, 0).$$

A voting situation may also be displayed in the same form as a profile, with number of voters, rather than the name of a voter, listed above a given ranking. We think of a voting situation as corresponding to a profile where we are not interested in which voters cast a given ballot. In other words, we treat two profiles as the same voting situation if voters simply swap ballots.

The probability of there being a tie in the Borda count depends on whether we are counting profiles or voting situations, as this probability can be expressed as either the number of profiles that produce ties over the number of possible profiles, or as the number of voting situations that produce ties over the number of possible voting situations. There are two main assumptions relevant to the problem of counting Borda ties, and which assumption we choose corresponds to our choice between counting voting situations and counting profiles [16]. If we assume *Independent Culture* (IC), then we assume that each profile is equally likely to occur. Since there are n ballots in a profile (one for each voter) and m! possible rankings for a ballot, the probability of seeing a given profile is always $\left(\frac{1}{m!}\right)^n$. For example, the probability of observing any m =3, n = 4 profile is just $\left(\frac{1}{6}\right)^4 = \frac{1}{1296}$, or approximately 0.08%.

In *Independent Anonymous Culture* (IAC), we assume that each possible voting situation is equally likely to occur. Thus in order to find the probability of observing a specific voting situation, we must count the number of total voting situations for m candidates and n voters. We do this using the following well-known combinatorial theorem.

Theorem 1.1. Consider all r-tuples $(a_1, a_2, \dots, a_r)$ such that $a_1, a_2, \dots, a_r \in \mathbb{Z}_{\geq 0}$ and $\sum_{i=0}^{r} a_i = b$. Then there exists $\binom{b+r-1}{r-1}$ distinct r-tuples.

Proof: Since each a_i is nonnegative, we can replace each a_i by a sequence of a_i 1s (which means no 1s at all in the case of $a_i = 0$). In this form, there are now $\sum_{i=0}^{r} a_i$ 1s, or b 1s, separated by r - 1 commas. We see that the placement of the commas now determines the value of each a_i when we convert the tuple back to its original form. An example of this is shown in Figure 1 below. Thus there are (b + r - 1) objects (1s + commas), and we can form any r-tuple (the original form) by choosing the position of the r - 1 commas. Thus, there are $\binom{b+r-1}{r-1}$ possible r-tuples, as desired. ■

$$\Pi = (1\ 1, \quad , \quad , 1 , 1, \quad) = (2, 0, 0, 1, 1, 0)$$

Figure 1. An example of the entries of 1's and commas that correspond to the voting situation from our earlier example.

Applying Theorem 1.1 and our definition of a voting situation, we see that the total number of voting situations in an election with m candidates and n voters is simply $\binom{n+m!-1}{m!-1}$. Therefore, if we assume IAC, the probability of a given voting situation occurring is

$$\frac{1}{\left(\frac{(n+m!-1)!}{\big((m!-1)!\big)(n+m!-1-(m!-1))!}\right)} = \frac{1}{\left(\frac{(n+m!-1)!}{\big((m!-1)!\big)(n!)}\right)} = \frac{\big((m!-1)!\big)(n!)}{(n+m!-1)!}.$$

For example, the IAC-probability for seeing a given m = 4, n = 4 voting situation is $\frac{(5!)(4!)}{9!} = \frac{1}{126}$, or 0.79%.

As we will see, the probability that an election will produce a Borda tie is different depending on whether we assume IC or IAC for our probability distribution (a simple demonstration of this difference can be done for the case of m = 3 and n = 2). With the exception of Sections 2.0 and 2.1, the remainder of the paper will be concerned with IAC, and therefore the problem of counting voting situations.

1.2 Permutahedra and the Borda count

In order to geometrically interpret the Borda count, we now assume that every finite set A of alternatives comes equipped with a single, fixed *reference enumeration,* where we take the reference enumeration of $\{a_1, a_2, \ldots, a_m\}$ to be the linear ordering $a_1 > a_2 > \ldots > a_m$. We denote the reference enumeration of $A = \{a_1, a_2, \ldots, a_m\}$ as $A_r = \langle a_1, a_2, \ldots, a_m \rangle$. For a_i and a_j as any two candidates in A_r, we write $a_i >_\sigma a_j$ if the ranking σ ranks alternative a_i above a_j. For a given ranking σ, the *rank* $\rho(a_j)$ is the number of alternatives a_k

in A_r satisfying $a_j >_\sigma a_k$, and a *rank vector* $\boldsymbol{\rho}(\sigma)$ is the m-tuple $(\rho(a_1), \rho(a_2), \ldots, \rho(a_m))$, listing ranks in reference enumeration order.

As an example for the lexicographic reference enumeration <p, q, r> of three alternatives, consider the 3rd ranking, q > p > r, taken from our list ⊛ . Then the rank $\rho(p) = 1$, since alternative p is only ranked above one other alternative, r. It follows that the rank vector $\boldsymbol{\rho}(3)$ for the 3rd ranking is (1, 2, 0), as $\rho(p) = 1$, $\rho(q) = 2$, and $\rho(r) = 0$. Note how $\boldsymbol{\rho}(3)$ corresponds to how many points p, q, and r receive, respectively, from a single ballot cast for the 3rd ranking.

From a geometric perspective, the function ρ assigns each ranking to a point in $\mathbb{R}^m$. These points are the vertices of the *m*-permutahedron, an (m – 1)-dimensional polytope (see Section 2.5 for definition of dimension and polytope) living in $\mathbb{R}^m$ with edges between points that differ only in the reversal of a single pair of alternatives. As seen in Figure 2, the 3-permutahedron is a regular hexagon in $\mathbb{R}^3$. Recently, Zwicker showed that the Borda count has an equivalent geometric form, called the Permuta-mean rule, whereby the winner of the election is the candidate atop the ranking on the permutahedron closest to the mean of all rank vectors of cast ballots (counting multiplicity) in the election [3]. Thus, an m-way tie corresponds to the mean point existing at the center of the permutahedron.

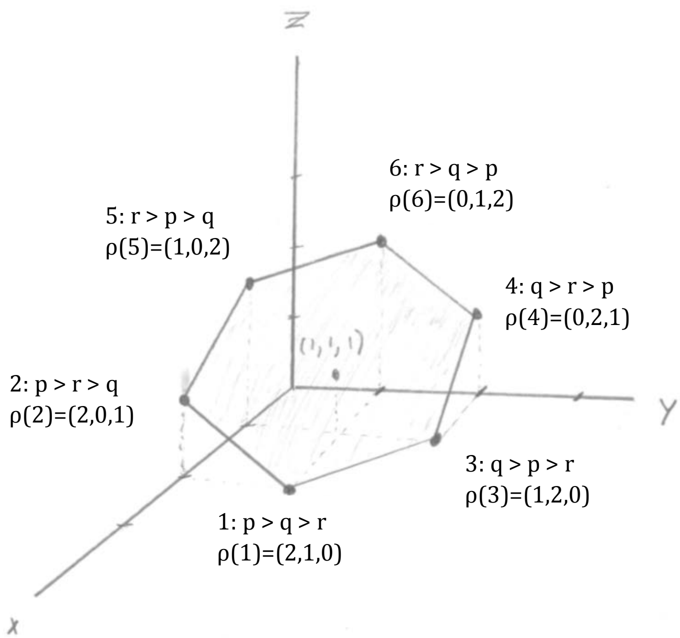

Figure 2. The 3-permutahedron is a regular hexagon centered at the point (1,1,1).

It should be noted that traditionally, the *m*-permutahedron has vertices that are permutations of the set {1, 2, . . . , m}, not {0, 1, . . ., m - 1}, and lives in the hyperplane $x_1 + x_2 + \ldots + x_m = \sum_{i=0}^{m-1} m - i$. We use our version of the *m*-permutahedron to correspond to our definition of the Borda count. However, we observe that our definition of the Borda count is equivalent to a version that assigns *scoring weights* (introduced in Section 1.3) from *m* down to 1, rather than *m - 1* down to 0. We will also discuss other variations of the Borda count in Section 1.3.

Although the 4-permutahedron lives in $\mathbb{R}^4$, it can still be visualized as a 3-dimensional polytope, the truncated octahedron. This is shown in Figure 3.

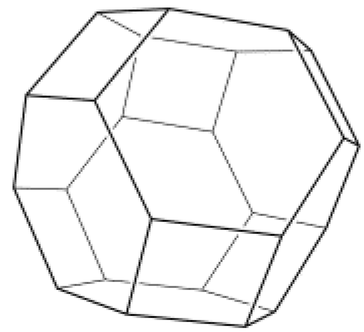

Figure 3. The 4-permutahedron is a truncated octahedron [2].

The 4-permutahedron has 14 faces (8 regular hexagonal and 6 square), 24 vertices, and 36 edges. The myriad symmetries in the truncated octahedron are central to the problem of counting the number of ways candidates can tie in the Borda count.

1.3 The origin-centered Borda count

Before we begin our discussion of counting ties, it is necessary to take a step back and briefly discuss how the Borda count fits into the bigger picture of methods used to determine election outcomes. First, we define a *voting rule*. For a finite set A of m alternatives and a finite set V of n voters, a *social choice correspondence* (or *voting rule*) is a function that assigns to each profile or voting situation in $L(A)^N$ a non-empty set of alternatives containing the winner(s) or social choice(s). The Borda count belongs to a particular set of voting rules called *scoring rules*, which can be defined in the following way:

Choose a vector $\mathbf{w} = \langle w_1, w_2, \dots, w_m \rangle$ of real-number *scoring weights,* satisfying $w_1 \geq w_2 \geq \dots \geq w_m$. Each voter awards w_1 points to a top choice, w_2 points to a second choice, etc. The winner is the alternative(s) with the greatest point total (sum of points over all voters).

Thus, in Section 1.0, we defined the Borda count as a scoring rule where the scoring weights are set as $w_1 = m - 1, w_2 = m - 2, \ldots, w_m = 0$. However, it is possible to adjust the scoring weights to obtain a scoring rule equivalent to this version of the Borda count. In general, we can prove that a scoring rule is equivalent to the Borda count as defined above iff it its scoring vector is a positive affine transform (see Section 2.3) of the scoring vector defined above. In other words, any scoring rule with evenly-spaced weights that are strictly decreasing yields the Borda count. One common version of the Borda count is a scoring rule with the scoring weights decreasing from m to 1 rather than from m - 1 to 0. However, we are interested in another version of the Borda count, which we'll call the *origin-centered* (*o-c*) Borda count. The o-c Borda count for m alternatives is the scoring rule with the vector of scoring weights $\langle w_i \rangle$ defined as follows:

(a) $$\langle w_i \rangle = \langle 2(j - i) + 1 \rangle_{i=1}^{m} \text{ where } m = 2j \text{ for some } j \in \mathbb{Z}_{>0}$$

(b) $$\langle w_i \rangle = \langle (j - i) + 1 \rangle_{i=1}^{m} \text{ where } m = 2j + 1 \text{ for some } j \in \mathbb{Z}_{\geq 0}$$

As an example, in a three-candidate election, $m = 2j + 1$with $j = 1$, so that the candidate in first place on a voter's ballot receives $(1 - 1) + 1 = 1$ point, the second-place candidate receives $(1 - 2) + 1 = 0$ points, and the candidate in last place receives $(1 - 3) + 1 = -1$ point. For four candidates, $m = 2j$ with $j = 2$, so that first place receives 3 points, second place receives 1 point, third place receives -1 point, and last place receives -3 points.

At first glance, it may seem that the only real advantage from using the o-c Borda count is the more natural position of the m-permutahedron, which is now centered at the origin. However, because of the equivalency between the Borda count and the Permuta-mean rule, the o-c Borda count also makes for greater combinatorial simplicity when it comes to counting the number of possible m-way ties for a given number of voters, as we will see later in Section 2.8.

1.4 Central voting situations and m-way ties in the Borda count.

Our motivation for counting m-way ties is relevant to the broader interests of studying voting theory. Among the properties a society might consider when selecting a voting rule is that rule's propensity to produce a unique winner [1]. A rule that is likelier to yield an m-way tie, i.e., a less decisive rule, is therefore less likely to produce a unique winner. While this occurrence may seem rare, and not worth considering for a large number of voters, it is certainly relevant when the number of voters is small. For instance, assuming IAC for three alternatives and two voters, there are 3 different voting situations that will produce a 3-way tie in the Borda count, corresponding to an $\frac{1}{7}$ (or 14.3%) probability that the system will produce a 3-way tie.

For m alternatives and n voters, a *central* voting situation is a voting situation $\mathbf{\Pi} = (\mathrm{n}_1, \mathrm{n}_2, .., \mathrm{n}_{\mathrm{m!}})$ that produces an m-way tie in the Borda count.[7]

Proposition 1.4.1. For m alternatives and n voters, a voting situation $\mathbf{\Pi} = (\mathrm{n}_1, \mathrm{n}_2, .., \mathrm{n}_{\mathrm{m!}})$ is central iff the (not necessarily distinct) rank vectors $\boldsymbol{\rho}(1), \boldsymbol{\rho}(2), \ldots, \boldsymbol{\rho}(\mathrm{m!})$, corresponding to cast ballots in a list, satisfy the following criterion:

[7] We note that this definition of central is different than the one used by Cervone et al [2] and Dai [9], although in both papers, the conditions for centrality are proven to be equivalent to our definition for the case of 3 alternatives.

$$\sum_{k=1}^{m!} n_k \boldsymbol{\rho}(k) = \frac{n}{m}\overrightarrow{W}$$

where $\overrightarrow{W}$ is the constant m-tuple of values corresponding to the sum W of the scoring weights w_i.

Thus for the standard definition of the Borda count, $W = \sum_{i=1}^{m} m - i$, while for the o-c Borda count, $W = \sum_{i=1}^{m} 2(j - i) + 1$ for m = 2j, and $W = \sum_{i=1}^{m} (j - i) + 1$ for m = 2j + 1.

Proof of Proposition 1.4.1:

(⇐): Let A_R = <$a_1, a_2, \ldots, a_m$> be a reference enumeration of m alternatives, and for an arbitrary voting situation $\boldsymbol{\Pi} = (n_1, n_2, \ldots, n_{m!})$ with n voters, let $s(a_i)$ denote the Borda score of alternative a_i for $1 \leq i \leq m$. Let $\boldsymbol{\rho}(k)$ denote the rank vector corresponding to the ranking on the k^{th} ballot in a specified list. Now assume

$$\sum_{k=1}^{m!} n_k \boldsymbol{\rho}(k) = \frac{n}{m}\overrightarrow{W}$$

Thus the sum of rank vectors is constant over all entries, since $\frac{n}{m}\overrightarrow{W}$ is constant, meaning that $s(a_1) = s(a_2) = \ldots = s(a_m)$. Thus there is an m-way Borda tie, so $\boldsymbol{\Pi}$ is central, as desired.

(⇒): Assume $\boldsymbol{\Pi}$ is central. Let σ be any ranking with rank vector $\boldsymbol{\rho}(\sigma)$. Then the sum of the ranks $\rho(a_i)$ (for $1 \leq i \leq m$) in the rank vector $\boldsymbol{\rho}(\sigma)$ is

$$\sum_{i=1}^{m} \rho(a_i) = \sum_{i=1}^{m} w_i.$$

Since there are n rank vectors (corresponding to a single vote for each of n voters), then the sum of Borda scores over all candidates is

$$n\sum_{i=1}^{m} w_i,$$

which is just

$$\sum_{i=1}^{m} s(a_i).$$

But $s(a_1) = s(a_2) = . . . = s(a_m)$ by our centrality assumption. So then

$$\sum_{i=1}^{m} s(a_i) = m * s(a_i) = n\sum_{i=1}^{m} w_i$$

Thus we may conclude that

$$s(a_i) = \frac{n}{m}\sum_{i=1}^{m} w_i.$$

Since each alternative receives a total Borda score of $\frac{n}{m}\sum_{i=1}^{m} w_i$, it follows that the sum of all rank vectors is

$$\sum_{k=1}^{m!} n_k \boldsymbol{\rho}(k) = \frac{n}{m}\overrightarrow{W}. \ \blacksquare$$

Corollary 1.4.2. In the o-c Borda count for m alternatives and n voters, a profile $\boldsymbol{\Pi} = (n_1, n_2, .., n_{m!})$ is central iff the (not necessarily distinct) rank vectors $\boldsymbol{\rho}(1)$, $\boldsymbol{\rho}(2)$, . . ., $\boldsymbol{\rho}(m!)$, corresponding to cast ballots in a list, satisfy the following criterion:

$$\sum_{k=1}^{m!} n_k \boldsymbol{\rho}(k) = \vec{0}$$

Proof of Corollary 1.4.2:

For the o-c Borda count, either

$$W = \sum_{i=1}^{m} 2(j - i) + 1\text{, where } m = 2j \text{ for some } j \in \mathbb{Z}_{>0}\text{, or}$$

$$W = \sum_{i=1}^{m} (j - i) + 1\text{, where } m = 2j + 1 \text{ for some } j \in \mathbb{Z}_{\geq 0}.$$

In either case, we can show that W = 0. The corollary follows. ■

Theorem 1.4.3. For m = 2j (j=1, 2, 3, . . .) alternatives and n voters, if a voting situation is central, then n = 2k (k = 1, 2, 3, . . .). ("A voting situation for an even number of alternatives can only produce an m-way Borda tie if there is an even number of voters.")

Proof of Theorem 1.4.3:

Let $A_R = \langle a_1, a_2, \ldots, a_m \rangle$ be a reference enumeration of m alternatives such that m = 2j for some $j \in \mathbb{Z}_{>0}$, and let $\boldsymbol{\prod} = (n_1, n_2, \ldots, n_{m!})$ be a central voting situation with n voters. Take any alternative a_k in A. Let $s(a_k)$ be a_k's o-c Borda score and let r_i denote the number of votes for ballots where a_k is ranked i^{th} for $0 \leq i \leq m$. Then

$$\sum_{i=1}^{m} r_i = n$$

Now, we know $\boldsymbol{\prod}$ is central iff $s(a_k) = 0$, by Corollary 1.4.2. So then

$$\sum_{i=1}^{m} r_i w_i = s(a_k) = 0,$$

where $w_i = 2(j - i) + 1$ and m = 2j for some $j \in \mathbb{Z}_{>0}$. Subtracting, we compute

$$\sum_{i=1}^{m} r_i - \sum_{i=1}^{m} r_i w_i = n$$

$$\sum_{i=1}^{m} r_i \left(1 - (2j - 2i + 1)\right) = n$$

$$2\sum_{i=1}^{m} r_i \,(i - j) = n$$

So then

$$\sum_{i=1}^{m} r_i \,(i - j) = \frac{n}{2}.$$

But we see that n/2 must be an integer, so we conclude that n is in fact even, as desired. ■

A version of this proof by Marchant, as pertains to profiles in IC, may be found in [4].

Section II:

Using Lattices to Count Ties: Random Walks and Ehrhart Theory

2.0 Counting m-way ties in the Borda count in different "cultures"

The problem of counting m-way ties in the Borda count is not new. Over the past fifteen years, research in this field has followed two different paths: one for counting profiles under IC assumptions, and one for counting voting situations under IAC assumptions. In 2001, French economist Thierry Marchant corrected a forty-year-old formula used to explain the magnetic properties of crystals with random lattices walks and applied it to count the number of profiles that produce 3-way Borda ties between 3 candidates as a function of n voters [4]. He then used lattice Green's functions to derive an approximate expression for 3-way ties as a function of n voters and compared values given by this expression to numerical results. More recently, Union College Professor William Zwicker, thesis student Ronghua Dai, and others used the Ehrhart Theory of quasi-polynomials and computer software to find the number of voting situations producing a 3-way tie between three candidates as a function of the number n of voters [2, 9]. They confirmed their result with brute force combinatorial methods. We now review and discuss both research into counting profiles and research into counting voting situations, and present a proof of Ehrhart's famous theorem.

