ON CONFORMAL TRANSFORMATIONS OF RIEMANNIAN SPACES WITH RECURRENT CONFORMAL CURVATURE

BY

TETURO MIYAZAWA

0. Introduction. It is well known that a Riemannian space is called symmetric in the sense of Cartan or recurrent if the curvature tensor satisfies $R^h{}_{ijk,l}=0$ or $R^h{}_{ijk,l}=\kappa_l R^h{}_{ijk}$ respectively, where comma denotes covariant differentiation with respect to the metric tensor g_{ij} of the space and κ_l is a non-zero vector. In previous papers [1, 2][1], we studied Riemannian spaces V_n $(n>3)$ which satisfy

$$C^h{}_{ijk,l}=0 \tag{0.1}$$

or

$$C^h{}_{ijk,l}=\kappa_l C^h{}_{ijk} \tag{0.2}$$

respectively, where κ_l is a non-zero vector and $C^h{}_{ijk}$ is the conformal curvature tensor, that is,

$$C^h{}_{ijk}\equiv R^h{}_{ijk}-\frac{1}{n-2}(R^h{}_k g_{ij}-R^h{}_j g_{ik}+R_{ij}\delta^h_k-R_{ik}\delta^h_j) + \frac{R}{(n-1)(n-2)}(\delta^h_k g_{ij}-\delta^h_j g_{ik}). \tag{0.3}$$

A Riemannian space defined by (0.1) has been called conformally symmetric by M. C. Chaki and B. Gupta [3]. We have called a Riemannian space defined by (0.2) a conformally recurrent space. Evidently, a symmetric space in the sense of Cartan is a conformally symmetric space, and a recurrent space is a conformally recurrent space. For brevity, we denote by CS_n-space or CK_n-space a Riemannian space defined by (0.1) or (0.2) respectively.

In §1 of this paper, we shall study conformal transformations of the CK_n-spaces, and in §2 we shall discuss infinitesimal conformal transformations in a CK_n-space. Throughout the paper, we suppose that the metric of the space considered is positive definite.

The present author wishes to express his grateful thanks to Prof. T. Adati for his invaluable instructions and discussion opportunities.

1. Conformal transformations of CK_n-spaces. Let V_n^* and V^n be Riemannian spaces. If the metric tensor g_{ij}^* of V_n^* is given by

1) Numbers in brackets refer to the references at the end of the paper.

(1. 1) $$g^*_{ij}=e^{2\sigma}g_{ij},$$
where g_{ij} is the metric tensor of V_n, then V^*_n is said to be a conformal transformation of V_n. By the conformal transformation (1. 1), as is well known, we have

(1. 2) $$C^{*h}{}_{ijk}=C^h{}_{ijk},$$
where the symbol * denotes the quantities of V^*_n.

Differentiating (1. 2) covariantly and making use of the relation
$$\begin{Bmatrix} h \\ i\ j \end{Bmatrix}^* = \begin{Bmatrix} h \\ i\ j \end{Bmatrix} + \delta^h_i\sigma_j+\delta^h_j\sigma_i-\sigma^h g_{ij} \qquad (\sigma_i\equiv\sigma_{,i},\quad \sigma^h\equiv g^{hi}\sigma_i),$$
we get

(1. 3) $$C^{*h}{}_{ijk;l}=C^h{}_{ijk,l}-2C^h{}_{ijk}\sigma_l-(C_{lijk}\sigma^h+C^h{}_{ljk}\sigma_i+C^h{}_{ilk}\sigma_j+C^h{}_{ijl}\sigma_k) +\sigma^a(\delta^h_l C_{aijk}+g_{il}C^h{}_{ajk}+g_{lj}C^h{}_{iak}+g_{kl}C^h{}_{ija}),$$
where semi-colon denotes covariant differentiation with respect to g^*_{ij}.

Now, we assume that both V_n and V^*_n are CK_n-spaces, then

(1. 4) $$C^h{}_{ijk,l}=\kappa_l C^h{}_{ijk},$$

(1. 5) $$C^{*h}{}_{ijk;l}=\kappa^*_l C^{*h}{}_{ijk}$$
for non-zero vectors κ_l and κ^*_l.

