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SOME PROPERTIES OF P-SASAKIAN MANIFOLDS

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Recently, I. Sato [4] defined the notion of (ϕ, ξ, η) structure of a differentiable manifold satisfying $\phi^2 = I - \eta \otimes \xi$ and $\eta(\xi) = 1$, where ϕ is a (1, 1)-tensor, ξ a vector field and η a 1-form on the manifold, and he called the manifold with such a structure an almost paracontact manifold and studied several properties of the manifold by an analogous manner to the case of an almost contact manifold [3]. Furthermore, he and K. Matsumoto defined and studied a P-Sasakian manifold and an SP-Sasakian manifold which are considered as special cases of an almost paracontact manifold and obtained several interesting results [5].

In this paper, we shall study P-Sasakian manifolds. We shall devote §1 to preliminaries. In §2, we shall give several theorems on P- and SP-Sasakian manifolds. We shall discuss in §3 conformally flat P-Sasakian Manifolds.

§1. PRELIMINARIES

Let M be a differentiable manifold of dimension n. If there exist in the M a mixed tensor field $\phi_i{}^h$, a contravariant vector field ξ^h and a covariant vector field η_i satisfying $\eta_a \xi^a = 1$, $\phi_i{}^a \phi_a{}^h = \delta_i^h - \eta_i \xi^h$, where Latin indices take values $1, 2, \cdots, n$, then such a manifold is said to have an almost paracontact structure $(\phi_i{}^h, \xi^h, \eta_i)$ and the manifold with such an almost paracontact structure is called an almost paracontact manifold.

It is showed that in an almost paracontact manifold there exists a positive definite Riemannian metric g_{ji}, which is called an associated Riemannian metric with the almost paracontact structure, such that $\eta_i = g_{ia}\xi^a$, $g_{ba}\phi_j{}^b\phi_i{}^a = g_{ji} - \eta_j\eta_i$. The set $(\phi_i{}^h, \xi^h, \eta_i, g_{ji})$ is called an almost paracontact Riemannian structure and the manifold with such an almost paracontact Riemannian structure is said to be an almost paracontact Riemannian manifold [4].

In an almost paracontact Riemannian manifold, the following relations hold good:

$$\eta_a\xi^a = 1, \quad \phi_i^{\ a}\eta_a = 0, \quad \phi_a^{\ h}\xi^a = 0,$$

(1.1)
$$\phi_i^{\ a}\phi_a^{\ h} = \delta_i^h - \eta_i\xi^h,$$
$$rank\ (\phi_i^{\ h}) = n-1 \ .$$

Moreover, if we define ϕ_{ji} by $\phi_{ji}=g_{ia}\phi_j^{\ a}$, then in addition to the above relations the followings are satisfied

(1.2)
$$\phi_{ji} = \phi_{ij} \quad ,$$
$$g_{ba}\phi_j^{\ b}\phi_i^{\ a} = g_{ji} - \eta_j\eta_i \quad .$$

Now, we consider an n-dimensional differentiable manifold with a positive definite metric g_{ji} which admits a unit covariant vector field η_i satisfying

(1.3)
$$\nabla_j\eta_i - \nabla_i\eta_j = 0 \quad ,$$
$$\nabla_k\nabla_j\eta_i = (-g_{ki} + \eta_k\eta_i)\eta_j + (-g_{kj} + \eta_k\eta_j)\eta_i,$$

where ∇_j denotes covariant differentiation with respect to the metric tensor g_{ji}. Furthermore, if we put

(1.4)
$$\xi^h = g^{ha}\eta_a, \quad \phi_i^{\ h} = \nabla_i\xi^h,$$

then it is easily verified that the manifold in consideration becomes an almost paracontact Riemannian manifold. Such a manifold is called a P-Sasakian manifold [5].