2.1 Counting profiles: random walks and Green's functions

In 2001, Marchant [4 - 5] used existing literature on the magnetic properties of crystals to study the IC-probability that a 3-candidate profile would produce a 3-way tie in an election. Marchant's results corrected and expanded previous work by C. Domb some four decades earlier [7].

In the context of crystallography, a lattice can take on a variety of structures, including triangular structure, as seen below in Figure 4.

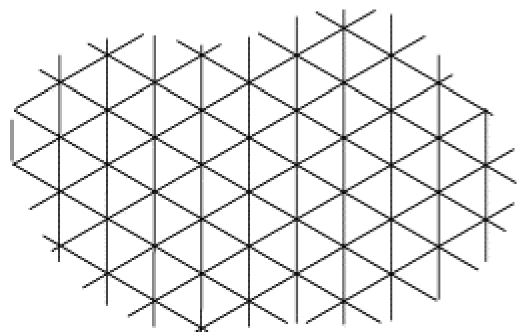

Figure 4. Triangular lattice [4].

A cycle in the above lattice is a path or "walk" that begins at a given start node x, travels along any number *l* of (not necessarily distinct) edges, and returns to the same node. For instance, the shortest cycle of non-zero length has length l = 2, when a cycle travels one edge away from x and then returns; and it easy to see that for a fixed x, the number r_2 of possible cycles for l = 2 is 6. In a cycle where l =3, the path is around a single triangle in the lattice, and r_3= 12 (there are six triangles surrounding each node, and each triangle can be travelled in two directions). In 1960, Domb claimed to find a general expression for r_l, that is, the number of possible cycles of length *l* for a given node on the triangular lattice, and he published his findings along with numerical values for $2 \leq l \leq 9$. While his numerical results were correct, his published formula was flawed due to a multiplicative factor that was lost in the typing process [5]. After communicating with Domb, Marchant published the correct expression and showed how it could be used to count the number of profiles producing m-way ties in the Borda count, where *l* = n, the number of voters in the election.

Although we do not go into detail here, the relationship between counting 3-way Borda ties and cycles on the triangular lattice is related to our notion of rank vectors and the permuta-mean rule. Notice that any collection of 6 triangles centered around a node x on the lattice in Figure 4 forms a regular hexagon (think: 3-permutahedron), and a line between x and an adjacent node corresponds to one of the six rank vectors. Therefore, taking a step between adjacent nodes on the lattice corresponds to a single ballot cast in the election. In the same way that the Permuta-mean rule says that the mean of rank vectors in an election is at the center of the m-permutahedron if and only if there is an m-way tie, a path on the lattice will end back at the start node x in the center of 6 triangles forming a hexagon (think: center of the 3-permutahedron) if and only if there is a 3-way tie. We will now present Marchant's main results and point the reader to [4 - 5] for the derivation.

Theorem 2.1.1 [Domb and Marchant]. For 3 candidates and n voters, the number of profiles that produce a 3-way Borda tie is

$$r_n = \sum_{s,t} \frac{n!}{s!\,t!} \sum_q 2^{s-q} \frac{(t+q)!}{\left(\left(\frac{t+q}{2}\right)!\right)^2} \frac{1}{q!\,(s-q)!},$$

where s, t = 0, 1, . . ., n; 2s + t = n; and $0 \le q \le s$ with (t + q) even. Since there are $(m!)^n$ ways to form a profile under IC assumptions for an election with m candidates and n voters, the above theorem leads to the following corollary.

Corollary 2.1.2 [Marchant]. For 3 candidates and n voters, the probability that a profile produces a 3-way Borda tie is

$$P_{3-\text{way tie},m=3}(n) = \frac{1}{6^n}\sum_{s,t}\frac{n!}{s!\,t!}\sum_{q}2^{s-q}\frac{(t+q)!}{\left(\left(\frac{t+q}{2}\right)!\right)^2}\frac{1}{q!\,(s-q)!},$$

where s, t = 0, 1, . . ., n; 2s + t = n; and 0 ≤ q ≤ s with (t + q) even. As a simple example, let's consider a 3-candidate election with n = 3 voters. Then according to our constraints, we must have, t = 0, 1, 2, 3 with 2s + t = 3 and t + q even. The only combination of s, t, and q where these conditions hold is: s = 1, t = 1, q = 1. So we compute

$$P_{3-\text{way tie},m=3}(3) = \left(\frac{1}{216}\frac{6}{(1)(1)}\right)\left(2^0\,\frac{2}{1^2}\frac{1}{(1)(1)}\right) = \frac{1}{18}, \text{ or } 5.55\%$$

In contrast, there are 2 voting situations with 3 voters and 3 candidates that result in a 3-way Borda tie (see Section 3.0 below), so the probability that a voting situation where m, n =3 results in a 3-way Borda tie is

$$2\,\frac{\big((m!-1)!\big)(n!)}{(n+m!-1)!} = 2\,\frac{(120)(6)}{40320} = \frac{1}{28}, \text{ or } 3.57\%$$

Shown below are some of Marchant's numerical results, which he obtained from Corollary 3.2, where we have multiplied by 6^n to obtain corresponding approximate r_n values (representing the total number of profiles with n number of voters that result in a 3-way Borda tie).

n	1	2	3	4	5	6	7	8	9	∞
r_n	0	6	12	90	360	2040	10080	54800	290640	---
$P_{\text{3-way tie, m = 3}}(n)$	0	0.167	0.056	0.069	0.046	0.044	0.036	0.033	0.029	0

Table 1. Approximate numerical results from Theorem 3.1 and Corollary 3.2 [4].

As seen above in Table 1, the probability that a profile produces a 3-way Borda tie does not decrease steadily as a function of voters. In fact, the probability increases from three to four voters, before decreasing in non-constant intervals for $4 \leq n \leq 9$. However, for $n \geq 10$, the probability of an m-way tie will decrease asymptotically as it approaches zero in the limit as n approaches infinity. This may be seen in Appendix I, where we show results from Marchant's Monte-Carlo simulations for up to 10 candidates and up to 100 voters. A function that approximates this asymptotic behavior well for $n \geq 10$ can also be derived using lattice Green's functions. We will now present the basic concepts behind this derivation. Since our analysis is only an outline, we direct the interested reader to Marchant's primary reference on random walks [17] and two other helpful resources [18 - 19].

Choose any starting node $\boldsymbol{x}$ on a d-dimensional lattice and let $\boldsymbol{l}$ be a d-dimensional vector. Now let $P_n(\boldsymbol{l})$ denote the probability that a random walk of n steps (where a step can be of any length) from $\boldsymbol{x}$ will conclude at the end point of the vector (relative to $\boldsymbol{x}$). It is convention that we may refer to this end node as $\boldsymbol{l}$ although we have already defined $\boldsymbol{l}$ as the vector itself. From any node $\boldsymbol{l'}$ on the lattice, the probability that a single step move will result at $\boldsymbol{l}$ is denoted p($\boldsymbol{l}$ - $\boldsymbol{l'}$). Therefore, the probability that a walk starting at $\boldsymbol{x}$ and ending at $\boldsymbol{l}$ in n + 1 steps is given by the simple recurrence relation:

$$P_{n+1}(\boldsymbol{l}) = \sum_{l'} p(\boldsymbol{l} - \boldsymbol{l'}) P_n(\boldsymbol{l'}), \qquad \text{Equation 2.1.4}$$

where the sum is taken over all nodes $\boldsymbol{l'}$ of the lattice. Since the above relation holds regardless of how we choose our starting point $\boldsymbol{x}$ (a property known as translational invariance), the above form can be reduced to a discrete convolution:

$$P_{n+1}(\boldsymbol{l}) = p(\boldsymbol{l} - \boldsymbol{l'}) * P_n(\boldsymbol{l'}), \qquad \text{Equation 2.1.5}$$

where $*$ denotes the convolution and we note that $P_n(\boldsymbol{l'})$ is itself a convolution. Furthermore, $P_{n+1}(\boldsymbol{l})$ may be written in terms of the probabilities of reaching each node on steps along the way to $\boldsymbol{l}$, i.e.,

$$P_{n+1}(l) = p_1 * p_2 * \;.\;.\;.\; * p_n * p_{n+1} * P_0, \qquad \text{Eq uation 2.1.6}$$

where we set $P_0 = \delta(\boldsymbol{x}) = \begin{cases} 1 \text{ for } \boldsymbol{l'} = \boldsymbol{x} \\ 0 \text{ otherwise} \end{cases}$ to denote that our starting point is at $\boldsymbol{x}$.

In order to simplify Equation 2.1.6, we must introduce the discrete Fourier transform (DFT). For a given sequence of N complex numbers X={x_0, x_1, . . ., x_{N-1}}, the DFT of X is the N-periodic sequence of complex numbers, whose kth entry is given as:

$$\tilde{x}_k = \sum_{n=0}^{N-1} x_n \cdot e^{-i2\pi kn/N}.$$

Now, according to the convolution theorem (the formal statement and proof of which may be found in [18]), the Fourier transform of a convolution of two functions is the product of the Fourier transforms of each function. Thus, Equation 2.1.6 may be written as

$$\widetilde{\mathrm{P}}_{\mathrm{n+1}}(\boldsymbol{k}) = \tilde{\mathrm{p}}_1\tilde{\mathrm{p}}_2 \ldots \tilde{\mathrm{p}}_{\mathrm{n+1}} = \prod_{\mathrm{i=1}}^{\mathrm{n+1}} \tilde{\mathrm{p}}_{\mathrm{i}}, \qquad \text{Equation 2.1.7}$$

where we have changed $\boldsymbol{l}$ to $\boldsymbol{k}$ since the domain of our function has changed after the DFT. At this point, we can introduce what is known as the 'structure function,' which describes the DFT of the probability function p for an individual step on a given lattice. For the triangular lattice shown in Figure 4, the structure function is given as

$$\lambda(\boldsymbol{k}) = \frac{1}{3}(\cos k_1 + cos k_2 + \cos(k_1 + k_2). \qquad \text{Equation 2.1.8}$$

We can now invert the DFTs in Equation 2.1.6 and use $\mathrm{P}_{\mathrm{n+1}}(\boldsymbol{l})$ to obtain the generating function,

$$\mathrm{P}(\boldsymbol{l}, \xi) = \sum_{\mathrm{i=0}}^{\infty} \mathrm{P}_{\mathrm{i}}(\boldsymbol{l})\xi^{\mathrm{i}} = \frac{1}{(2\pi)^{\mathrm{d}}} \int \ldots \int_{B} \frac{\mathrm{e}^{-\mathrm{i}\boldsymbol{l}\cdot\boldsymbol{k}}\mathrm{d}^{\mathrm{d}}\boldsymbol{k}}{1 - \xi\lambda(\boldsymbol{k})}, \qquad \text{Equation 2.1.9}$$

where d is the dimension of the lattice (2 in this case) and $B = [-\pi, \pi]^{\mathrm{d}}$ is called the *first Brillouin zone*. The function on the right in Equation 2.1.9 is called the Green's function of our lattice, or the lattice Green's function (LGF). Setting $\boldsymbol{l}$ equal to the 0 vector to limit the above function to the probability that our path is a cycle, Zumofen and Blumen [20] showed that we can substitute our structure function (Equation 2.1.7) into the LGF to get

$$P(\mathbf{0},\xi) = \frac{6}{\pi\xi\sqrt{a+1}\sqrt{b-1}} K\left(\sqrt{\frac{2(b-a)}{(a+1)(b-1)}}\right), \qquad \text{Equation 2.1.10}$$

where $a = \frac{3}{\xi} + 1 - \sqrt{3 + \frac{6}{\xi}}$, $b = \frac{3}{\xi} + 1 + \sqrt{3 + \frac{6}{\xi}}$, and K is itself a function known as the complete elliptic integral of the first kind. For the right conditions of ξ (specifically, $\xi \approx 1$), we can expand Equation 2.1.10 and substitute back into the left side of the equality in Equation 2.1.9 to see that, for n sufficiently close to infinity,

$$P_{n+1}(\mathbf{0}) \approx \frac{\sqrt{3}}{2\pi n}.$$

Thus, this expression explains the asymptotic behavior of P as a function of n, and as demonstrated in Appendix I, actually fits the Monte Carlo results for $n \geq 10$ fairly well. Marchant attempted to use the above method to derive LGF's and approximate probability functions for elections with 4 or more candidates, where one can imagine that the 2-dimensional triangular lattice would become a 3-dimensional lattice with truncated octahedrons divided into 12 subsections with a node at the center. In the end, Marchant was unable to integrate the LGF for more than three alternatives [4].

2.2 Counting voting situations: an introduction to Ehrhart theory

Whereas random walks and lattice Green's functions have proved most useful for finding the probability of a 3-way Borda tie assuming IC, French mathematician Eugene Ehrhart's theory of quasi-polynomials (polynomials with periodic coefficients) has done equally well assuming IAC. Developed in the 1960s, Ehrhart theory is used to count the number of integer lattice points contained within a region in Euclidean space called a polytope, when the polytope is dilated by an integer factor, i.e., when every element of the polytope is multiplied by an integer [8]. However, the utility of Ehrhart polynomials for voting theory was only realized in 2008, when Lepelley, Louichi, and Smaoui published some initial results on the Borda count and Plurality voting [10]. In essence, a region in space can be parameterized with linear constraints such that the integer points within the region represent specific voting situations. Hence the applicability of Ehrhart theory, with the dilation factor becoming the number of voters in the electorate. This connection paved the way to further research. Later in 2008, Union College thesis student Ronghua Dai used Ehrhart theory to find an expression for the number of 2 and 3-way Borda ties as a function of n number of voters in the case of 3 alternatives [9]. We now discuss the basics of Ehrhart theory and present Dai's results.

2.3 Introduction to affine geometry I

In order to properly understand Ehrhart Theory, we must first introduce some basic definitions from affine geometry. Let S be a subset of Euclidean space and take any $s_1, s_2, \ldots, s_k$ in S. Let $\lambda_1, \lambda_2, \ldots, \lambda_k$ be any nonnegative real numbers such that $\sum_{i=1}^{k} \lambda_i = 1$. Then S is *convex* if $\sum_{i=1}^{k} \lambda_i s_i \in$ S always holds.[8] Geometrically, S is convex if for any two points s_i and s_j in S, all of the points on the straight line between s_i and s_j are also in S. For example, in Figure 5, the set A is convex while the set B is not.

[8] It is sufficient to define convexity for k = 2, but this more general statement follows as a consequence.

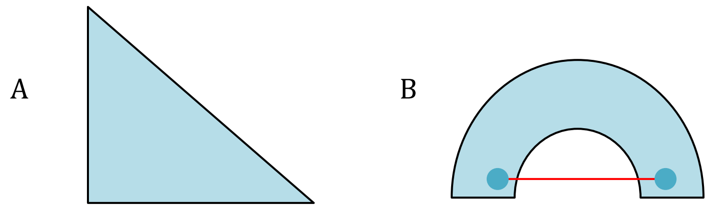

Figure 5. For the above sets A and B living in an arbitrary plane, A is convex while B is not, as evidenced by the red line connecting two points in B, where part of the line is outside B.

Next, let $X = \{x_1, x_2, x_3, \ldots\}$ be a set of points in Euclidean space. Then the *convex hull* of X, denoted CH(X), is the (unique) minimal convex set containing X. Equivalently,

$$\mathrm{CH}(X) = \left\{ \sum_i \alpha_i x_i \,\middle|\, \forall i\colon \alpha_i \in \mathbb{R}, \alpha_i \geq 0 \text{ and } \sum_i \alpha_i = 1 \right\}.$$

By way of example, let X be the set of three points in an arbitrary plane shown in Figure 6. Then the set C = CH(X), while D ≠ CH(X), although it is still convex.

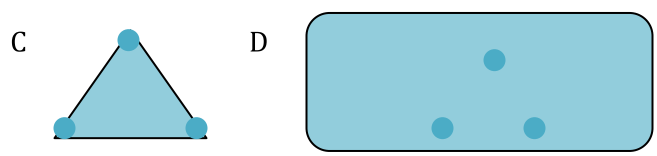

Figure 6. Left: C is the convex hull of the 3 points in X. Right: D is convex, but D ≠ CH(X).

The definition for the *affine hull of* X, denoted Aff(X), is the same as the definition for CH(X), except that the α_i scalar coefficients may be negative.

$$\mathrm{Aff}(X) = \left\{ \sum_i \alpha_i x_i \,\middle|\, \forall i\colon \alpha_i \in \mathbb{R} \text{ and } \sum_i \alpha_i = 1 \right\}.$$

Geometrically, this means that the affine hull of two points is the line going through the points; the affine hull of three non-collinear points is the plane containing the points; and the affine hull of four points in $\mathbb{R}^3$, not lying in the same plane, is all of $\mathbb{R}^3$.

An *affine transform* is a function f: $\mathbb{R} \to \mathbb{R}$ of the form $x \mapsto ax + b$ where a and b are real constants, and a *positive affine transform* adds the restriction $a > 0$.

2.4 Pick's Theorem

A *convex lattice polygon* P is a convex polygon in $\mathbb{R}^2$ whose corners are integer lattice points. We define $nP = \{na \mid a \in P\}$, where n is a positive integer called the *dilation* factor, and we say nP is the polygon P dilated by a factor of n. Let A(P) denote the area of P with dilation factor of 1 (i.e., no dilation), let B(P) denote the number of lattice points on the boundary of P (also not dilated), and let $L_p(n)$ denote the total number of lattice points in nP (including boundary points). By way of example, consider the following convex polygon "P_5" in $\mathbb{R}^2$ with different dilation factors.

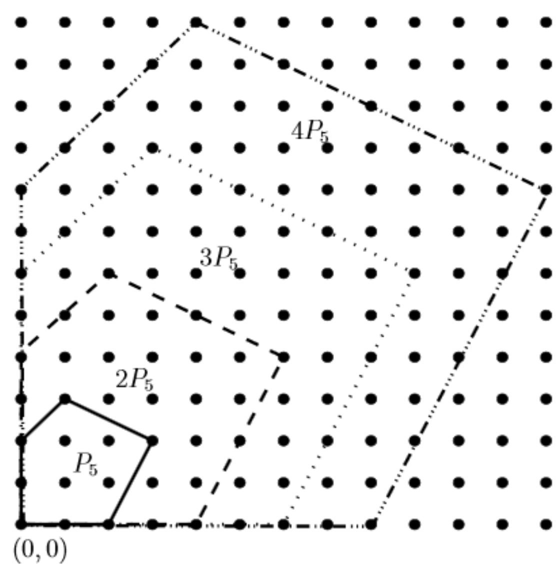

Figure 7. Convex lattice polygon in $\mathbb{R}^2$ dilated by factors of $1 \le n \le 4$ [21].

Taking the above lattice to be the first quadrant of $\mathbb{R}^2$ with x and y as the horizontal and vertical axes, respectively, we may define P_5 with the following system of linear equations:

$$\begin{aligned} x &\geq 0 \\ y &\geq 0 \\ y &\leq x + 2 \\ y &\leq -\frac{1}{2}x + 3.5 \\ y &\geq 2x - 3 \end{aligned}$$

Note that this may be written more simply in matrix form as:

$$\begin{bmatrix} -1 & 0 \\ 0 & -1 \\ -1 & 1 \\ \frac{1}{2} & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \leq \begin{bmatrix} 0 \\ 0 \\ 2 \\ 3.5 \\ 3 \end{bmatrix}$$

Now, as seen in Figure 7, P_5 is a pentagonal convex lattice polygon with area $A(P_5)$ that we may find by separately considering the two lattice triangles (one on top and one on the right side) and the lattice square. Summing these three parts, we get $A(P_5) = A_{\square} + A_{\triangle upper} + A_{\triangle side} = 4 + 1.5 + 1 = 6.5$. By inspection, we also see that the number of lattice points on the boundary of P_5 is $B(P_5) = 7$ and the number of total lattice points including the boundary is $L_{P5}(1) = 11$.