Substituting (1. 4) and (1. 5) in (1. 3) and using (1. 2), we have

(1. 6) $$(\kappa^*_l-\kappa_l)C^h{}_{ijk}=-2C^h{}_{ijk}\sigma_l-(C_{lijk}\sigma^h+C^h{}_{ljk}\sigma_i+C^h{}_{ilk}\sigma_j+C^h{}_{ijl}\sigma_k) +\sigma^a(\delta^h_l C_{aijk}+g_{il}C^h{}_{ajk}+g_{lj}C^h{}_{iak}+g_{kl}C^h{}_{ija}).$$
Contraction with respect to h and l in (1. 6) gives

(1. 7) $$(\kappa^*_a-\kappa_a)C^a{}_{ijk}=(n-3)\sigma_a C^a{}_{ijk}$$
by virtue of
$$C^a{}_{ajk}=C^a{}_{iak}=C^a{}_{ija}=0 \quad\text{and}\quad C^h{}_{ijk}+C^h{}_{jki}+C^h{}_{kij}=0.$$
Transvecting (1. 7) with σ^i, we get

(1. 8) $$(\kappa^*_a-\kappa_a)C^a{}_{bjk}\sigma^b=0.$$

On the other hand, transvection (1. 6) with σ^l gives
$$(\kappa^*_a-\kappa_a+2\sigma_a)\sigma^a C^h{}_{ijk}=0.$$
Hence we find either

(1. 9) $$C^h{}_{ijk}=0$$
or

(1. 10) $$(\kappa^*_a-\kappa_a)\sigma^a=-2\sigma_a\sigma^a.$$

We consider the case when (1. 10) holds good. Transvecting (1. 6) with $(\kappa^*_h-\kappa_h)\sigma^i$, we have
$$(\kappa^*_l-\kappa_l)(\kappa^*_h-\kappa_h)C^h{}_{ijk}\sigma^i=-2(\kappa^*_h-\kappa_h)C^h{}_{ijk}\sigma^i\sigma_l-(\kappa^*_h-\kappa_h)\sigma^h C_{lijk}\sigma^i$$
$$-(\kappa^*_h-\kappa_h)C^h{}_{ljk}\sigma_i\sigma^i-(\kappa^*_h-\kappa_h)C^h{}_{ilk}\sigma^i\sigma_j-(\kappa^*_h-\kappa_h)C^h{}_{ijl}\sigma^i\sigma_k$$
$$+(\kappa^*_h-\kappa_h)C^h{}_{ajk}\sigma^a\sigma_l+(\kappa^*_h-\kappa_h)C^h{}_{iak}\sigma^i\sigma^a g_{lj}+(\kappa^*_h-\kappa_h)C^h{}_{ija}\sigma^a\sigma^i g_{kl}.$$
Substituting (1. 7), (1. 8) and (1. 10) in this equation, we get
$$\sigma_a\sigma^a\sigma_b C^b{}_{ljk}=0.$$

Hence we find either

(1.11) $$\sigma_a\sigma^a=0,\ \text{that is,}\ \sigma=\text{constant}$$

or

(1.12) $$\sigma_b C^b{}_{ljk}=0.$$

When (1.12) holds good, (1.6) becomes

$$(\kappa_l^*-\kappa_l)C^h{}_{ijk}=-2C^h{}_{ijk}\sigma_l-C_{lijk}\sigma^h-C^h{}_{ljk}\sigma_i-C^h{}_{ilk}\sigma_j-C^h{}_{ijl}\sigma_k.$$

Transvecting this equation with σ_h and using (1.12), we have

$$\sigma_a\sigma^a C_{lijk}=0.$$

Hence we find either (1.9) or (1.11). In the case (1.11), the equation (1.6) becomes

$$(\kappa_l^*-\kappa_l)C^h{}_{ijk}=0,$$

from which follows (1.9) or

$$\kappa_l^*=\kappa_l.$$

Thus we have

Theorem 1.1. If a CK_n-space is transformed into another CK_n-space by a conformal transformation (1.1), then the following cases occur:

(1) the space is conformally flat,

(2) σ=constant and the recurrence vectors coincide.

Since a recurrent space is a CK_n-space, from this theorem we have the following

Corollary. If a Riemannian space is transformed into another Riemannian space by a conformal transformation (1.1) as follows:

a CK_n-space $\longrightarrow$ a recurrent space,

or

a recurrent space $\longrightarrow$ a CK_n-space or a recurrent space,

then the space is conformally flat or σ=const. and the recurrence vectors coincide.

Now, if σ=constant and the space is not conformally flat, then (1.3) can be written as

$$C^{*h}{}_{ijk;l}=C^h{}_{ijk,l}.$$

Consequently, a CK_n-space may be transformed into a CK_n-space by a conformal transformation (1.1).