In a P-Sasakian manifold, the following relations hold good:

(1.5)
$$R_{kji}^{\ \ \ a}\eta_a = g_{ki}\eta_j - g_{ji}\eta_k, \quad R_i^{\ a}\eta_a = -(n-1)\eta_i \ ,$$

where $R_{kji}^{\ \ \ h}$ and R_{ji} are the curvature tensor and the Ricci tensor respectively,

$$R_{lkj}^{\ \ \ a}\phi_{ai} + R_{lki}^{\ \ \ a}\phi_{ja} = \phi_{lj}(g_{ki} - 2\eta_k\eta_i) + \phi_{li}(g_{kj} - 2\eta_k\eta_j)$$
$$- \phi_{kj}(g_{li} - 2\eta_l\eta_i) - \phi_{ki}(g_{lj} - 2\eta_l\eta_j),$$

(1.6)
$$R_{kjih} - R_{kjba}\phi_i^{\ b}\phi_h^{\ a} = (g_{ji}g_{kh} - g_{ki}g_{jh}) - (\phi_{ji}\phi_{kh} - \phi_{ki}\phi_{jh})$$
$$-2(g_{kh}\eta_j\eta_i - g_{jh}\eta_k\eta_i + g_{ji}\eta_k\eta_h - g_{ki}\eta_j\eta_h),$$

$$R_{ji} - R_{jcba}\phi^{cb}\phi_i^{\ a} = (n-2)g_{ji} - \phi\phi_{ji} - (2n-3)\eta_j\eta_i,$$

where we have put $\phi^{ji} = g^{ja}\phi_a^{\ i}$, $\phi = g^{ba}\phi_{ba}$.

Let us consider an n-dimensional differentiable manifold with a positive definite metric g_{ji} which admits a unit covariant vector field η_i satisfying

(1.7)
$$\nabla_j\eta_i = -g_{ji} + \eta_j\eta_i \quad ,$$

then we can easily show by putting $\xi^h = g^{ha}\eta_a$ and $\phi_{ji} = \nabla_j\eta_i$ that the manifold in consideration is a P-Sasakian manifold. Such a manifold is called an SP-Sasakian manifold [5].

Let P be a point of M and M_P be the tangent space of M at P. In the M_P, the set of vector v such that $\eta(v)=0$ spans an $(n-1)$-dimensional subspace V_P of M_P. When $\phi_{ji} = \nabla_j\eta_i = g_{ji} - \eta_j\eta_i$, $\phi v = v$ $(v \in V_P)$. Therefore we only deal with the case of (1.7).

§2. SOME THEOREMS ON P- AND SP-SASAKIAN MANIFOLDS

I. Sato [4] introduced four tensor fields $N_{ji}{}^h$, N_{ji}, $N_i{}^h$ and N_i in an almost paracontact Riemannian manifold with local coordinate $\{x^h\}$ as follows:

$$
\begin{aligned}
&N_{ji}{}^h = \phi_{ji}{}^h - (\partial_j\eta_i - \partial_i\eta_j)\xi^h,\\
(2.1)\qquad &N_{ji} = \phi_j{}^a(\partial_a\eta_i - \partial_i\eta_a) - \phi_i{}^a(\partial_a\eta_j - \partial_j\eta_a),\\
&N_i{}^h = \mathfrak{L}(\xi)\phi_i{}^h, \quad N_i = \mathfrak{L}(\xi)\eta_i,
\end{aligned}
$$

where $\partial_j = \frac{\partial}{\partial x^j}$, $\phi_{ji}{}^h$ is the Nijenhuis tensor of $\phi_i{}^h$ and $\mathfrak{L}(\xi)$ means the Lie derivative with respect to the vector field ξ^h. $N_{ji}{}^h$ is so-called torsion tensor field of the almost paracontact structure. Concerning these four tensor fields, he proved the following

LEMMA 2.1 [4]. *If any one of N_{ji} and $N_{ji}{}^h$ vanishes, then N_i vanish. If $N_{ji}{}^h$ vanishes, then all the other tensors N_i, N_{ji} and $N_i{}^h$ vanish.*

The torsion tensor $N_{ji}{}^h$ can be written in an almost paracontact manifold, by a straightfoward culculation, in the form

$$
(2.2)\qquad N_{ji}{}^h = \phi_j{}^a(\nabla_a\phi_i{}^h - \nabla_i\phi_a{}^h) - \phi_i{}^a(\nabla_a\phi_j{}^h - \nabla_j\phi_a{}^h) + \eta_i(\nabla_j\xi^h) - \eta_j(\nabla_i\xi^h).
$$

Now we assume that the manifold is a P-Sasakian one. Substituting (1.3) and (1.4) into (2.2) and making use of (1.1) and (1.2), we find

$$N_{ji}{}^h = 0 \ .$$

Thus we have

THEOREM 2.1. *A P-Sasakian manifold has the vanishing torsion tensor $N_{ji}{}^h$.*

Let M be an n-dimensional differentiable manifold with an almost paracontact structure (ϕ, ξ, η) and R be a real line. We construct a product manifold $M \times R$. If we denote the tangent space of $M \times R$ at a point (P, Q), $(P \in M,\ Q \in R)$ by T, then the tangent space M_P of M at P may be naturally identified with a subspace of T.