In 1889, Austrian mathematician George Alexander Pick formulated the following result:

Pick's Theorem[9] (or Ehrhart's Theorem for $\mathbb{R}^2$, integer vertices). Let P be a convex lattice polygon, $n \in \mathbb{N}$. Then

$$L_p(n) = A(P)\, n^2 + \frac{B(P)}{2} n + 1.$$

We omit a proof of Pick's Theorem here, but direct the reader to [23]. Returning to the example of the pentagonal convex lattice polygon from Figure 5, we can now solve for the number of lattice points for the largest shown dilation of $4P_5$ as follows:

$$L_{p_5}(4) = (6.5)\, 4^2 + \frac{7}{2}(4) + 1 = 119.$$

More generally, we get the following quadratic polynomial for the number of lattice points as a function of n dilations:

$$L_{p_5}(n) = 6.5n^2 + 3.5n + 1$$

2.5 Introduction to affine geometry II

Over a half-century ago, Ehrhart generalized Pick's theorem to the case of rational vertices and higher dimensions. However, for us to expand some of the concepts from the lattice polygon case, we must first introduce some more definitions from affine geometry. For the rest of the paper, when we refer to *dimension*, we mean *affine dimension*, defined as follows:

A subset X of Euclidean space has *affine dimension* d where d + 1 is the minimal size of a subset X' of X for which Aff(X') = Aff(X).

[9] Pick's original theorem did not actually concern the dilated polygon, but the step from Pick's original polygon to the dilated polygon can be easily made, e.g., [22].

For instance, a plane is 2-dimensional since it can only be formed by 3 or more points; a line is 1-dimensional since it can only be formed by 2 or more points; and a single point is 0-dimensional. A *hyperplane* is an affine subspace (closed under affine transforms) of $\mathbb{R}^d$ with dimension d – 1, or *co-dimension*[10] 1. Thus, in $\mathbb{R}^2$, a hyperplane is a line (e.g., one of the lines from the system of linear equations for the P_5 example); in $\mathbb{R}^3$, a hyperplane is a plane; and in $\mathbb{R}^d$, a hyperplane is the set of solutions to a single linear equation:

$$\vec{a} \cdot \vec{x} = a_1x_1 + a_2x_2 + \ldots + a_dx_d \geq b \qquad \left(b \in \mathbb{R},\ \ a \neq \vec{0}\right).$$

A *Closed half-space* is the set of all points on or to one side of a hyperplane. Equivalently, it is the solution set to a single linear inequality. A *polytope* is a bounded intersection of finitely many closed half spaces. Equivalently, a polytope is the convex hull of finitely many points of $\mathbb{R}^d$. For instance, P_5 (from Section 2.4) is a polytope because it is the bounded intersection of five closed half spaces (think 5-sided polygon), where each closed half space is the set of all points on or to one side of 1-dimensional hyperplanes, or lines. In terms of convex hulls, P_5 is a polytope because it the convex hull of the set of points {(0, 0), (2, 0), (3, 2), (1, 3), (0, 2)}.

The following is a famous example of a 2-dimensional polytope living in $\mathbb{R}^3$. Consider the intersection of the set of closed half spaces: $x, y, z \geq 0$, with $x + y + z \leq 1$ and $x + y + z \geq 1$. This may be represented as CH{(1, 0, 0), (0, 1, 0), (0, 0, 1)}, or with matrix notation as

[10] In general, the co-dimension of an affine subspace V living in an affine space W is Dim(W) - Dim(V).

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 1 & 1 \\ -1 & -1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \leq \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}$$

The intersection of the five closed half spaces forms a 2-dimensional polytope known as the *2-simplex,* shown below in Figure 8.

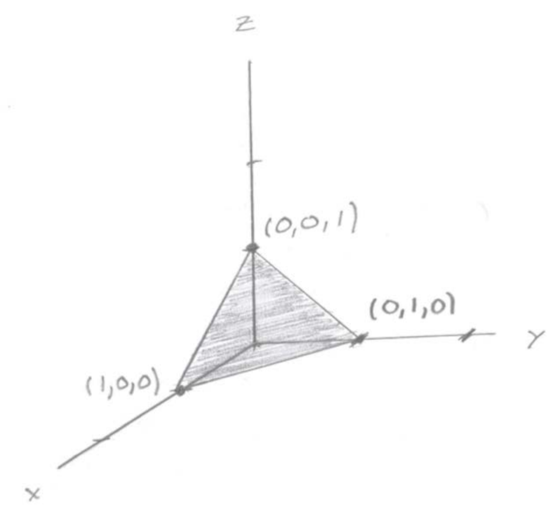

Figure 8. The standard 2-Simplex in $\mathbb{R}^3$.

More generally, we form the n-simplex as follows: Let $x_i \geq 0$ (i = 1, …, n + 1)

$$\sum_{i=1}^{n+1} x_i \geq 1 \qquad\qquad \sum_{i=1}^{n+1} x_i \leq 1.$$

If for some $V \subset \mathbb{Z}^{\mathbf{d}}$, a polytope P = CH(V), then P is called an *integral polytope*. If $V \subset \mathbb{Q}^d$, then P is called a *rational* polytope. Note that the standard 2-Simplex shown above is an integral polytope, as it is the convex hull of the integer points (1, 0, 0), (0, 1, 0), and (0, 0, 1). Also, in our earlier example, P_5 is an integral polytope, since all vertices are on integer lattice points.

2.6 Quasi-Polynomials, Ehrhart's Theorem, and McMullen's Theorem

Ehrhart theory allows us to count integer points in rational polytopes, not just integral polytopes. Take for example the following rational polytope, which we'll call P_Δ, on an integer lattice.

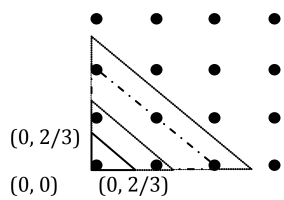

Figure 9. A rational polytope in $\mathbb{R}^2$, the triangle with vertices (0,0), (2/3, 0), and (0, 2/3), is seen dilated by factors of 2, 3, and 4.

In Figure 9, $L_{P_\Delta}(1) = 1$, or in other words, the original triangle only contains one lattice point (at the origin); $L_{P_\Delta}(2) = 3$; $L_{P_\Delta}(3) = 6$; and $L_{P_\Delta}(4) = 6$. Notice how $L_{P_\Delta}(n)$ increases, but then stalls when $L_{P_\Delta}(3) = L_{P_\Delta}(4) = 6$. Rational polytopes can pick up integer lattice points in fits and starts as they dilate, but although this behavior may appear random, we will soon see that it actually has periodic features. Accordingly, we use functions known as quasi-polynomials, a generalized form of polynomials, to calculate $L_p(n)$.

A function $f\colon \mathbb{N} \to \mathbb{N}$ is a *quasi-polynomial* if there exists an integer N > 0 and polynomials $f_0, f_1, \ldots, f_{N-1}$ such that

$$f(n) = f_i(n) \quad \text{if } n \equiv i \bmod N,$$

where the (non-unique) integer *N* is called the *quasi-period* of f. Equivalently, we can write

$$f(n) = c_d(n)n^d + c_{d-1}(n)n^{d-1} + \ldots + c_0(n),$$

where $c_i(n)$ is a periodic function with integral period. For all $0 \le i \le d$, the quasi-period N must divisible by the minimal period of each function $c_i(n)$, although its non-uniqueness is due to the fact that it need not be least common multiple (lcm) [24].

To illustrate how we interpret the above definitions, we will consider the following quasi-polynomial of degree 2, which we note does not correspond to any figure in this paper.

$$f(n) = \ll 1, 2 \gg n^2 + \ll 3, 4, 5 \gg n + \ll 6 \gg$$

Here, the fact that the quadratic coefficient <<1, 2>> has two entries represents that $c_2(n)$ has period 2. Thus $c_2(n) = 1$ if $n \equiv 0 \bmod 2$, and $c_2(n) = 2$ if $n \equiv 1 \bmod 2$. Similarly, $c_1(n)$ has period 3 and $c_0(n)$ has period 1, i.e., $c_0(n) = 6$ all of the time. By way of example, for n = 5, we would get $c_2(5) = 2$, $c_1(5) = 5$, and $c_0(5) = 6$. In the context of the first definition, we see that N = 6 is a quasi-period for f(n), as it is divisible by each of the coefficient functions' periods of 2, 3, and 1. And we can see that there exists polynomials:

$$f_0(n) = n^2 + 3n + 6$$
$$f_1(n) = 2n^2 + 4n + 6$$
$$f_2(n) = n^2 + 5n + 6$$
$$f_3(n) = 2n^2 + 3n + 6$$
$$f_4(n) = n^2 + 4n + 6$$
$$f_5(n) = 2n^2 + 5n + 6$$

where $f(n) = f_i(n)$ if $n \equiv i \bmod 6$. For the remainder of this discussion, we will use the second definition of quasi-polynomials, rather than the definition that makes use of quasi-period. We are now ready to state the generalized version of Ehrhart's Theorem.

Ehrhart's Theorem: Let nP be a d-dimensional (affine dimension d) rational polytope P in $\mathbb{R}^k$ dilated by a factor of some $n \in \mathbb{N}$. Then the number of lattice points $L_p(n)$ is always a degree d quasi-polynomial, called the Ehrhart quasi-polynomial. In particular, if P is an integral polytope, $L_p(n)$ is a polynomial.[11]

In the late 1970s, British mathematician Peter McMullen expanded on Ehrhart's work by further interpreting the coefficients of the Ehrhart quasi-polynomial. First, let the *i-index* of a rational d-dimensional polytope P be the smallest number $s_i \in \mathbb{Z}_{>0}$, $i \in \mathbb{Z}_{\geq 0}$, such that for each i-dimensional face F_i of P, the affine hull of s_iF_i contains at least one integer point. For this definition, we say the entire polytope P is a d-dimensional face of itself. Note that if $i \geq j$, then $s_i | s_j$ [6, 25]. To better understand these definitions, consider the 2-dimensional rational polytope, $P = CH\{(\frac{1}{2}, \frac{3}{4}, \frac{-1}{12}), (\frac{1}{6}, \frac{-1}{12}, \frac{5}{12}), (\frac{-2}{3}, \frac{5}{6}, \frac{1}{6})\}$ and its corresponding affine hull.

[11] Ehrhart also showed that the minimum quasi-period of $L_p(n)$ divides the lcm of the denominators of the coordinates of P's vertices [8, 24].

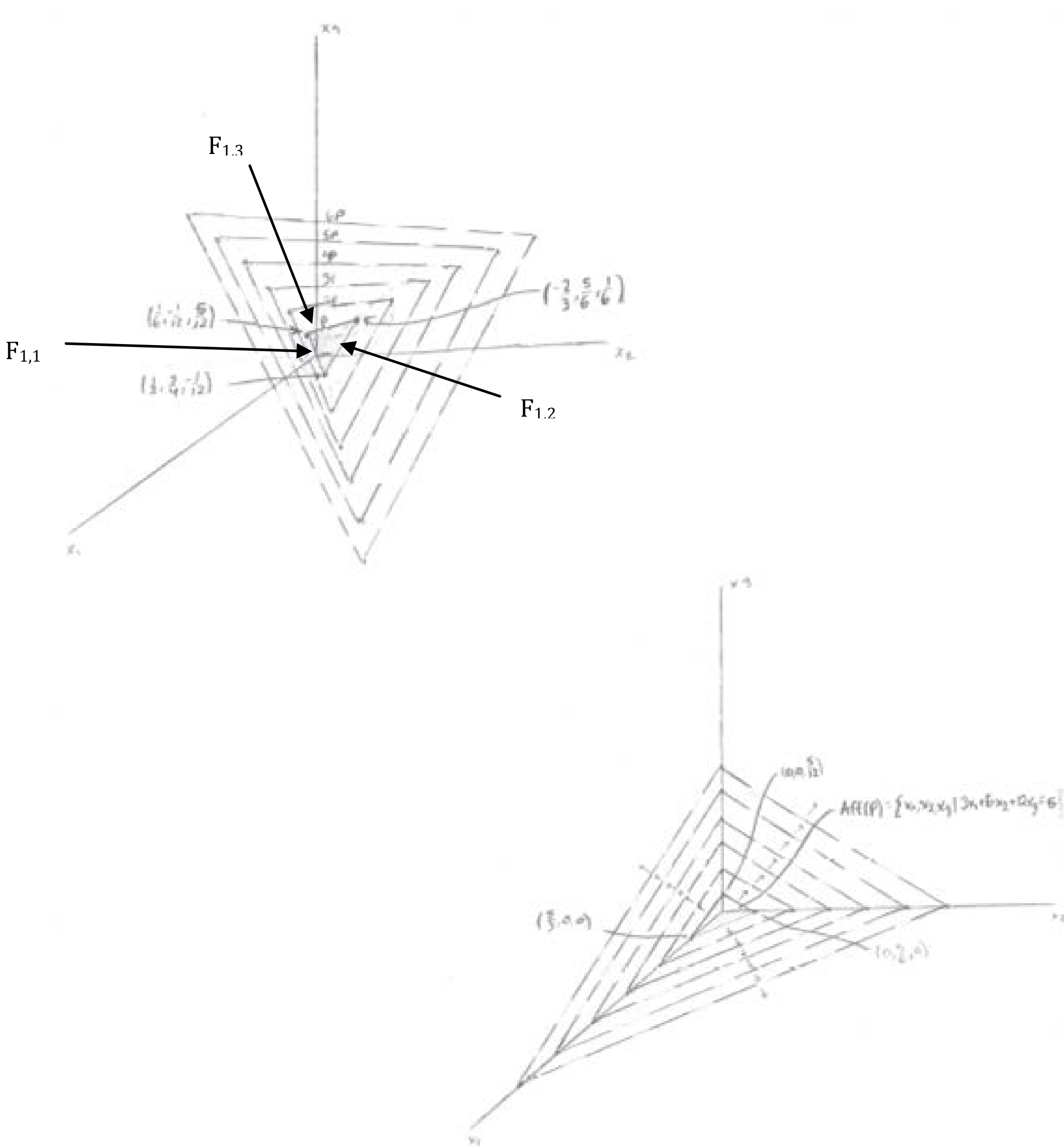

Figure 10. Upper left: the 2-dimensional rational polytope P= CH$\{(\frac{1}{2}, \frac{3}{4}, \frac{-1}{12}), (\frac{1}{6}, \frac{-1}{12}, \frac{5}{12}), (\frac{-2}{3}, \frac{5}{6}, \frac{1}{6})\}$, dilated by factors of $1 \le n \le 6$, with three 1-dimensional faces, $F_{1,1}$, $F_{1,2}$, $F_{1,3}$. Lower right: the intersection of corresponding affine hulls and the first octant for each nP.

Here, P = F_2 is necessarily the only 2-dimensional face of the polytope. From the three points that build P, we find that the affine hull of F_2 is the set of points $\{x_1, x_2, x_3 \mid 3x_1 + 6x_2 + 12x_3 = 5\}$. An easy check to see if the affine hull contains any integer lattice points may be done by calculating the greatest common denominator (gcd) of the coefficients of the x_i; there exists such an integer solution if and only if the gcd of the coefficients divides the constant term (5 in this case). Observe that by manipulating denominators, we can assume that the coefficients and the constant term can always be chosen to be integers if the initial polytope is rational. We leave the proof that the hyperplane $\{x_1, x_2, \ldots, x_d \mid \forall i: \alpha_i, b \in \mathbb{Z} \text{ and } \alpha_1 x_1 + \alpha_1 x_1 + \ldots + \alpha_d x_d = b\}$ contains integer points iff $\gcd(\alpha_1, \alpha_2, \ldots, \alpha_d) \mid b$ to the reader. Note that this method can only be used to confirm the existence of integer points in affine hulls of co-dimension 1.

Since $\gcd(3, 6, 12) = 3$, and $3 \nmid 5$, we know that $\mathrm{Aff}(F_2)$ does not contain any integer lattice points. Furthermore, the affine hull of $2F_2 = \{x_1, x_2, x_3 \mid 3x_1 + 6x_2 + 12x_3 = 10\}$ and $3 \nmid 10$. However, continuing in this way, we see that the affine hull of $3F_3$ does contain integer points (e.g., (1, 0, 1)) as $3 \mid 15$. Thus, since there is only one 2-dimensional face and 3 is the smallest integer s_2 such that $\mathrm{Aff}(s_2F_2)$ contains integer lattice points, we conclude that the 2-index for P is $s_2 = 3$. We also conjecture that 6 is the smallest integer s_1 such that $\mathrm{Aff}(s_1F_{1,1})$, $\mathrm{Aff}(s_16F_{1,2})$, and $\mathrm{Aff}(s_16F_{1,3})$ all contain integer points, so we conclude that the 1-index for P is $s_1 = 6$.[12] Finally, we observe that the 0-index of P is simply the least common denominator (lcm) of the denominators of the coordinates of the three points (as a 0-dimensional face and the corresponding affine hull are the same point in P). By inspection, we find that the 0-index is $s_0 = 12$. We see that $s_2 \mid s_1 \mid s_0$, as noted above. We are now ready to state McMullen's theorem.

[12] This conjecture is based on our only finding integer points contained within each of the three 1-dimensional face's affine hulls dilated by a factor of 6 (or a multiple thereof). Recall that we cannot apply the same method used for affine hull of the 2-dimensional face, since the affine hulls of the 1-dimensional faces of P live in $\mathbb{R}^3$ and therefore do not have co-dimension 1.

McMullen's Theorem: Let P be a rational d-dimensional polytope in $\mathbb{R}^k$ with

$$L_P(n) = c_d(n)n^d + c_{d-1}(n)n^{d-1} + \dots + c_0(n),$$

as the Ehrhart quasi-polynomial. Given a dimension i, with $0 \le i \le d$, let s_i be the i-index of P. Then s_i is a period of $c_i(t)$ [6, 25].

One consequence of McMullen's theorem is that if P is full-dimensional (d = k), then it follows that $c_d(n)$ has period 1, since the affine hull of P is all of R^k, which necessarily contains integer points. In fact, Ehrhart showed that for full-dimensional polytopes, $c_d(n)$ = vol(P) [8, 26]. We will now prove Ehrhart's Theorem and McMullen's Theorem in parallel.

2.7 Proof of Ehrhart's Theorem, McMullen's Theorem

We will follow the form and substance of Sam and Woods [6], who employed simpler but less powerful tools than Ehrhart used in his original proof, which involved generating functions [8, 26]. When possible, we will fill in details omitted by the original paper. First, we must prove a series of lemmas.