Thus, considering Theorem 1.1, we have

Theorem 1.2. In order that a CK_n-space which is not conformally flat is transformed into another CK_n-space by a conformal transformation (1.1), it is necessary and sufficient that σ in (1.1) is constant.

Next, we assume that V_n is a CK_n-space and V_n^* is a CS_n-space. Then, regarding κ_l^* as zero identical in the proof of Theorem 1.1, we find either $C^h{}_{ijk}=0$ or σ=constant and $\kappa_l=\kappa_l^*=0$. However, since κ_l is a non-zero vector, the space must be conformally flat.

Hence, we have

Theorem 1.3. If a CK_n-space is transformed into a CS_n-space or a CS_n-space is transformed into a CK_n-space by a conformal transformation, then the space is conformally flat.

Since a symmetric space in the sense of Cartan is a CS_n-space, from this theorem we have the following

Corollary. If a Riemannian space is transformed into another Riemannian space by a conformal transformation as follows:

a CK_n-space $\longrightarrow$ a symmetric space,

a CS_n-space $\longrightarrow$ a recurrent space,

a recurrent space $\longrightarrow$ a CS_n-space or a symmetric space,

or

a symmetric space $\longrightarrow$ a CK_n-space or a recurrent space,

then the space must be conformally flat.

If V_n and V_n^* are both CS_n-spaces, then regarding κ_l and κ_l^* as both zero identical in the proof of Theorem 1.1, we find either $C^h{}_{ijk}=0$ or σ=constant. Thus we have

Theorem 1.4. If a CS_n-space is transformed into another CS_n-space by a conformal transformation (1.1), then the space is conformally flat or σ=constant.

Corollary. If a Riemannian space is transformed into another Riemannian space by a conformal transformation (1.1) as follows:

a CS_n-space $\longrightarrow$ a symmetric space,

or

a symmetric space $\longrightarrow$ a CS_n-space or a symmetric space,

then the space is conformally flat or σ=constant.

Theorem 1.5. In order that a CS_n-space which is not conformally flat is transformed into another CS_n-space by a conformal transformation (1.1), it is necessary and sufficient that σ in (1.1) is constant [1].

2. Infinitesimal conformal transformations in CK_n-spaces. Let us suppose that a Riemannian space V_n admits an infinitesimal conformal transformation defined by a vector field v^i. Then, denoting by $\pounds$ the Lie derivative with respect to the field v^i, we have [4]:

(2.1) $$\pounds g_{ij}=2\varphi g_{ij},$$

(2.2) $$\pounds\left\{ {h \atop i\ j} \right\}=\delta_i^h\varphi_j+\delta_j^h\varphi_i-\varphi^h g_{ij}\quad (\varphi_i\equiv\varphi_{,i},\quad \varphi^h\equiv g^{hi}\varphi_i),$$

(2.3) $$\pounds R^h{}_{ijk}=\delta_j^h\varphi_{i,k}-\delta_k^h\varphi_{i,j}+\varphi^h{}_{,j}g_{ik}-\varphi^h{}_{,k}g_{ij},$$

(2.4) $$\pounds C_{ij}=\varphi_{i,j}\quad \left(C_{ij}\equiv-\frac{1}{n-2}R_{ij}+\frac{R}{2(n-1)(n-2)}g_{ij}\right),$$

(2.5) $$\pounds C^h{}_{ijk}=0.$$

Prof. T. Adati and Mr. S. Yamaguchi [5] studied infinitesimal conformal transformations in a recurrent space. Their proof for a theorem in their paper [5] suggests that the following Theorem 2.1 may be also true. So that we are greatly indebted to them. However, we shall prove our theorem by a more simple method. It is an expansion of their theorem.

Theorem 2.1. If a CK_n-space admits an infinitesimal conformal transformation, then the space is conformally flat or the transformation is homothetic.

PROOF. Let V_n be a CK_n-space. Then, since

(2.6) $$C^h{}_{ijk,l}=\kappa_l C^h{}_{ijk}$$

for a non-zero vector κ_l, we have from (2.5)