Lemma 4.5: Define the *falling factorial* $i^{\underline{d}} := i(i-1)(i-2)\dots(i-(d-1))$. Then

$$P(n) \text{ asserts that } \sum_{i=0}^{n} i^{\underline{d}} = \frac{1}{d+1}(n+1)^{\underline{d+1}}.$$

We'll prove the lemma by induction on n. For the base case, let n = 0. Then on the left hand side (LHS), we compute:

$$\sum_{i=0}^{n} i^{\underline{d}} = 0^{\underline{d}} = 0(0-1)\dots(0-d+1) = 0.$$

And on the right hand side (RHS), we compute:

$$\frac{1}{d+1}(0+1)^{\underline{d+1}} = \left(\frac{1}{d+1}\right)1^{\underline{d+1}} = \left(\frac{1}{d+1}\right)(1)(0)\,.\,.\,.(1-\ d+1) = 0\text{, as desired.}$$

Thus P(n) holds for n = 0. Next we'll assume that that P(n) holds for n and we'll show that P(n + 1) holds. We compute:

$$\sum_{i=0}^{n+1} i^{\underline{d}} = \sum_{i=0}^{n} i^{\underline{d}} + (n+1)^{\underline{d}} = \frac{1}{d+1}(n+1)^{\underline{d+1}} + (n+1)^{\underline{d}}$$

And we want to show: $\frac{1}{d+1}(n+1)^{\underline{d+1}} + (n+1)^{\underline{d}} = \frac{1}{d+1}(n+2)^{\underline{d+1}}$. We expand the LHS:

$$\frac{1}{d+1}(n+1)^{\underline{d+1}} + (n+1)^{\underline{d}} = \left[\frac{1}{d+1}(n+1)(n)\,.\,.\,.\,\big(n-(d-1)\big)\right] + \big[(n+1)(n)\,.\,.\,.\,\big(n-(d-2)\big)\big].$$

And similarly, for the RHS:

$$\frac{1}{d+1}(n+2)^{\underline{d+1}} = \frac{1}{d+1}(n+2)(n+1)\,.\,.\,.\big(n-(d-2)\big).$$

Now let $a = \frac{1}{d+1}(n+2)(n+1)\,.\,.\,.\big(n-(d-2)\big)$.

Then substituting,, we see that the LHS becomes:

$$\frac{(n-(d-1))}{(n+2)}a + \frac{(d+1)}{(n+2)}a,$$

$$= \frac{(n-d+1+d+1)}{(n+2)}a = \frac{(n+2)}{(n+2)}a = a\text{, as desired.}$$

Thus we have proved that P(n+1) holds, and the lemma follows. ■

Lemma 4.6: Let $f(n) = c_d(n)n^d + c_{d-1}(n)n^{d-1} + \ldots + c_0(n)$ be a quasi-polynomial of degree d, where $c_i(t)$ is a periodic function of period s_i, for each i. Define $F: \mathbb{Z}_{\geq 0} \to \mathbb{Q}$ by

$$F(n) = \sum_{i=0}^{\left\lfloor \frac{an}{b} \right\rfloor} f(i),$$

where a, b ∈ $\mathbb{Z}$ and $\lfloor * \rfloor$ is the greatest integer ("floor") function. Let $S_i = \frac{s_i b}{\gcd(s_i, a)}$. Then:

(1) $F(n) = C_{d+1}(n)n^{d+1} + C_d(n)n^d + \ldots + C_0(n)$ is a quasi-polynomial of degree d + 1;

(2) $\operatorname{lcm}\{S_d, S_{d-1}, \ldots, S_i\}$ is a period of $C_i(t)$, for $0 \leq i \leq d$; and

(3) C_{d+1} has period 1.

Before we prove the lemma, we'll consider an example given by Sam and Woods [6]. Suppose we have the very simple quasi-polynomial,

$$f(n) = \ll \frac{1}{2}, 0 \gg n,$$

and we would like to evaluate the sum,

$$F(n) = \sum_{i=0}^{\left\lfloor \frac{3n}{2} \right\rfloor} f(i).$$

Then the period of $c_1(n)$ is 2 (as there are two spaces in the brackets before n) and the period of $c_0(n)$ is 1 (as $c_0(n) = 0$ all of the time). Thus $s_1 = 2$ and $s_0 = 1$, and since a = 3, b = 2, we get:

$$\mathrm{S}_1 = \frac{(2)(2)}{1} = 4 \text{ and } \mathrm{S}_0 = \frac{1(2)}{1} = 2.$$

From part (3) of Lemma 4.6, we know that the period of $C_{d+1}(n)$, or the period of the coefficient of the quadratic term in this case, is 1; the period of the $C_1(n)$ term is $\mathrm{lcm}(S_1) = \mathrm{lcm}\{4\} = 4$; and the period of the $C_0(n)$ term is $\mathrm{lcm}\{S_1, S_0\} = \mathrm{lcm}\{4, 2\} = 4$. In fact,

$$F(n) = \sum_{i=0}^{\left\lfloor \frac{3n}{2} \right\rfloor} f(i) = \sum_{j=0}^{\left\lfloor \frac{3n}{4} \right\rfloor} j,$$

where we were allowed to cut the number of terms in half and substitute in j because f(i) alternates between 0 for i odd, and i/2 for i even. This gives

$$\sum_{j=0}^{\left\lfloor \frac{3n}{4} \right\rfloor} j = \frac{1}{2}\left(\left\lfloor \frac{3n}{4} \right\rfloor\right)\left(\left\lfloor \frac{3n}{4} \right\rfloor + 1\right) = \frac{1}{2}\left(\left\lfloor \frac{3n}{4} \right\rfloor + 1\right)^{2}$$

$$= \frac{9}{32}n^2 + \ll \frac{3}{8}, \frac{-3}{16}, 0, \frac{3}{16} \gg n + \ll 0, \frac{-3}{32}, \frac{-1}{8}, \frac{-3}{32} \gg.$$

Observe that the periodicities of the coefficients of this quasi-polynomial agree with the above calculations we made according to the lemma. In particular, we see the importance of using $\mathrm{lcm}\{S_d, S_{d-1}, \ldots, S_i\}$ to find the period of $C_i(n)$; while s_1 was the only periodic term for f(n), its periodicity also affected the C_0 term of F(n). We will now prove the lemma.

For $d, s, j \in \mathbb{Z}_{\geq 0}$, let φ be the periodic function

$$\varphi_{s,j}(n) = \begin{cases} 1 \text{ if } n \equiv j \pmod{s} \\ 0 \text{ otherwise} \end{cases}$$

and g be the quasi-polynomial

$$g_{d,s,j}(n) = \varphi_{s,j}(n)\prod_{k=0}^{d-1}\left(\frac{n-j}{s}-k\right).$$

Thus, in our last example, we had $\ll \frac{1}{2}, 0 \gg n = \ll 1, 0 \gg \left(\frac{n-0}{2} - 0\right) = g_{1,2,0}(n)$. Since we may always write the function f(n) as a linear combination of these quasi-polynomials (for various degree d, period s, and j) , we need only prove that

$$G_{d,s,j}(n) = \sum_{i=0}^{\left\lfloor\frac{an}{b}\right\rfloor} g_{s,j}(i)$$

is a quasi-polynomial of degree d+1 with period $S = \frac{sb}{\gcd(s,a)}$ and leading term periodicity 1. Now, for some $m \in \mathbb{Z}_{\geq 0}$ with $n \equiv j \pmod{s}$, we may reduce g to a regular polynomial in m by setting n = ms + j so that

$$g_{d,s,j}(ms+j) = \varphi_{s,j}(ms+j)\prod_{k=0}^{d-1}\left(\frac{(ms+j)-j}{s}-k\right)$$

$$= 1\prod_{k=0}^{d-1}(m-k) = m(m-1)\ldots\big(m-(d-1)\big) = m^{\underline{d}}.$$

Therefore, for any $k \in \mathbb{Z}_{+}$, we have

$$\sum_{i=0}^{k} g_{s,j}(i) = \sum_{m=0}^{\left\lfloor\frac{k-j}{s}\right\rfloor} g_{s,j}(ms+j)$$

$$= \sum_{m=0}^{\left\lfloor\frac{k-j}{s}\right\rfloor} m^{\underline{d}}$$

$$= \frac{1}{d+1}\left(\left\lfloor \frac{k-j}{s} \right\rfloor + 1\right)^{\underline{d+1}} \qquad \text{by Lemma 4.5.}$$

For a, b ∈ ℤ, we may substitute so that k = $\left\lfloor \frac{an}{b} \right\rfloor$. Thus,

$$G_{d,s,j}(n) = \frac{1}{d+1}\left(\left\lfloor \frac{\left\lfloor \frac{an}{b} \right\rfloor - j}{s} \right\rfloor + 1\right)^{\underline{d+1}}$$

is a quasi-polynomial with degree d + 1, as desired. To check that $G_{d,s,j}(n)$ has period S = $\frac{sb}{\gcd(s,a)}$ and a leading coefficient with periodicity 1, we will substitute n = mS + k and show that the resultant expression reduces to a polynomial in m:

$$G_{d,s,j}(mS+k) = G_{d,s,j}\left(m\frac{sb}{\gcd(s,a)} + k\right)$$

$$\frac{1}{d+1}\left(\left\lfloor \frac{\left\lfloor \frac{a\left(m\frac{sb}{\gcd(s,a)}+k\right)}{b} \right\rfloor - j}{s} \right\rfloor + 1\right)^{\underline{d+1}} = \frac{1}{d+1}\left(\left\lfloor \frac{\frac{ams}{\gcd(s,a)} + \left\lfloor \frac{ak}{b} \right\rfloor - j}{s} \right\rfloor + 1\right)^{\underline{d+1}},$$

where we have removed the floor function for $\frac{ams}{\gcd(s,a)}$, since this is necessarily an integer.

Simplifying further, we get

$$G_{d,s,j}(mS+k) = \frac{1}{d+1}\left(\frac{am}{\gcd(s,a)} + \left\lfloor \frac{\left\lfloor \frac{ak}{b} \right\rfloor - j}{s} \right\rfloor + 1\right)^{\underline{d+1}},$$

a polynomial in m with leading coefficient that does not depend on k and thus has period 1. The lemma follows. ■

We will now proceed to prove Ehrhart's Theorem and McMullen's Theorem in parallel, by induction on polytope dimension d. For the base case, let d = 0. Then a rational 0-dimensional polytope is just a point in $\mathbb{Q}^k$. Let D be the smallest positive integer such that DP is an integer point. Then it follows that

$$L_P(n) = c_0(n), \text{where } c_0(n) = \begin{cases} 1 \text{ if } D \mid n \\ 0 \text{ otherwise} \end{cases}.$$

In other words, the number of integer points in the polytope will always be either 1 or 0, as the polytope itself is just a single point; it is only an integer point if the dilation factor n is some multiple of D. Therefore, $L_p(n)$ is a degree 0 quasi-polynomial that reduces to a polynomial (periodicity 1) when D = 1, so Ehrhart's Theorem for the base case follows. Furthermore, the 0-index of P is $s_0 = D$, since the affine hull of a point is just that point, and D is the smallest positive integer to produce an integer point. And since the period of $c_0(n)$ is D, then McMullen's Theorem for the base case follows as well.

We will now assume that these two theorems hold for all d' < d. For the inductive step, we will first prove three claims.

Claim 1: Without loss of generality (WLOG), P is full-dimensional (dim(P) = k).

Proof of Claim 1: Let s' be the smallest positive integer such that Aff(s'P) contains integer points. Then $s' \mid s_i$ for each i, as previously noted. Now let V = Aff(s'P). While we do not provide a proof, there exists an affine transformation T: V → $\mathbb{R}^{\dim(P)}$ that maps V ∩ $\mathbb{Z}^k$ (the integer lattice points in V) bijectively onto $\mathbb{Z}^{\dim(P)}$, and we say P' = T(s'P). Since V contains s'P and $\mathbb{R}^{\dim(P)}$ contains T(s'P), then in particular, the bijective mapping T preserves integer points in the dilated polytopes. For example, consider the polytope $P = CH\{(3,0,0),(0,3,0),(0,0,3)\}$. P is not full dimensional, as it is a 2-dimensional polytope in $\mathbb{R}^3$. In addition, we see that s' = 1, since Aff(P) (and in fact P itself) contains integer points. In this case, we can produce a bijective affine transformation T(1P) by simply projecting P onto the x_1-x_2 plane as follows:

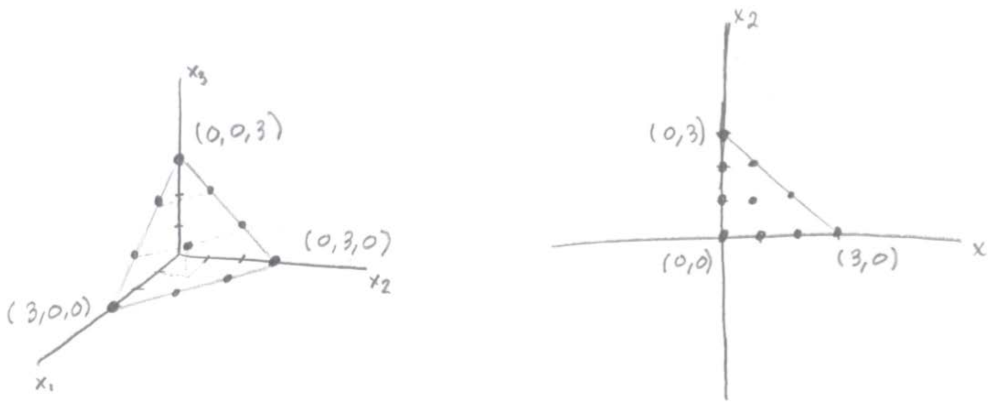

Figure 11. 1P (left) and T(1P) = P' (right).

Although the above example is quite simple, it demonstrates a one-to-one mapping from $V \cap \mathbb{Z}^3$ onto $\mathbb{Z}^2$, and in particular, as seen in Figure 11, $L_P(1) = L_{P'}(1) = 10$.

If we can prove the theorem for our full-dimensional polytope P', we can prove it for P, since

$$L_P(n) = \begin{cases} L_{P'}\left(\frac{n}{s'}\right) & \text{if } s' \mid n \\ 0 & \text{otherwise} \end{cases}.$$

Thus we may assume WLOG that P is full-dimensional (dim(P) = k).

Claim 2: WLOG, P = CH{0, Q}, where Q is a (d - 1)-dimensional rational polytope.

Proof of Claim 2: WLOG, we may translate some rational d-dimensional polytope by an integer vector so that it does not contain the origin, as such a translation will clearly not alter the number of integer lattice points in the polytope. As we will see, $L_P(n)$ may be expressed as sums and differences of the number of integer points in polytopes of the form {0,Q} (including lower dimensional Q), using inclusion-exclusion methods to properly count the intersection of faces. To accomplish this, we define two types of faces:

1. The collection $\mathcal{F}_v$ of faces F of P that are "visible": a facet (i.e., a (d -1)-dimensional face) is visible if

$$\forall a \in F, \forall \lambda \text{ s.t. } 0 < \lambda < 1 \, [\lambda a \notin P],$$

and a lower dimensional face is visible if every facet that it is contained in is visible. Geometrically, a face is visible if a line from the origin to any point on the face does not contain any other point in the polytope.

2. The collection $\mathcal{F}_h$ of faces F of P that are "hidden": a facet is "hidden" if it is not visible, and a lower dimensional face is hidden if every facet that it is contained in is hidden. Geometrically, a face is hidden if some point on the face is 'behind' another point on the polytope, i.e., a line from the origin to some point on the face contains another point in the polytope.

For instance, consider the simple example where P = CH{(1,1),(3,1),(3,2)}.

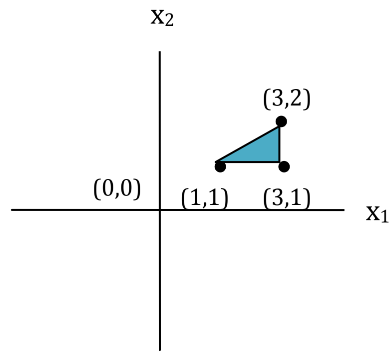

Figure 12. The 2-dimensional polytope P = CH{(1,1),(3,1),(3,2)}.

We see that CH{(1,1),(3,2)}, a 1-dimensional facet of P, contains the point (3,2), and if we choose $\lambda = 1/2$, then we get $(1/2)(3, 2) = (3/2, 1)$, which is in P. Thus this facet is not visible, so it is hidden. We can use the same point (3, 2) and the same λ to show that CH{(3,1),(3,2)} is hidden as well. However, by inspection we see that for each point a in the facet CH{(1,1),(3,1)}, λa will not be contained in P (since any λ will send the x_2 coordinates of points in CH{(1,1),(3,1)} to some $x_2 < 1$, and therefore out of P). Thus CH{(1,1),(3,1)} is visible. From a geometric stand point, we see that a line from the origin to almost every point in CH{(1,1),(3,2)} and CH{(3,1),(3,2)} must pass through P, but a line from the origin to any point in CH{(1,1),(3,1)} will not. In addition, the only 0-dimensional face that is hidden is the point on top of the triangle, since both facets containing it are hidden. The other 0-dimensional faces are neither visible, nor hidden.

For a face F of P, let P_F = CH(0,F). Then inclusion-exclusion provides the following:

$$L_P(n) = \sum_{F \in \mathcal{F}_H} (-1)^{d-1-\dim(F)} L_{P_F}(n) - \sum_{F \in \mathcal{F}_V} (-1)^{d-1-\dim(F)} \left(L_{P_F}(n) - L_F(n)\right).$$

While we omit it here, a proof of this identity using a topological approach was included in the original paper by Sam and Woods [6]. We will, however, include and explain a helpful example from the same paper. Consider the following generic 2-dimensional polytope P (bold) in $\mathbb{R}^2$ (the plane of the paper), which has been translated such that it does not include the origin.

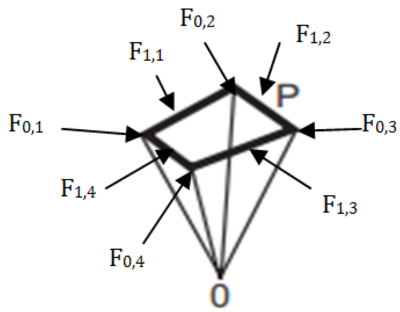

Figure 13. Arbitrary 2-dimensional polytope P (bold) shown with 4 1-dimensional facets and 4 0-dimensional faces labeled. 4 2-dimensional convex hulls are formed from the origin and facets [6].

In the above figure, we see right away that $F_{1,1}$ and $F_{1,2}$ are hidden, so $F_{0,2}$ is also hidden. $F_{1,3}$ and $F_{1,4}$ are visible, so $F_{0,4}$ is visible. Since $F_{0,1}$ and $F_{0,3}$ are contained in both hidden and visible facets, we consider these faces neither hidden nor visible. We can now interpret the above inclusion-exclusion formula for P as follows:

$$L_P(n) = \sum_{F \in \mathcal{F}_H} (-1)^{d-1-\dim(F)} L_{P_F}(n) - \sum_{F \in \mathcal{F}_V} (-1)^{d-1-\dim(F)} \left(L_{P_F}(n) - L_F(n)\right)$$

$$= \left[L_{P_{F_{1,1}}}(n) + L_{P_{F_{1,2}}}(n) - L_{P_{F_{0,2}}}(n)\right]$$

$$- \left[\left(L_{P_{F_{1,3}}}(n) - L_{F_{1,3}}(n)\right) + \left(L_{P_{F_{1,4}}}(n) - L_{F_{1,4}}(n)\right) - \left(L_{P_{F_{0,4}}}(n) - L_{F_{0,4}}(n)\right)\right].$$

Geometrically, this looks like:

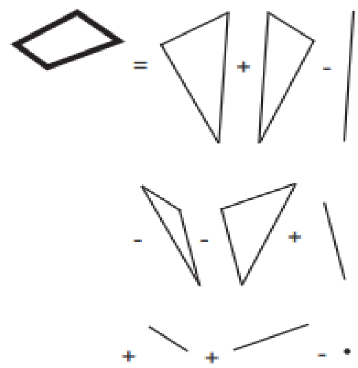

Figure 14. A polygon decomposition: a geometric interpretation of the above inclusion-exclusion formula [6].

For an example of such inclusion-exclusion counting, let's assume that the face $F_{0,2}$ is actually a point on the integer lattice. Then in the above figure and formula, we see that $F_{0,2}$ is included once by $P_{F_{1,1}}$, once by $P_{F_{1,2}}$, excluded once by $P_{F_{0,2}}$, and does not appear anywhere else. Thus $F_{0,2}$ is only counted once, as desired. As another example, we know that the origin is an integer point, but one that is outside of P by assumption. The origin is included a total of 3 times, once each by $P_{F_{1,1}}$, $P_{F_{1,2}}$, and $P_{F_{0,1}}$; and it is excluded a total of 3 times, once each by $P_{F_{1,3}}$, $P_{F_{1,4}}$, and $P_{F_{0,4}}$. As the inclusions and exclusions cancel, the origin is not counted at all, as desired.

Returning to the proof of our claim, it remains to be shown that each of the decomposed parts of the polytope P (represented by terms in the inclusion-exclusion formula) can be dilated by an integer such that their affine hulls contain an integer point. We know that for any face F of P, the i-dimensional faces F' of P_F are either faces of P or contain the origin, depending on whether F is hidden or visible. If F' is a face of P, then Aff(s_iF') contains integer points by definition of s_i. In the other case, if F' contains the origin, then Aff(s_iF') contains integer points because the origin itself is an integer point.

Thus, in either case, F' satisfies the assumptions of the Ehrhart's and McMullen's theorems, so we have addressed the first two terms of the inclusion-exclusion sum. For the final term, $\sum_{F \in \mathcal{F}_v} (-1)^{d-1-\dim(F)} L_F(n)$, we are dealing only with faces of P of dimension d - 1 or lower. Here, according to the inductive hypothesis, the conditions of the theorems are also satisfied. Therefore, if we prove the theorems for a polytope P' of the form conv{0, Q}, where Q is a (d - 1)-dimensional rational polytope, then the theorems follow for P as well, since we have shown that the number of integer points in P can always be considered as sums and differences of the number of integer points in different P'.

Claim 3: WLOG, we may assume P = conv{0, Q}, where Q is a (d - 1)-dimensional rational polytope living in the hyperplane $x_d = q$, where $q \in \mathbb{Q}_{>0}$.

Proof of Claim 3: We say that a transformation x' = Ax is *unimodular* if A is a square integer matrix with det(A) = ±1. Furthermore, a linear transformation sends a lattice to itself iff the transformation is unimodular. While we omit a proof of this here, the interested reader can find this proof and a general introduction to lattice transformations in [27]. Our ability to always perform a unimodular transformation proves the claim.

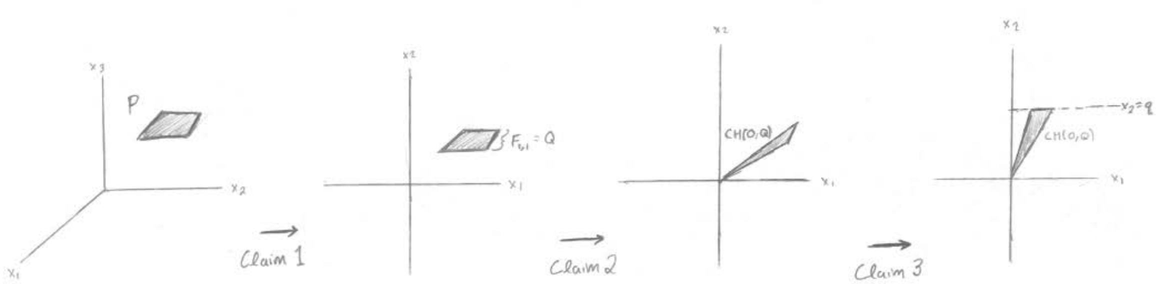

Figure 15. An example of an application of claims 1 - 3. From left to right: a 2-dimensional polygon in $\mathbb{R}^3$ is projected onto $\mathbb{R}^2$ via an affine transform T, with integer points preserved; inclusion-exclusion is applied using CH(0, Q), where here Q is the 1-dimensional facet $F_{1,1}$ of P; a unimodular transformation sends CH(0,Q) to the 1-dimensional hyperplane $x_2 = q$, where $q \in \mathbb{Q}_{>0}$, and integer points are preserved.

We will now proceed to prove Ehrhart's Theorem and McMullen's Theorem for P = conv{0, Q}, where Q is a (d - 1)-dimensional rational polytope living in the hyperplane x_d = q, with q ∈ $\mathbb{Q}_{>0}$. Since q is a positive rational, then WLOG, P lives in the hyperplane $x_d = \frac{a}{b}$, where a, b are positive integers and gcd(a, b) = 1. Since faces of Q are in fact faces of P, then for an i-dimensional face of Q, Aff(s_iF) contains integer points. Now let $\overline{Q} = \frac{b}{a}Q$, so then $\overline{Q}$ lives in the hyper plane x_d = 1. Then for an i-dimensional face $\overline{F}$ of $\overline{Q}$, Aff($s_i \frac{a}{b}\overline{F}$) contains integer points. So we have that the integer points in a dilated P, denoted by the intersection nP ∩ $\mathbb{Z}^d$, is the disjoint union

$$\bigcup_{i=0}^{\left\lfloor \frac{na}{b} \right\rfloor} i\overline{Q} \cap \mathbb{Z}^d,$$

which means that the number of integer points in P is just the sum

$$L_P(n) = \sum_{i=0}^{\left\lfloor \frac{na}{b} \right\rfloor} L_{\overline{Q}}(i).$$

Since we know that $L_{\overline{Q}}(i)$ is a (d - 1)-degree quasi-polynomial by the inductive hypothesis, we may conclude that $L_P(n)$ is a quasi-polynomial of degree d by Lemma 4.6. Furthermore, the S_i in the statement of Lemma 4.6 are given by

$$S_i = \frac{\left(s_i \frac{a}{b}\right) b}{\gcd\left(s_i \frac{a}{b}, a\right)} = \frac{as_i}{a} = s_i.$$

And since $s_d \mid s_{d-1} \mid . . . \mid s_0$, then $s_i = \mathrm{lcm}(s_d, s_{d-1}, . . ., s_i)$, which is a period of the i^{th} coefficient of $L_P(n)$ by Lemma 4.6. We also see that if P is an integral polytope, then $s_d = s_{d-1} = . . . = s_0 = 1$, so 1 is a period of each coefficient of $L_P(n)$, making $L_P(n)$ simply a polynomial. Thus the inductive step holds and we are done. ■

2.8 Computer generation of the Ehrhart quasi-polynomial counting 3-way ties

For his 2008 undergraduate senior thesis, Dai modeled voting situations that produce a 3-way Borda tie between 3 candidates as a system of linear constraints. In order to understand his methodology, we revisit to some notation that we mentioned briefly in Section 1.4. Let A_R = <p, q, r> be a reference enumeration of candidates in an election with s(p), s(q), and s(r) denoting the Borda scores for p, q, and r, respectively. Then there is a 3-way tie when s(p) = s(q) = s(r). Now, using the same list of rankings from Section 1.1, consider the voting situation

$$\prod \ = \ (n_1, n_2, n_3, n_4, n_5, n_6),$$

where $n_1 + n_2 + n_3 + n_4 + n_5 + n_6 = n$. With the o-c Borda count, we see that $s(p) = n_1 + n_2 - n_4 - n_6$ (i.e., p collects 1 point from each of the voters casting ballots where p is ranked first, no points from voters who cast ballots where p is ranked second, and -1 point from each of the voters casting ballots where p is ranked last). Similarly, $s(q) = n_3 + n_4 - n_2 - n_5$, and $s(r) = n_5 + n_6 - n_1 - n_3$. And since s(p) = s(q) = s(r), then $n_1 + n_2 - n_4 - n_6 = n_3 + n_4 - n_2 - n_5 = n_5 + n_6 - n_1 - n_3$. We can now set up our system of linear constraints as follows:

(1) $n_1 + n_2 + n_3 + n_4 + n_5 + n_6 = n$
(2) $n_1 + n_2 - n_4 - n_6 = n_3 + n_4 - n_2 - n_5$
(3) $n_3 + n_4 - n_2 - n_5 = n_5 + n_6 - n_1 - n_3$
(4) $n_1 \geq 0$
(5) $n_2 \geq 0$
⋮
(9) $n_3 \geq 0$.

Since each of the n_i is nonnegative, then the number of integer solutions to the above system of linear equations equals the number of voting situations for n voters that will produce a 3-way Borda tie between 3-candidates. However, we are not quite at the final form of the system of linear equations that will be used by the computer software. In order to get there, we will write (1), (2), and (3) using linear inequalities and also re-arrange terms in (2) and (3). Thus the first three lines of the above system become:

$$
\begin{aligned}
&(1)\ n_1 + n_2 + n_3 + n_4 + n_5 + n_6 \le n \\
&(2)\ n_1 + n_2 + n_3 + n_4 + n_5 + n_6 \ge n \\
&(3)\ n_1 + 2n_2 - n_3 - 2n_4 + n_5 - n_6 \le 0 \\
&(4)\ n_1 + 2n_2 - n_3 - 2n_4 + n_5 - n_6 \ge 0 \\
&(5)\ n_1 - n_2 + 2n_3 + n_4 - 2n_5 - n_6 \le 0 \\
&(6)\ n_1 - n_2 + 2n_3 + n_4 - 2n_5 - n_6 \ge 0,
\end{aligned}
$$

which may be written in matrix form as

$$
\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ -1 & -1 & -1 & -1 & -1 & -1 \\ 1 & 2 & -1 & -2 & 1 & -1 \\ -1 & -2 & 1 & 2 & -1 & 1 \\ 1 & -1 & 2 & 1 & -2 & -1 \\ -1 & 1 & -2 & -1 & 2 & 1 \end{bmatrix}
\begin{bmatrix} n_1 \\ n_2 \\ n_3 \\ n_4 \\ n_5 \\ n_6 \end{bmatrix}
\le
\begin{bmatrix} n \\ -n \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix},
$$

a 3-dimensional polytope living in $\mathbb{R}^6$. Dai used computer software LattE, short for Lattice point Enumeration, to find all nonnegative integer solutions for the above system and therefore the number of voting situations vs(n) that produce an m-way Borda tie for 3 candidate elections [28]. Consistent with Ehrhart's Theorem, his result was a quasi-polynomial, specifically

$$
\mathrm{vs}(n) = \frac{n^3 + 9n^2 + \ll 42,15 \gg n + \ll 72,25,-88,9,56,7 \gg}{72},
$$

with degree 3 and minimal periods of 1, 1, 2, and 6 for $c_3(n)$, $c_2(n)$, $c_1(n)$, and $c_0(n)$, respectively.

Section III

Beyond m = 3: Using Permutahedra to Count Borda Ties

3.0 Brute force combinatorics: confirming the computer result

Dai used basic combinatorial methods to calculate the same quasi-polynomial function vs(n) from Section 2.8. Since we cannot currently compute the Ehrhart quasi-polynomial for 4-way Borda ties between 4 alternatives with the LattE software, this brute force combinatorial approach initially seemed the most relevant for our goal of expanding beyond the case of m = 3. However, our current inability to use combinatorial methods to find a quasi-polynomial expression that counts 4-way Borda ties between 4 alternatives demonstrates that the tremendous increase in complexity from 3 to 4 alternatives is not limited to Ehrhart theory. Indeed, our research has focused on pinning down exactly how complex the 'brute force' approach becomes when increasing the number of alternatives from 3 to 4, as much as it has been concerned with actually counting ties. Although we have not reached our ultimate goal of counting all 4-way Borda ties between 4 alternatives, we have found some interesting connections and results by expanding on Dai's combinatorial methods. Therefore, we will begin this section by discussing the key parts of Dai's counting techniques.

First, recall from Section 1.4 that a voting situation $\boldsymbol{\prod} = (n_1, n_2, .., n_{m!})$ is central if it produces an m-way tie in the Borda count. We can think of all central voting situations for m alternatives as forming an ordering in the following way:

Let $\prod_1 = (n_1, n_2, \ldots, n_{m!})$, $\prod_2 = (n_1', n_2', \ldots, n_{m!}')$ be any two central voting situations. Then $\prod_1 > \prod_2$ iff

1) $n_i \geq n_i'$ for all $0 \leq i \leq m!$, and

2) $n_i > n_i'$ for at least one i with $0 \leq i \leq m!$

We say that a voting situation is *elementary* if it is a minimal central voting situation in the above ordering.[13] Put differently, a voting situation is elementary if it is central and contains no smaller central voting situations in the ordering. For the case of m = 3, there are two 'types' of elementary voting situations (we will discuss what 'type' means in the next section): reversals and cycles. As shown below in Figure 16, an elementary reversal is a voting situation with 2 voters, where the rank vectors[14] of cast ballots are antipodal points on the permutahedron, i.e., points opposite each other on the permutahedron with mean at the center.

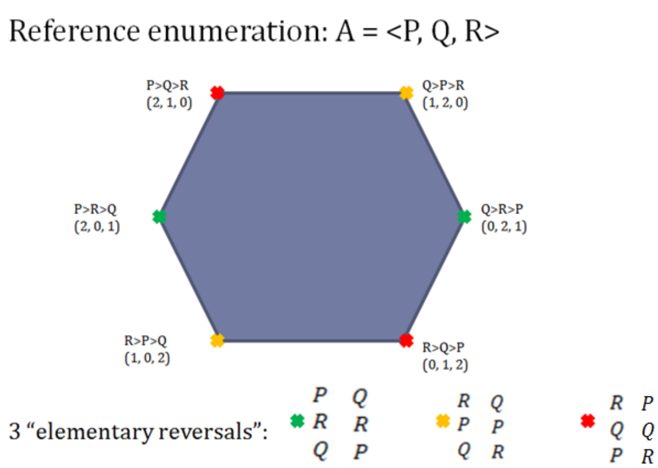

Figure 16. The three elementary reversals for m = 3.

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[13] As with our definition of central, our definition of elementary is also different from that used by Cervone et al and Dai, but equivalent for the case of m = 3 [2, 9].

[14] Recall from Section 1.2 that a rank vector $\boldsymbol{\rho}(\sigma)$ for a ranking σ of m alternatives $\{a_1, a_2, \ldots, a_m\}$ is the m-tuple of ranks $\rho(a_i)$ in reference enumeration order, e.g., for reference enumeration $A_R = \langle p, q, r \rangle$ and a ranking $\sigma = q > p > r$, we get the rank vector $\boldsymbol{\rho}(\sigma) = (\rho(p), \rho(q), \rho(r)) = (1, 2, 0)$.

As seen in Figure 16, an elementary reversal in a 3-candidate election is formed when two voters agree on a second place candidate but disagree on first and third choice candidates. Next, we form an elementary cycle for $m = 3$, also known as a Condorcet[15] cycle (or Condorcet paradox) for $m = 3$, by having three voters cast ballots in the following way:

Say a ballot is cast for $(p > q > r)$. Then another cast ballot sends p to third place, and q and r each move up one space to first and second place, respectively $(q > r > p)$. Finally, a third cast ballot sends q to last place, and moves r and p up to first and second place, respectively $(r > p > q)$.

Equivalently, a cycle can be formed using rank vectors. For a rank vector of standard Borda scoring weights (w_1, w_2, w_3) corresponding to a cast ballot, another ballot is cast with rank vector $(w_1 + 1 \bmod 3, w_2 + 1 \bmod 3, w_3 + 1 \bmod 3)$, and a third ballot is cast with rank vector $(w_1 + 2 \bmod 3, w_2 + 2 \bmod 3, w_3 + 2 \bmod 3)$. This definition can be adjusted for non-standard Borda weights (e.g., for the o-c Borda count). As shown in Figure 17, there are two elementary cycles for $m = 3$.

---

[15] Condorcet cycles are named for 18th Century French mathematician and political scientist Marquis de Condorcet, who developed a voting rule in which Condorcet cycles also produce ties. Condorcet mysteriously died soon after being imprisoned in the aftermath of the French Revolution [29].

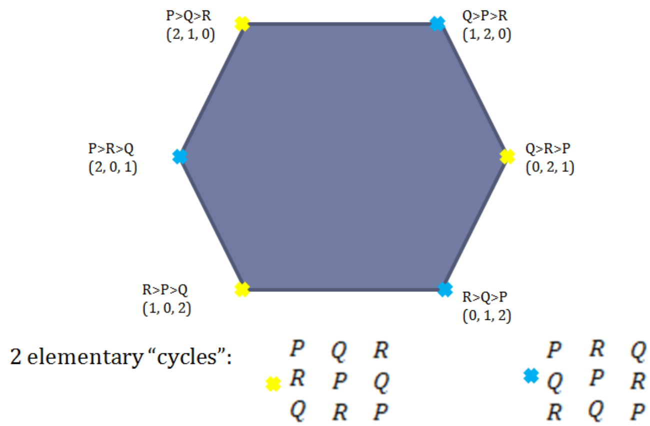

Figure 17. 2 elementary cycles (reference enumeration A_R = <P, Q, R>).

Dai's goal was to count the number of central voting situations for 3 candidates (and thus the number of 3-way Borda ties between 3 candidates) by counting the number of positive linear combinations of elementary voting situations. His main challenge with this approach was the problem of double-counting. For instance, a voting situation where each of 6 voters casts a ballot with a different ranking clearly produces a 3-way tie - in terms of the Permuta-mean rule, the mean of the 6 different rank vector coordinates on the permutahedron is at the center. Of course, we only want to count this central voting situation once, but we see that it can be created in two ways from elementary voting situations: as either a sum of the 3 reversals or a sum of the 2 cycles.

Dai's crucial tool for overcoming this issue was proving that every voting situation was central if and only if it was created by a unique nonnegative (integer) linear combination of the three reversals and only one of the two cycles. In other words, a central voting situation created by a nonnegative linear combination of reversals and a single cycle could *only* be created by that linear combination; conversely, any such linear combination of reversals and a single cycle created a central voting situation. While we omit the proof here, we direct the interested reader to his 2008 thesis [8]. With this powerful result in hand, Dai was able to make use of Theorem 1.1 by modeling each central voting situation as a 4-tuple with entries corresponding to the number of reversals and one of the cycles in the voting situation (multiplying by a factor of 2 at the end to account for central voting situations constructed from each of the two cycles). Dai divided the voting situations into 6 cases: where the number of voters $n = 3k$ (k odd); $n = 3k$ (k even); $n = 3k + 1$ (k odd); $n = 3k +1$ (k even); $n = 3k + 2$ (k odd); $n = 3k + 2$ (k even). For each case, Dai obtained a different polynomial function of n, and all six polynomials were then combined into one quasi-polynomial. Since Dai's quasi-polynomial was formed from 6 polynomials, we see that it had quasi-period 6. As expected, this quasi-polynomial was the same one (vs(n)) given at the end of Section 2.8.

3.1 m = 4: permuting voting situations of the same 'type'

In order to extend Dai's methods to the case of 4 candidates, we must also find the set of elementary voting situations that will produce all central voting situations via a unique nonnegative integer linear combination. Only in this way we can make use of the power of Theorem 1.1. But where to begin?

Dai began by classifying elementary voting situations into two categories, cycles and reversals, and this classification seems very natural. One type corresponds to 2 voters, the other to 3. Furthermore, in a geometrical context, we see that any two antipodal points on the 3-permutahedron can be mapped to two other antipodal points with rotations. Likewise, three rank vector points corresponding to a cycle can be mapped to the other three-point cycle in the same way. However, in the case of the 4-permutahedron (4 candidates), classifying - and even finding - elementary voting situations becomes far more complicated.