(2.7) $$\pounds C^h{}_{ijk,l}=(\pounds\kappa_l)C^h{}_{ijk}.$$

Substituting (2.2), (2.5) and (2.7) in the identity [4]

$$\pounds C^h{}_{ijk,l}-(\pounds C^h{}_{ijk})_{,l}=C^a{}_{ijk}\pounds\left\{{h \atop l\ a}\right\}-C^h{}_{ajk}\pounds\left\{{a \atop l\ i}\right\}-C^h{}_{iak}\pounds\left\{{a \atop l\ j}\right\}-C^h{}_{ija}\pounds\left\{{a \atop l\ k}\right\},$$

we have

(2.8) $$C^h{}_{ijk}\pounds\kappa_l=-2C^h{}_{ijk}\varphi_l-(C_{lijk}\varphi^h+C^h{}_{ljk}\varphi_i+C^h{}_{ilk}\varphi_j+C^h{}_{ijl}\varphi_k)$$
$$+\varphi^a(\delta^h_l C_{aijk}+g_{il}C^h{}_{ajk}+g_{lj}C^h{}_{iak}+g_{kl}C^h{}_{ija}).$$

Contraction with respect to h and l in (2.8) gives

(2.9) $$C^a{}_{ijk}\pounds\kappa_a=(n-3)\varphi_a C^a{}_{ijk},$$

and consequently, by transvection with φ^i we have

(2.10) $$\varphi^i C^a{}_{ijk}\pounds\kappa_a=0.$$

On the other hand, transvecting (2.8) with φ^l, we get

$$C^h{}_{ijk}\varphi^l\pounds\kappa_l=-2\varphi_a\varphi^a C^h{}_{ijk}.$$

Hence we find either

(2.11) $$C^h{}_{ijk}=0$$

or

(2.12) $$\varphi^a\pounds\kappa_a=-2\varphi_a\varphi^a.$$

We consider the case when (2.12) holds good. Transvecting (2.8) with $\varphi^i\pounds\kappa_h$ and making use of (2.9), (2.10) and (2.12), we have

$$\varphi_a C^a{}_{ljk}\varphi_b\varphi^b=0.$$

Hence we find either

(2.13) $$\varphi_a\varphi^a=0, \text{ that is, } \varphi=\text{constant}$$

or

(2.14) $$\varphi_a C^a{}_{ljk}=0.$$

In the case (2.14), (2.8) becomes

$$C^h{}_{ijk}\pounds\kappa_l=-2C^h{}_{ijk}\varphi_l-(C_{lijk}\varphi^h+C^h{}_{ljk}\varphi_i+C^h{}_{ilk}\varphi_j+C^h{}_{ijl}\varphi_k).$$

Transvecting this equation with φ_h and using (2.14), we get

$$C_{lijk}\varphi_a\varphi^a=0.$$

Hence we find either (2.11) or (2.13). Q.E.D.

In the case when κ_l in (2.6) is equal to zero identically, that is, in the case when the space is a CS_n-space, we find that the above proof also holds good. Hence we have

Theorem 2.2. If a CS_n-space admits an infinitesimal conformal transformation, then the space is conformally flat or the transformation is homothetic.

The infinitesimal homothetic transformation in a compact Riemannian space is always a motion [6]. Hence we have the following theorems:

Theorem 2.3. If a compact CK_n-space admits an infinitesimal conformal transformation, then the space is conformally flat or the transformation is a motion.

Theorem 2.4. If a compact CS_n-space admits an infinitesimal conformal transformation, then the space is conformally flat or the transformation is a motion.

Furthermore, since a recurrent space is a CK_n-space and a symmetric space is a CS_n-space, when the space is recurrent or symmetric, we can obtain similar theorems in that case.

REFERENCES

[1] T. Adati and T. Miyazawa: On conformally symmetric spaces, Tensor, N.S., 18 (1967), 335–342.

[2] T. Adati and T. Miyazawa: On a Riemannian space with recurrent conformal curvature, Tensor, N.S., 18 (1967), 348–354.

[3] M. C. Chaki and B. Gupta: On conformally symmetric spaces, Indian J. Math., 5 (1963), 113–122.

[4] K. Yano: The theory of Lie derivatives and its applications, North-Holland Publishing Co., Amsterdam, (1957).

[5] T. Adati and S. Yamaguchi: On some transformations in Riemannian recurrent spaces, TRU Math., 3 (1967), 9–12.

[6] K. Yano and S. Bochner: Curvature and Betti numbers, Princeton Univ. Press, (1953).

[7] T. Miyazawa and S. Yamaguchi: Some theorems on K-contact metric manifolds and Sasakian manifolds, TRU Math., 2 (1966), 46–52.

[8] T. Miyazawa: On Riemannian spaces admitting some recurrent tensors, Toyo Univ. Eng. Dep't Research Reports, 3 (1967), 5–13.

FACULTY OF ENGINEERING,
TOYO UNIVERSITY