For $m = 4$, there are 12 pairs of antipodal points on the 4-permutahedron, so we can think of these pairs as the analog of reversals for $m = 3$. As an example, for the reference enumeration $A_R = \langle p, q, r, s \rangle$ of alternatives in an election, the voting situation where one voter casts a ballot for $(p > q > r > s)$ and one voter casts a ballot for $(s > r > q > p)$ results in a four-way Borda tie. This voting situation and all other reversals for $m = 4$ are color-coded on the permutahedron below in Figure 18.

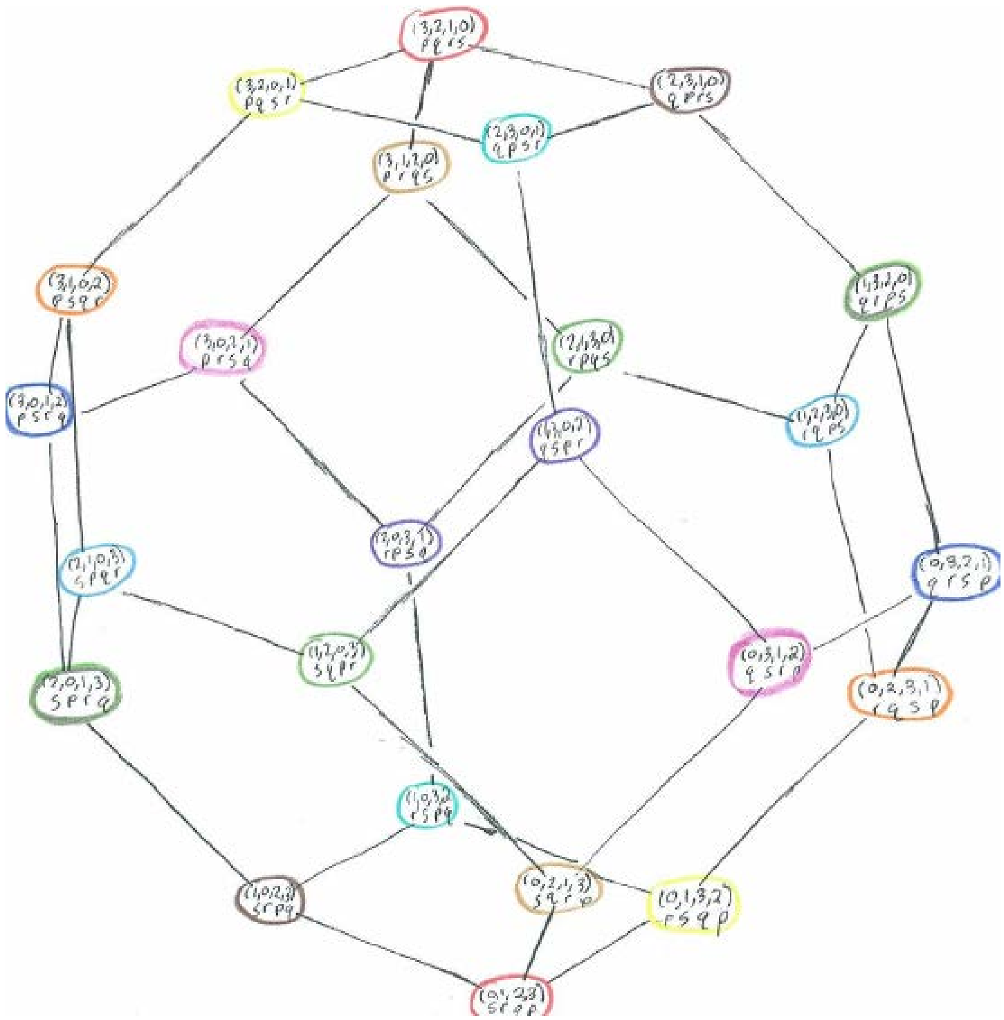

Figure 18. The 4-permuahedron with all twelve elementary reversals color coded. For example, the voting situation corresponding to single votes for the ballots with rankings (p > q > r > s) and (s > r > q > p) is circled in red.

Now, we know that there are no elementary voting situations for four alternatives and three voters by Theorem 1.4.3. However, by inspection and computational methods (discussed below in Section 3.2) we have found that there are numerous elementary voting situations for m = 4 with 4, 6, 8, 10, and 12 voters; and there may be elementary profiles for 14 or more voters. Furthermore, even two elementary voting situations with the same number of voters may have very different geometrical representations on the 4-

permutahedron. For an example of this, we direct the reader to compare Appendix IIa, Appendix 2b, and Appendix 2c, where we have included color-coded sketches of the 4-permutahedron with all elementary voting situations for 4 voters. In addition, while there are no elementary voting situations with an odd number of voters, there are elementary voting situations with an even number of voters and an odd number of distinct rankings due to repeated ballots cast. One example of this for A_R = <p, q, r, s> is the voting situation where 2 votes are cast for (s > r > q > p), and 1 vote each is cast for (q > s > p > r), (q > p > r > s), (p > r > s > q), and (p > r > q > s).

The new complexity for m = 4 motivates our formal analysis of what it means for two elementary voting situations (and more generally, any two voting situations) to be of the same type. Our discussion of types of voting situations will be organized into five parts. First, we will discuss our notion of *type* as being the generalization of Dai's distinction between elementary reversals and elementary cycles for 3 alternatives. Here, we will also define *rr-type*, which we conjecture may be equivalent to our general notion of type. Second, we will define what it means for two voting situations to be of the same *alternative type* (or just *a-type*). Next, we will define the *antipodal map*, followed by what it means for two voting situations to be of the same *weight* type (or just *w-type*). Finally, we will discuss some of the connections and results that we have found by classifying voting situations into different versions of type. We conjecture that all voting situations generated by the antipodal map and permutations of alternatives may be equivalent to the set of all voting situations of the same rr-type (generated by rotations and reflections). In addition, we direct the reader to Appendix III for a brief discussion of how we can define *voter type* (or just *v-type*) to describe the relationship between voting situations and profiles. The reader should also note that throughout this section, we will repeatedly refer to two examples: one in which we turn a 4-voter elementary profile $\mathbb{P}$ into another 4-voter elementary profile $\mathbb{P}'$, and a second in which we turn an 8-voter elementary voting situation $\mathbb{P}^*$ into another 8-voter elementary profile, $\mathbb{P}^{*'}$.[16]

3.1a Overview of ***type*** and ***rr-type***

Working from Dai's distinction between elementary reversals and cycles, we think of two voting situations $\prod_1, \prod_2$ as being of the same *type* if a transformation of the permutahedron, where rank vectors from $\prod_1$ are sent to rank vectors from $\prod_1$, preserves adjacent vertices. Now, we will classify two voting situations to be of the same *rr-type* (or just *type*) if and only if one's rank vector locations on the permutahedron can be obtained from the other's by a combination of rotations and reflections in (m - 1)-dimensional space (so not the original affine space that the m-permutahedron lives in, but the affine space in which the m-permutahedron is full-dimensional). For m = 3, we only need rotations to go between two elementary voting situations of the same type, i.e., while preserving adjacency. A pair of vertices on the 3-permutahedron corresponding to an elementary reversal can be rotated in the plane of the hexagon by multiples of 120° to obtain either of the other two elementary reversals (or the original reversal). Similarly, a set of three vertices on the 3-permutahedron corresponding to an elementary cycle can be rotated in the plane of the hexagon by multiples of 60° to obtain the other elementary cycle (or the original elementary cycle). However, as we will see, the step from 3 alternatives to 4

[16] At times we may also re-introduce labeled voters (and therefore profiles, rather than voting situations), but we will make clear when we are doing this. Recall that any profile(s) can be turned into a voting situation if we simply choose to ignore the identities of the voters.

alternatives requires both rotations and reflections be used to obtain all elementary voting situations of the same type. In order to rotate and reflect the m-permutahedron, we need to define an *alternative permutation*, an *antipodal mapping*, and a *weight permutation*.

3.1b *Alternative permutations, alternative permutation matrices,* and *alternative type*

Let A = {a_1, a_2, …, a_m} be a set of m alternatives and V = {v_1, v_2, …, v_n} be a set of n voters. An *alternative permutation* π_a: A →A, written

$$\pi_a = \begin{pmatrix} a_1 & a_2 & \cdots & a_m \\ \pi(a_1) & \pi(a_2) & \ldots & \pi(a_m) \end{pmatrix},$$

is a permutation on the set of alternatives. For example, consider the set of four alternatives {a_1, a_2, a_3, a_4}. Then the alternative permutation $\pi_a = \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ a_1 & a_3 & a_2 & a_4 \end{pmatrix}$ is the (bijective) function from A to A where

$$\pi_a(a_1) = a_1,$$

$$\pi_a(a_2) = a_3,$$

$$\pi_a(a_3) = a_2,$$

$$\pi_a(a_4) = a_4.$$

Simply put, we have switched alternatives a_2 and a_3. Now, an *alternative permutation matrix* P_{π_a} is the m *x* m square matrix

$$P_{\pi_a} = \begin{bmatrix} p_{1,1} & p_{1,2} & \cdots & p_{1,m} \\ p_{2,1} & \ddots & & \\ \vdots & & \ddots & \\ p_{m,1} & & & p_{m,m} \end{bmatrix}$$

where $p_{h,k} = \begin{cases} 1 \; if \; a_h = \pi(a_k) \\ 0 \; otherwise \end{cases}$. So for the above alternative permutation π_a, we see that $p_{1,1} = 1$, $p_{2,3} = 1$, $p_{3,2} = 1$, $p_{4,4} = 1$, and all remaining entries are 0. This gives the following matrix.

$$P_{\pi_a} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

To see how this matrix acts on a profile, consider the following scenario for 4 alternatives and 4 voters. Let A_r = <a_1, a_2, a_2, a_4> be an alternative reference enumeration and V_r = <v_1, v_2, v_3, v_4> be a voter reference enumeration.

$$\mathbb{P} = \begin{matrix} v_1 & v_2 & v_3 & v_4 \\ a_1 & a_2 & a_3 & a_4 \\ a_2 & a_1 & a_4 & a_3 \\ a_3 & a_4 & a_2 & a_1 \\ a_4 & a_3 & a_1 & a_2 \end{matrix}$$

From the (elementary central) profile $\mathbb{P}$, we can form a corresponding m *x* n integer matrix $S_{\mathbb{P}}$ in either of two equivalent ways. In the first method, we can consider the j^{th} column of the matrix $S_{\mathbb{P}}$ to be the rank vector for the ballot cast by the the j^{th} voter in our voter reference enumeration. In the second method, we can consider the i^{th} row of $S_{\mathbb{P}}$ to be the *score vector* of the i^{th} alternative in our alternative reference enumeration, where a score vector is defined as follows:

Let a_i be the i^{th} alternative in an alternative reference enumeration. Given a profile, the score vector $sc(a_i)$is the n-tuple such that the j^{th} entry is equal to the Borda points awarded by the j^{th} voter in the voter reference enumeration. Thus for the above reference enumerations and profile $\mathbb{P}$, $sc(a_2)$ = (2, 3, 1, 0), as a_2 receives two points from v_1's second place vote, three points from v_2's first place vote, 1 point from v_3's third place vote, and no points from v_4's last place vote.

Using either the score or rank vector method described above, we now form the *score-rank* matrix $S_{\mathbb{P}}$:

$$S_{\mathbb{P}} = \begin{bmatrix} 3 & 2 & 0 & 1 \\ 2 & 3 & 1 & 0 \\ 1 & 0 & 3 & 2 \\ 0 & 1 & 2 & 3 \end{bmatrix}$$

Let $\mathbb{P}_1$, $\mathbb{P}_2$ be two profiles for m voters and n alternatives, with S_1, S_2 as corresponding score-rank matrices. Then we say $\mathbb{P}_1$ and $\mathbb{P}_2$ are of the same *alternative type (*or just *a-type)* if there exists an alternative permutation matrix $P_{\pi_a} \in \mathbb{M}_{m \times m}$ such that $P_{\pi_a} S_1 = S_2$. For instance, let's consider the above profile $\mathbb{P}$ with score-rank matrix $S_{\mathbb{P}}$, and the above alternative permutation π_a with alternative permutation matrix P_{π_a}. We compute:

$$P_{\pi_a} S_{\mathbb{P}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 2 & 0 & 1 \\ 2 & 3 & 1 & 0 \\ 1 & 0 & 3 & 2 \\ 0 & 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 2 & 0 & 1 \\ 1 & 0 & 3 & 2 \\ 2 & 3 & 1 & 0 \\ 0 & 1 & 2 & 3 \end{bmatrix}.$$

This second score-rank matrix, which we will call $S_{\mathbb{P}'}$, has a corresponding profile $\mathbb{P}'$ that may be obtained from the original reference enumeration. For instance, using the rank vector method, we observe from the first column that the first voter cast a ballot ranking a_1 first, a_2 third, a_3 second, and a_4 last. Continuing in this way, we get

$$\mathbb{P}' = \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \\ a_1 & a_3 & a_2 & a_4 \\ a_3 & a_1 & a_4 & a_2 \\ a_2 & a_4 & a_3 & a_1 \\ a_4 & a_2 & a_1 & a_3 \end{array}.$$

Both of these elementary profiles may be seen on the 4-permutahedron in Appendix IIb (with $a_1 = p$, $a_2 = q$, $a_3 = r$, and $a_4 = s$). Note that a profile/voting situation $\mathbb{P}$ is central iff the sum of scores in a given row of $S_{\mathbb{P}}$ is the same for all rows.

By looking at the geometrical representation of all points on the 4-permutahedron (Appendix IIb), we see that by switching a_2 and a_3, we have rotated sets of rank vectors (corresponding to voting situations) on the 4-permutahedron by 90° about a plane formed by 2 of its 3 axes (e.g., in Appendix IIb, yellow coordinates go to purple coordinates). However, we can also see that a permutation acting on the individual rank vectors does not rotate each these coordinates by the same amount. We leave open the question of whether an alternative permutation always produces rotations of the permutahedron, and if it does always produce rotations, then what these rotations might look like. However, we note that by finding out what a single transposition (such as the one sending $\mathbb{P}$ to $\mathbb{P}'$) does to points on the 4-permutahedron, we can use symmetry to draw more general conclusions about the geometrical interpretation of alternative permutations.

3.1c The antipodal map

We can also use an alternative permutation so that the set of rank vector coordinates given by $\mathbb{P}$ can be mapped to a set containing the antipodal rank vectors, where the alternative permutation simply switches a_3 and a_4. We originally conjectured that every such 'antipodal mapping' (formally defined in a moment) - which preserves adjacent points on the permutahedron and thus clearly falls within our notion of type - could be achieved with alternative permutations. However, we soon proved our intuition wrong by finding a counterexample. For A_R = <p, q, r, s> and n = 8, consider the following elementary voting situation:

$$\mathbb{P}^* = \begin{array}{cccc} \mathbf{2} & \mathbf{2} & \mathbf{3} & \mathbf{1} \\ r & q & s & s \\ p & r & p & r \\ q & p & q & q \\ s & s & r & p \end{array},$$

where bold numbers above the ballot rankings represent the number of votes for each ballot. Then $\mathbb{P}^*$ has score-rank matrix[17]

$$\mathrm{S}_{\mathbb{P}^*} = \begin{array}{cccc} \mathbf{2} & \mathbf{2} & \mathbf{3} & \mathbf{1} \\ 2 & 1 & 2 & 0 \\ 1 & 3 & 1 & 1 \\ 3 & 2 & 0 & 2 \\ 0 & 0 & 3 & 3 \end{array}.$$

Now, we'll define the *antipodal map* as the permutation of scoring weights

$$\pi_{ant} = \begin{pmatrix} 3 & 2 & 1 & 0 \\ 0 & 1 & 2 & 3 \end{pmatrix}.$$

We will soon provide a more complete definition for how this permutation acts on voting situation, but for now, we'll simply say that the antipodal map sends every individual scoring weight (an integer) in a score-rank matrix to a new scoring weight given by π_{ant}. Thus 3s and 0s switch, and 2s and 1s switch. Equivalently, it sends each rank vector

[17] Since we are not labeling voters in this example, we can only form a unique score-rank matrix by fixing some ordering of the columns (instead of referring to specific voters). Ultimately, however, if we are only interested in voting situations (i.e., for IAC-assumptions), the ordering of the columns does not matter. In other words, as long we keep track of the number of votes for a given rank vector (column), we can form various equivalent score-rank matrices.

(column) in a score-rank matrix to its antipodal rank vector. For $\mathbb{P}^*$, the antipodal map takes $S_{\mathbb{P}^*}$ and creates the new score-rank matrix

$$S_{\mathbb{P}^{*\prime}} = \begin{array}{cccc} \mathbf{2} & \mathbf{2} & \mathbf{3} & \mathbf{1} \\ 1 & 2 & 1 & 3 \\ 2 & 0 & 2 & 2, \\ 0 & 1 & 3 & 1 \\ 3 & 3 & 0 & 0 \end{array}$$

which in turn corresponds to the new elementary voting situation

$$\mathbb{P}^{*\prime} = \begin{array}{cccc} \mathbf{2} & \mathbf{2} & \mathbf{3} & \mathbf{1} \\ s & s & r & p \\ q & p & q & q. \\ p & r & p & r \\ r & q & s & s \end{array}$$

However, there is an easy way to see that this application of the antipodal map does not correspond to any alternative permutation. By inspection, we see that in $\mathbb{P}^*$, r gets ranked 1^{st} twice, 2^{nd} three times, and last 3 times; in $\mathbb{P}^{*\prime}$, meanwhile, no alternative gets ranked in this way. Thus r cannot be permuted into any other alternative to produce $\mathbb{P}^{*\prime}$, although as we said before, it is clear that $\mathbb{P}^*$ and $\mathbb{P}^{*\prime}$ are of the same type. Since we cannot always use alternative permutations to obtain the antipodal map, which is just a reflection of the permutahedron, we will define a new mechanism in the context of permutation matrices.

3.1d *Weight type*

In order to get from $\mathbb{P}^*$ to $\mathbb{P}^{*\prime}$ with permutation matrices, we have to introduce some more definitions. For A = $\{a_1, a_2, \ldots, a_m\}$, V = $\{v_1, v_2, \ldots, v_n\}$, and Borda scoring weights $w_1, w_2, \ldots, w_m$ (e.g., m - 1, m - 2, . . ., 0), we'll define a *weight permutation* π_w: V $\rightarrow$V, written

$$\pi_w = \begin{pmatrix} w_1 & w_2 & \dots & w_m \\ \pi(w_1) & \pi(w_2) & \dots & \pi(w_m) \end{pmatrix},$$

as a permutation on the set of scoring weights. Therefore, the antipodal map π_{ant} is just a special case of a weight permutation with π(3) = 0, π(2) = 1, π(1) = 2, and π(0) = 3.

A *weight permutation matrix* P_{π_w} is the m *x* m square matrix

$$P_{\pi_w} = \begin{bmatrix} p_{1,1} & p_{1,2} & \cdots & p_{1,m} \\ p_{2,1} & \ddots & & \\ \vdots & & \ddots & \\ p_{m,1} & & & p_{m,m} \end{bmatrix}$$

where $p_{h,k} = \begin{cases} 1 \; if \; p_h = \pi(p_k) \\ 0 \; otherwise \end{cases}$. So for the antipodal map π_{ant}, we get

$$P_{\pi_w} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

We'll denote the product of a weight permutation matrix P_{π_w} and a voting situation Π as

$$P_{\pi_w} * \Pi = \begin{bmatrix} p_{1,1} & p_{1,2} & \cdots & p_{1,m} \\ p_{2,1} & \ddots & & \\ \vdots & & \ddots & \\ p_{m,1} & & & p_{m,m} \end{bmatrix} * \begin{matrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & \ddots & & \\ \vdots & & \ddots & \\ a_{m,1} & & & a_{m,n} \end{matrix},$$

where each $a_{i,j}$ is an alternative in the list form of the voting situation[18]. Then

$$P_{\pi_w} * \Pi = \Pi' = \begin{matrix} a'_{1,1} & a'_{1,2} & \cdots & a'_{1,n} \\ a'_{2,1} & \ddots & & \\ \vdots & & \ddots & \\ a'_{m,1} & & & a'_{m,n} \end{matrix},$$

[18] For repeated ballots, we can either list the repeated rank vectors separately in this matrix or just keep track of which column in the matrix corresponds to more than one vote.

where we multiply in standard matrix form such that $(1)(a_{i,j}) = a'_{i,j}$ and all $(0)(a_{i,j})$ are ignored. For instance, with the antipodal map weight permutation and the voting situation $\mathbb{P}^*$, we get the product

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} * \begin{matrix} r & r & q & q & s & s & s & s \\ p & p & r & r & p & p & p & r \\ q & q & p & p & q & q & q & q \\ s & s & s & s & r & r & r & p \end{matrix} = \begin{matrix} s & s & s & s & r & r & r & p \\ q & q & p & p & q & q & q & q \\ p & p & r & r & p & p & p & r' \\ r & r & q & q & s & s & s & s \end{matrix}$$

which is just

$$\mathbb{P}^{*\prime} = \begin{matrix} \mathbf{2} & \mathbf{2} & \mathbf{3} & \mathbf{1} \\ s & s & r & p \\ q & p & q & q. \\ p & r & p & r \\ r & q & s & s \end{matrix}$$

Now let $\mathbb{P}_1$, $\mathbb{P}_2$ be any two voting situations with score-rank matrices S_1 and S_2, respectively. Then we'll say $\mathbb{P}_1$ and $\mathbb{P}_2$ are of the same *weight*-type, or *w-type*, iff there exists a weight permutation matrix $P_{\pi_w} \in \mathbb{M}_{m \times m}$ such that $P_{\pi_w} * \mathbb{P}_1 = \mathbb{P}_2$.

3.1e Connections between types

There are a couple of key observations that we can make now. Surprisingly (at least to us at first), the set of all voting situations of the same *a*-type is not always equal to the set of all voting situations of the same *w*-type. We have shown this in our most recent example, where $\mathbb{P}^*$ and $\mathbb{P}^{*\prime}$ are of the same w-type, but not the same a-type. Furthermore, alternative permutations always preserve centrality (i.e., if one voting situation is central, then every other voting situation of the same a-type is also central). In contrast, w-type permutations do not always preserve centrality, although we can easily show that the antipodal map is a special case where centrality is always preserved. We also notice that even though we cannot get from $\mathbb{P}^*$ to $\mathbb{P}^{*\prime}$ by permuting individual alternatives, we can get from $\mathbb{P}^*$ to $\mathbb{P}^{*\prime}$ by permuting entire rows of alternatives in the list form of the voting situation, specifically, with the row permutation

$$\pi_r = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix},$$

which switches the first and fourth rows, and the second and third rows of the voting situation. Furthermore, while we *can* get from $\mathbb{P}^*$ to $\mathbb{P}^{*\prime}$ by permuting individual weights, we cannot get from $\mathbb{P}^*$ to $\mathbb{P}^{*\prime}$ by permuting rows in the score-rank matrix. Further inspection suggests that this contrast might be suggestive of a more general claim about weight permutations corresponding to permuting rows of a voting situation, and alternative permutations corresponding to permuting rows of a score-rank matrix. Presently, we have not been able to prove or disprove this connection.

All of this begs the question, when are two voting situations of the same rr-type - when can we be sure that one is simply a combination of reflections and rotations of the other? At the moment, we conjecture that voting situations of the same alternative type are always generated by some form of rotation, which would then make them of the same rr-type. But we know from the $\mathbb{P}^*/\mathbb{P}^{*\prime}$ example that these are not sufficient to produce all voting situations of the same rr-type. If our conjecture about alternative permutations corresponding to rotations is true, then the reason for this is simple. While specific sets of rank vectors (representing profiles/voting situations) can sometimes be mapped to their antipodal coordinates with a permutation of alternatives, every point on the entire permutahedron could only be rotated with an alternative permutation. Since rotations are orientation preserving, alternative permutations do not result in reflections. Thus, obtaining all reflections would require the antipodal map. While we conjecture that all profiles/voting situations generated by alternative permutations and the antipodal map are of the same rr-type, we do not know if these sets are in fact equivalent. Are there some voting situations that are of the same w-type, but are not antipodal maps of each other and not of the same a-type, that nonetheless satisfy our general definition of rr-type? We believe that future work directed at answering questions like these will not only aid us in counting ties with combinatorial methods, but will also contribute to the general study of profiles, voting situations, and the Borda count.

3.2 Searching for elementary profiles with Python

Besides classifying elementary voting situations (and voting situations in general), we are also interested in counting elementary voting situations. With the help of Cornell physics graduate Neil Sexton, we have used the computer language Python to develop computer programs for counting elementary voting situations with different specified numbers of voters for 4-candidate elections. We have attached a copy of the three versions (we will explain the reason 3 versions shortly) of the code for 8 voters in Appendix IV.

Each program works by first listing all 24 permutations of standard Borda scoring weights 3, 2, 1, and 0 as rank vectors. For a pre-determined n (number of voters), the program checks which n sums of rank vectors yield the constant entries given by Proposition 1. 4. 1. As an example for 8 voters, the program checks what combinations of 8 rank vectors sum to the vector (12, 12, 12, 12). The program then eliminates combinations that contain smaller central voting situations in the reference ordering described in Section 3.0. When we actually run the program, it first prints the list of all m! possible rank vectors, followed by tuples containing the numbers on the list between 0 and m! - 1 which correspond to rank vectors in an elementary voting situation. Finally, the program prints a final count for the number of elementary voting situations. For instance, the output for 2 voters looks like

```
0 (0, 1, 2, 3)
1 (0, 1, 3, 2)
2 (0, 2, 1, 3)
3 (0, 2, 3, 1)
4 (0, 3, 1, 2)
5 (0, 3, 2, 1)
6 (1, 0, 2, 3)
7 (1, 0, 3, 2)
8 (1, 2, 0, 3)
9 (1, 2, 3, 0)
10 (1, 3, 0, 2)
11 (1, 3, 2, 0)
12 (2, 0, 1, 3)
13 (2, 0, 3, 1)
14 (2, 1, 0, 3)
15 (2, 1, 3, 0)
16 (2, 3, 0, 1)
17 (2, 3, 1, 0)
18 (3, 0, 1, 2)
19 (3, 0, 2, 1)
20 (3, 1, 0, 2)
21 (3, 1, 2, 0)
22 (3, 2, 0, 1)
23 (3, 2, 1, 0)

{(1, 22), (4, 19), (5, 18), (2, 21), (3, 20), (7, 16), (8, 15), (9, 14), (6, 17), (11, 12), (10, 13), (0, 23)}
The number of elementary voting situations for 2 voters is 12
```

In this way, we counted the number of elementary voting situations for m = 4 and n ≤ 12. By adding an extra condition into the code, we also counted the number of elementary voting situations with no repeated ballots for m = 4 and n ≤ 12. Finally, by taking out the conditions that eliminate smaller central voting situations in the ordering, we counted the total number of central voting situations for m = 4 and n ≤ 12. Our results are shown below in Table 3.2.

n	2	4	6	8	10	12
all central voting situations	12	114	1328	12981	100476	638126
all elementary voting situations	12	36	532	2076	5664	5720
elementary voting situations with non-repeating ballots	12	36	220	96	0	0

Table 2. Number of central voting situations, and number of elementary voting situations with and without repeated ballots for m = 4 and n ≤ 12.

As seen in the table, there are no elementary voting situations with repeated ballots for 2 voters and 4 voters. The number of voting situations with no ballot cast more than once peaks at 220 for 6 voters, before dropping to 96 for 8 voters and 0 for 10 or more voters. Meanwhile, the number of central voting situations appears to increase by roughly an order of magnitude per two voters (at least until n = 10). While we do not present it here, Zwicker has used Ramsey theory to show that there is an upper bound on the number of elementary voting situations for a given number of alternatives.

In the future, when we fully describe voting situations of the same rr-type in the context of alternative and weight permutations, we hope to count the number of different rr-types for a given number of voters by building these conditions into the program. We already know by inspection that there is only 1 rr-type for 2-voter elementary voting situations (reversals, Figure 18 in Section 3.1), and there are 3 rr-types for 4-voter elementary voting situations, all of which are shown in Appendix II.

We also note that the utility and efficiency of our program is limited by the skill of the programmer (the author) and time. In the future, a skilled programmer can likely merge separate codes for different numbers of voters, so that a user-friendly command can determine which n the user is interested in at the moment of running the code.

3.3 Conclusions and future work

The shift from 3-candidate to 4-candidate elections corresponds an enormous increase in the complexity of the problem of determining the probability of getting an all-way tie in the Borda count. The complexity of this problem has so far hindered four different approaches at solving it: one aimed at getting an exact IC-probability function, one trying to derive an approximate IC-probability function, and two different methods meant to obtain a quasi-polynomial expression for counting ties under IAC assumptions.

Fifteen years ago, Marchant [4, 5] used the theory of random walks - relevant to the study of magnetic properties of crystals - to obtain an expression for the IC-probability of an election resulting in a 3-way Borda tie between 3 candidates as a function of n, the number of voters. He then use Fourier transforms and lattice Green's functions to derive a formula that approximated the IC-probability of a 3-way Borda tie between 3 candidates, and found that the results from his two formulas were similar for $n \geq 10$. However, due to the complexity of the problem, Marchant was neither able to find an exact formula using random walks, nor an approximate formula using lattice Green's functions, for 4-way Borda ties between 4 voters.

In 2008, Dai and Zwicker [2, 9] used two approaches, Ehrhart theory and 'brute force' combinatorial methods, to get a quasi-polynomial expression counting the total number of voting situations, assuming IAC, which produce 3-way Borda ties between 3-candidates as a function of

n. For the Ehrhart theory approach, they modeled the conditions for a 3-way Borda tie as a system of linear constraints, and used computer software to enumerate the total number of integer points within the polytope bounded by these constraints. For their combinatorial approach, they counted the total number of central voting situations for $m = 3$ as special linear combinations of elementary voting situations. However, the limits of computing power meant that they could not extend the Ehrhart theory approach to 4-way Borda ties between 4 candidates.

In the spirit of our predecessors, our research has been both theoretical and computational. On the theoretical side, we have classified different versions of 'type' based on different kinds of permutations. While we were unsuccessful in extending Dai's combinatorial approach to 4 candidates, we found some interesting connections between rank vector representations of profiles/voting situations on the 4-permutahedron. We believe that an understanding of the connections between different types of profiles/voting situations may point future 'brute force' combinatorial efforts in the right direction. On the computational side, we have written computer programs that count the number of central and elementary voting situations for $m = 4$ and $n \leq 12$. As computing power (and programming aptitude!) increases, these programs have the potential to extend our understanding of type even beyond $m = 4$.

The ultimate prize in this field would be finding a general theorem that gives the total number of all-way Borda ties as a function of both m, the number of candidates, and n, the number of voters (we're interested in IAC assumptions, but the utility of such an expression for an IC probability distribution would be equally profound). Whether such an all-encompassing formula even exists is unknown. What is known, however, is that there is still much work to be done on the problem of counting all-way Borda ties. We hope that our review and research will aid and inspire future efforts.

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Appendix I:

Marchant's Monte Carlo approximations of IC-probabilities for all-way Borda ties [4].

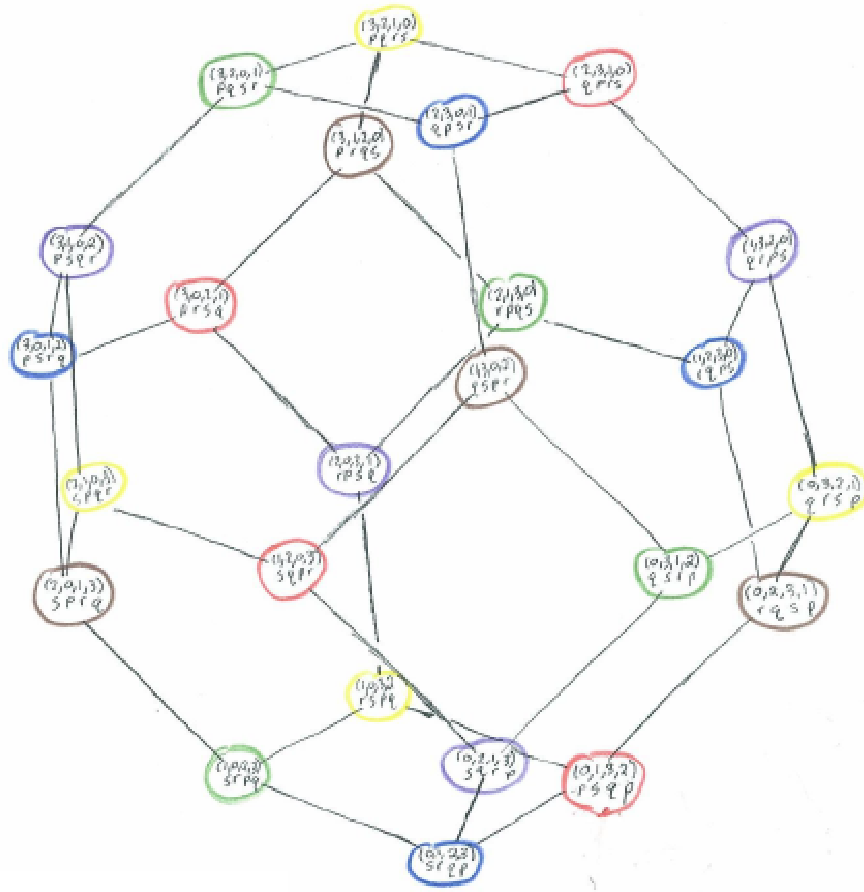
Table 1. Probabilities of k-ties with the Borda method. Five decimal places entries are truncated numbers. Others are exact. Italic entries denote probabilities estimated by a Monte-Carlo method

							n							
k	2	3	4	5	6	7	8	9	10	11	12	13	14	15
2	0.5	0	0.375	0	0.3125	0	0.27344	0	0.24609	0	0.22559	0	0.20947	0
3	0.16667	0.05556	0.06944	0.04630	0.04372	0.03601	0.03263	0.02884	0.02627	0.02392	0.02204	0.02040	0.01900	0.01777
4	0.04167	0	0.00750	0	0.00364	0	0.00238	0	*0.00148*	0	*0.00121*	0	*0.00088*	0
5	0.00833	0.00042	*0.00059*	*0.00029*	*0.00020*	*0.00012*	*0.00010*	*0.00011*	*0.00009*	*0.00008*	*0.00007*	*0.00004*	*0.00004*	*0.00003*
6	0.00139	0	*0.00005*	0	*0.00002*	0	*0.00001*	0	*0.00000*	0	*0.00000*	0		0
7	0.00020	0.00000	*0.00000*	*0.00000*										
8	0.00002	0		0		0		0		0		0		0
9	0.00000													
10	0.00000	0		0		0		0		0		0		0

							n							
k	16	17	18	19	20	30	40	50	60	70	80	90	100	infinity
2	0.19638	0	0.18547	0	0.17620	0.14446	0.12537	0.11228	0.10258	0.09503	0.08893	0.08387	0.07959	0
3	0.01670	0.01574	0.01489	0.01413	0.01344	0.00904	0.00681	0.00546	0.00456	0.00391	0.00342	0.00305	0.00274	0
4	*0.00092*	0	*0.00072*	0	*0.00061*	*0.00038*	*0.00024*	*0.00022*	*0.00015*	*0.00010*	*0.00006*	*0.00008*	*0.00008*	0
5	*0.00006*	*0.00005*												0
6		0		0										0

Appendix II: Color-coded elementary voting situations on the 4-Permutahedron for n = 4.

Appendix IIa

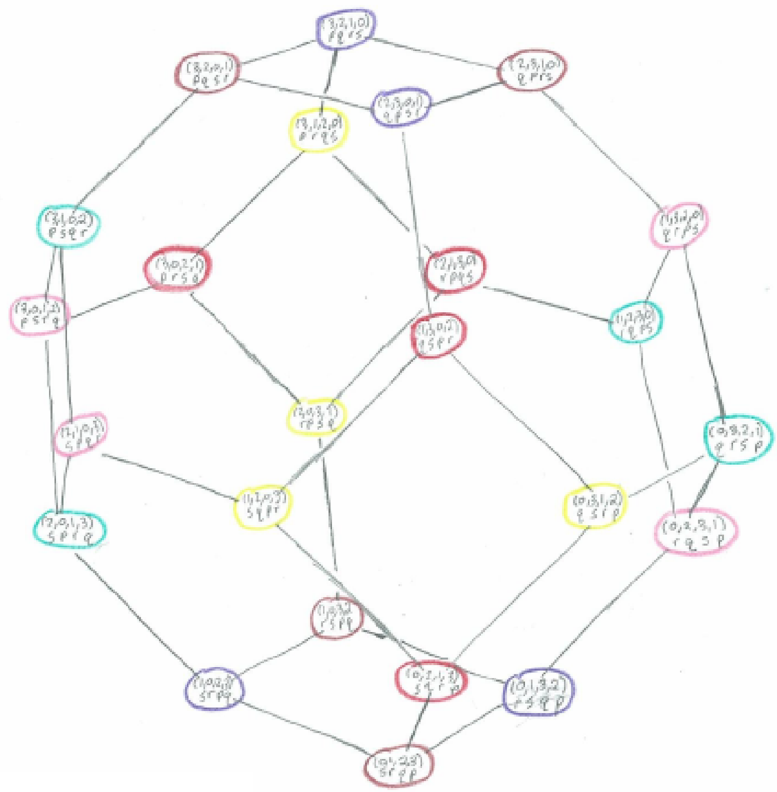


6 elementary cycles containing 4 points are formed by taking non-antipodal opposite points spaced far apart and twisting.

Rankings (YELLOW):

p	r	s	q
q	s	p	r
r	p	q	s
s	q	r	p

Appendix IIb

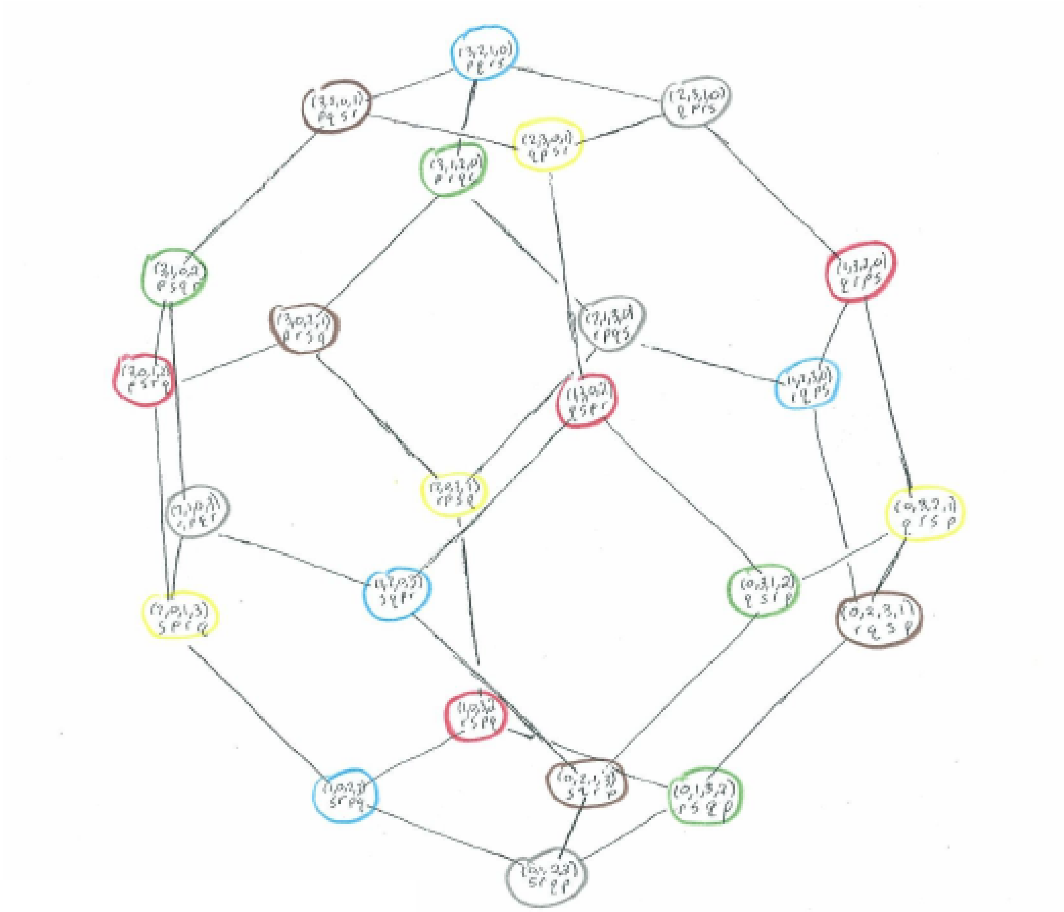


6 elementary cycles containing 4 points are formed by taking points on opposite squares and twisting.

Rankings:

p	**q**	**r**	**s**
q	**p**	**s**	**r**
r	**s**	**p**	**q**
s	**r**	**q**	**p**

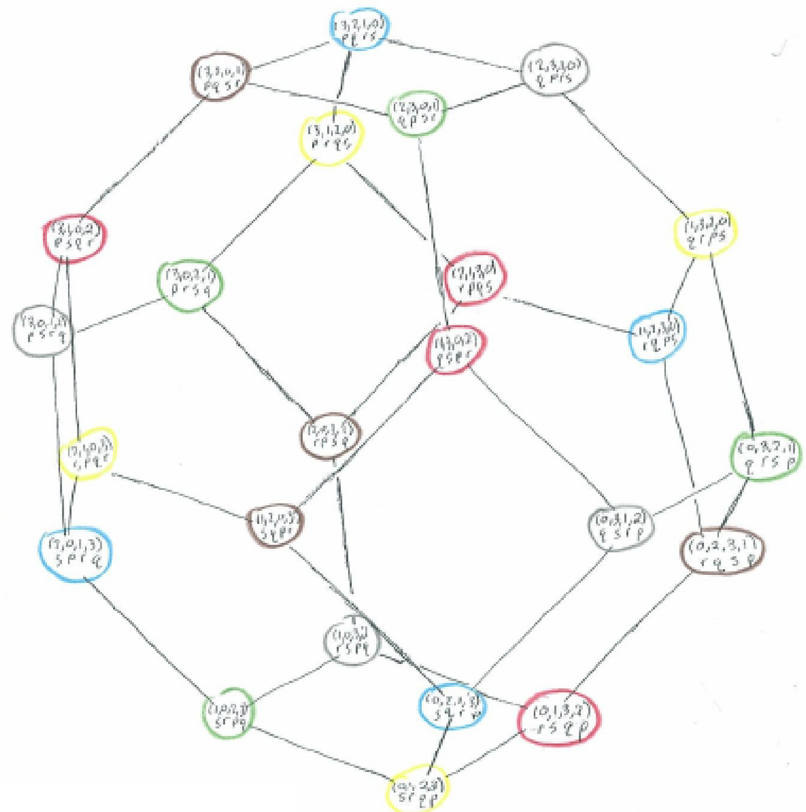
Appendix IIc



. 6 [or 24 in total counting next 3 figures] elementary cycles containing 4 points are formed by taking points on opposite hexagons and twisting.

Rankings:

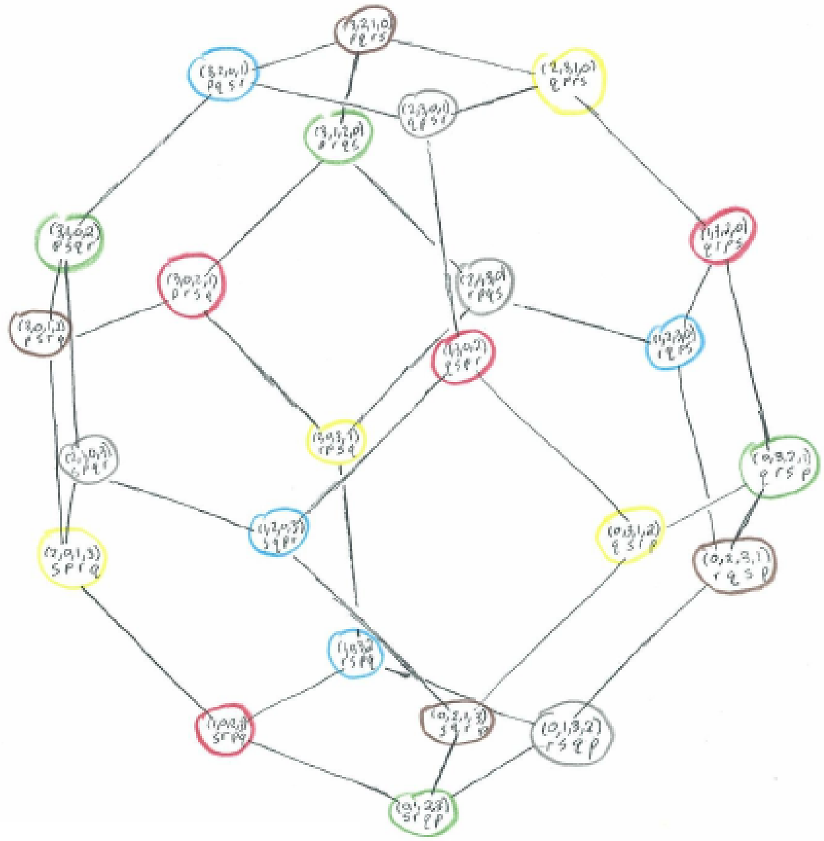
p	r	s	s
q	q	q	r
r	p	p	p
s	s	r	q



6 elementary cycles containing 4 points are formed by taking points on opposite hexagons and twisting.

Rankings:

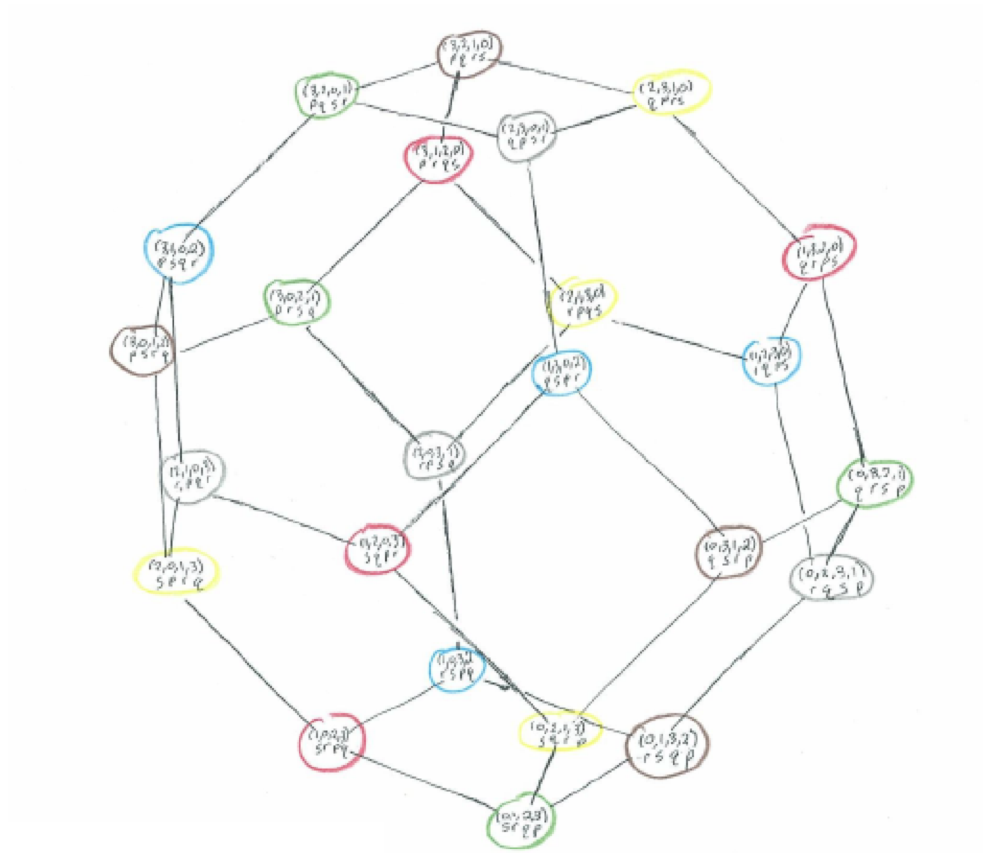
p	r	s	s
q	q	q	p
r	p	r	r
s	s	p	q



6 elementary cycles containing 4 points are formed by taking points on opposite hexagons and twisting.

Rankings:

p	p	s	r
q	s	q	q
r	r	r	s
s	q	p	p



6 elementary cycles containing 4 points are formed by taking points on opposite hexagons and twisting.

Rankings:

p	**p**	**q**	**r**
q	**s**	**s**	**s**
r	**r**	**r**	**q**
s	**q**	**p**	**p**

Appendix III: Voter permutations, voter permutation matrices, and voter type

We can use matrices to formally define the relationship between profiles and voting situations. For a set of alternatives A = $\{a_1, a_2, \ldots, a_m\}$ and a set of voters V = $\{v_1, v_2, \ldots, v_n\}$, we'll define a *voter permutation* π_v: V →V, written

$$\pi_v = \begin{pmatrix} v_1 & v_2 & \ldots & v_n \\ \pi(v_1) & \pi(v_2) & \ldots & \pi(v_n) \end{pmatrix},$$

as a permutation on the set of voters. Then a *voter permutation matrix* P_{π_v} is the n *x* n square matrix

$$P_{\pi_v} = \begin{bmatrix} p_{1,1} & p_{1,2} & \cdots & p_{1,n} \\ p_{2,1} & \ddots & & \\ \vdots & & \ddots & \\ p_{n,1} & & & p_{n,n} \end{bmatrix}$$

where $p_{h,k} = \begin{cases} 1 \; if \; v_h = \pi(v_k) \\ 0 \; otherwise \end{cases}$. We can now state the following definition:

Let $\mathbb{P}_1$, $\mathbb{P}_2$ be any two profiles with score-rank matrices S_1 and S_2, respectively. Then we'll say $\mathbb{P}_1$ and $\mathbb{P}_2$ are the same voting situation iff there exists a voter permutation matrix $P_{\pi_v} \in \mathbb{M}_{n \times n}$ such that $S_1 P_{\pi_v} = S_2$. For instance, consider the profile $\mathbb{P}$ from example in Section 3.1b,

$$\mathbb{P} = \begin{matrix} v_1 & v_2 & v_3 & v_4 \\ a_1 & a_2 & a_3 & a_4 \\ a_2 & a_1 & a_4 & a_3, \\ a_3 & a_4 & a_2 & a_1 \\ a_4 & a_3 & a_1 & a_2 \end{matrix}$$

with the voter permutation $\pi_v = \begin{pmatrix} v_1 & v_2 & v_3 & v_4 \\ v_4 & v_2 & v_1 & v_3 \end{pmatrix}$. So for the voter permutation matrix,

we see that $p_{1,3} = 1$, $p_{2,2} = 1$, $p_{3,4} = 1$, $p_{4,1} = 1$, and all other entries are 0. Thus we compute

$$S_1 P_{\pi_v} = \begin{bmatrix} 3 & 2 & 0 & 1 \\ 2 & 3 & 1 & 0 \\ 1 & 0 & 3 & 2 \\ 0 & 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 3 & 2 & 1 \\ 2 & 0 & 1 & 3 \\ 3 & 1 & 0 & 2 \end{bmatrix}.$$

This corresponds to the new profile,

$$\mathbb{P}' = \begin{matrix} v_1 & v_2 & v_3 & v_4 \\ a_4 & a_2 & a_1 & a_3 \\ a_3 & a_1 & a_2 & a_4, \\ a_1 & a_4 & a_3 & a_2 \\ a_2 & a_3 & a_4 & a_1 \end{matrix}$$

and it is easy to show that $\mathbb{P}$ and $\mathbb{P}'$ are the same voting situation in tuple form.

Appendix IV: the Computer Code

For purposes of brevity, we will only present three versions of the code: one which counts elementary voting situations for n = 8 (Appendix IVa), one which counts elementary voting situations with no repeated ballots for n = 8 (Appendix IVb), and one which counts central voting situations for n = 8 (Appendix IVc). Equivalent versions of the code for other values of n may be obtained by modifying the conditions.

Appendix IVa: All elementary voting situations for n = 8.

```
#This program counts the total number of elementary voting situations for 8 voters.

import itertools
import operator

elementary = []
k = list(itertools.permutations(range(4)))
for index, x in enumerate(k):
    print(index, x)

def tuple_sum(k,*args):
    t_sum = (0,)*len(k[0])
    for a in args:
        t_sum = tuple(map(operator.add, t_sum, k[a]))
    return t_sum

def idx_sum_not(not_val, *args):

    for a in range(len(args)-1):
        for b in range(a+1, len(args)):
            if args[a]+args[b] == not_val:
                return False

    return True

for i in range(len(k)):
    for j in range(i,len(k)):
        for l in range(j,len(k)):
            for m in range(l,len(k)):
                for n in range (m,len(k)):
                    for o in range (n,len(k)):
                        for p in range (o,len(k)):
                            for q in range (p,len(k)):
                                if tuple_sum(k,i,j,l,m,n,o,p,q) == (12, 12, 12, 12)\
                and idx_sum_not(23, i,j,l,m,n,o,p,q)\ #do not count voting situations that contain reversals
                and set(tuple(itertools.permutations([i,j,l,m,n,o,p,q], 4))).isdisjoint(set(((5, 7, 14, 23),\
#do not count voting situations that countain 4-voter elementary voting situations
(4, 8, 13, 21),(4, 12, 13, 17),(0, 7, 17, 22),(1, 4, 18, 23),(0, 11, 14, 21),(4, 7, 17, 18),(3, 8, 13, 22),\
(3, 10, 12, 21),(2, 11, 13, 20),(7, 8, 9, 22),(7, 10, 11, 18),(2, 3, 18, 23),(1, 10, 15, 20),(5, 6, 16, 19),\
(1, 4, 20, 21),(2, 9, 12, 23),(1, 8, 17, 19),(6, 8, 11, 21),(2, 3, 19, 22),(0, 9, 16, 18),(0, 14, 15, 17),\
(0, 5, 19, 22),(1, 6, 16, 23),(5, 9, 12, 20),(2, 12, 15, 17),(4, 6, 15, 22),(3, 11, 14, 18),(7, 9, 10, 20),\
(0, 5, 20, 21),(6, 8, 9, 23),(2, 10, 15, 19),(3, 13, 14, 16),(5, 12, 13, 16),(1, 14, 15, 16),(6, 10, 11, 19)))):
                                    elementary.append((i,j,l,m,n,o,p,q))
elementary_unique = set(tuple(sorted(t)) for t in elementary)
print(elementary_unique)
print("The #of elementary voting situations for n = 8 is", len(elementary_unique))
```

Appendix IVb: All elementary voting situations with no repeated ballots for n = 8.

```
#This program counts the total number of elementary voting situations with no repeated ballots for 8 voters.

import itertools
import operator

elementary = []
k = list(itertools.permutations(range(4)))
for index, x in enumerate(k):
    print(index, x)

def tuple_sum(k,*args):
    t_sum = (0,)*len(k[0])
    for a in args:
        t_sum = tuple(map(operator.add, t_sum, k[a]))
    return t_sum

def idx_sum_not(not_val, *args):

    for a in range(len(args)-1):
        for b in range(a+1, len(args)):
            if args[a]+args[b] == not_val:
                return False

    return True

for i in range(len(k)):
    for j in range(i+1,len(k)):
        for l in range(j+1,len(k)):
            for m in range(l+1,len(k)):
                for n in range (m+1,len(k)):
                    for o in range (n+1,len(k)):
                        for p in range (o+1,len(k)):
                            for q in range (p+1,len(k)):
                                if tuple_sum(k,i,j,l,m,n,o,p,q)\
                    == (12, 12, 12, 12)\
                    and idx_sum_not(23, i,j,l,m,n,o,p,q)\ #do not count voting situations that contain reversals
                    and set(tuple(itertools.permutations([i,j,l,m,n,o,p,q], 4))).isdisjoint(set(((5, 7, 14, 23),\
#do not count voting situations that countain 4-voter elementary voting situations
(4, 8, 13, 21),(4, 12, 13, 17),(0, 7, 17, 22),(1, 4, 18, 23),(0, 11, 14, 21),(4, 7, 17, 18),(3, 8, 13, 22),\
(3, 10, 12, 21),(2, 11, 13, 20),(7, 8, 9, 22),(7, 10, 11, 18),(2, 3, 18, 23),(1, 10, 15, 20),(5, 6, 16, 19),\
(1, 4, 20, 21),(2, 9, 12, 23),(1, 8, 17, 19),(6, 8, 11, 21),(2, 3, 19, 22),(0, 9, 16, 18),(0, 14, 15, 17),\
(0, 5, 19, 22),(1, 6, 16, 23),(5, 9, 12, 20),(2, 12, 15, 17),(4, 6, 15, 22),(3, 11, 14, 18),(7, 9, 10, 20),\
(0, 5, 20, 21),(6, 8, 9, 23),(2, 10, 15, 19),(3, 13, 14, 16),(5, 12, 13, 16),(1, 14, 15, 16),(6, 10, 11, 19))))\
                    and i != j != l != m != n != o != p != q: #do not include repeated ballots
                                    elementary.append((i,j,l,m,n,o,p,q))
elementary_unique = set(tuple(sorted(t)) for t in elementary)
print(elementary_unique)
print("The # of elementary voting situations with no repeated ballots for n = 8 is", len(elementary_unique))
```

Appendix IVc: All central voting situations for n = 8.

```
#This program counts the total number of central voting situations for 8 voters.

import itertools
import operator

elementary = []
k = list(itertools.permutations(range(4)))
for index, x in enumerate(k):
    print(index, x)

def tuple_sum(k,*args):
    t_sum = (0,)*len(k[0])
    for a in args:
        t_sum = tuple(map(operator.add, t_sum, k[a]))
    return t_sum

def idx_sum_not(not_val, *args):

    for a in range(len(args)-1):
        for b in range(a+1, len(args)):
            if args[a]+args[b] == not_val:
                return False

    return True

for i in range(len(k)):
    for j in range(i,len(k)):
        for l in range(j,len(k)):
            for m in range(l,len(k)):
                for n in range (m,len(k)):
                    for o in range (n,len(k)):
                        for p in range (o,len(k)):
                            for q in range (p,len(k)):
                                if tuple_sum(k,i,j,l,m,n,o,p,q) == (12, 12, 12, 12):
                                            elementary.append((i,j,l,m,n,o,p,q))
elementary_unique = set(tuple(sorted(t)) for t in elementary)
print(elementary_unique)
print("The # of central voting situations for n = 8 is", len(elementary_unique))
```