Now, denoting the unit vector of R by ζ, we define a linear map $F: T \to T$ by

$$(2.3) \qquad \begin{aligned} &F(X) = \phi X, \quad \text{if} \quad X \in M_p, \quad \eta(X) = 0 \ , \\ &F(\xi) = \zeta \ , \quad F(\zeta) = \xi, \end{aligned}$$

then we can easily see that $F^2(X) = X$, $F \neq I$ hold good for any vector X of T. Therefore, F gives an almost product structure on T. As $P \in M$ and $Q \in R$ are arbitrary we see that an almost product structure F can be defined over $M\times R$ by means of the almost paracontact structure (ϕ, ξ, η).

About the integrability of such a structure F, I. Sato proved the following

LEMMA 2.2[4]. *Let M be a differentiable manifold with an almost paracontact structure (ϕ, ξ, η). Then, the almost product structure F over $M\times R$ defined by (2.3) is completely integrable if and only if $N_{ji}{}^h=0$ holds good over the whole M.*

Thus, from Theorem 2.1 and above Lemma 2.2 we have

THEOREM 2.2 *Let M be a P-Sasakian manifold with the structure (ϕ, ξ, η). Then, the almost product structure F over $M\times R$ defined by (2.3) is completely integrable.*

We assume that a P-Sasakian manifold has the vanishing Ricci curvature tensor R_{ji}. Then from (1.5) we have

$$-(n-1)\eta_i = 0,$$

which is inconsistent with such an assumption that the vector η_i is a unit vector. Thus we have

THEOREM 2.3 *In a P-Sasakian manifold, the Ricci curvature tensor can not vanish. Especially, a P-Sasakian manifold can not be flat.*

We assume that a P-Sasakian manifold is an Einstein one, then we have

$$(2.4) \qquad R_{ji} = \frac{R}{n} g_{ji} \ .$$

Substituting (2.4) into (1.5), we find

$$R = -n(n-1).$$

Thus we have

LEMMA 2.3. *If a P-Sasakian manifold is an Einstein one, the scalar curvature* *has a negative constant value* $-n(n-1)$. *Especially, if a P-Sasakian manifold is of constant curvature, the scalar curvature has a negative constant value* $-n(n-1)$.

If we assume that a P-Sasakian manifold is of constant curvature, then from Lemma 2.3 we have

(2.5) $\quad R_{kjih} = -(g_{ji}g_{kh} - g_{ki}g_{jh}),$

from which follows (2.4).

Substituting (2.4) and (2.5) into $(1.6)_3$, we get on account of (1.2)

$$\phi\phi_{ji} = (n-1)(g_{ji} - \eta_j\eta_i).$$

Contraction above equation with g^{ji} gives

$$\phi^2 = (n-1)^2 \quad .$$

Hence we find

$$\phi_{ji} = -g_{ji} + \eta_j\eta_i,$$

that is, the manifold is an SP-Sasakian one.

Thus we have

THEOREM 2.4. *If a P-Sasakian manifold is of constant curvature, the manifold is an SP-Sasakian one.*

The equations $(1.2)_1$ and $(1.4)_2$ show that η_i is a gradient vector of a scalar $\eta = \eta(x)$, that is to say,

(2.6) $\quad \eta_i = \dfrac{\partial\eta}{\partial x^i} \quad .$

Thus, in a P-Sasakian manifold there exists a family of hypersurfaces $\eta(x^1, x^2, \cdots, x^n)$ = constant to which the vector ξ^h is normal. On the other hand, from $(1.1)_1$ and $(1.4)_2$ we have

$$\xi^a\nabla_a\xi^h = 0 \quad ,$$

from which we find that the curves generated by ξ^h are all geodesics.

Thus we have

THEOREM 2.5. *A P-Sasakian manifold contains a family of hypersurfaces* $\eta(x)$ = *constant satisfying* (2.6) *whose orthogonal trajectories are geodesics.*

Contracting $(1.3)_2$ with g^{ji}, we get

$$\nabla_k\phi = 0,$$

from which follows ϕ= constant. Hence we have

LEMMA 2.4. *In a P-Sasakian manifold, ϕ is a constant.*

We shall represent one of hypersurfaces $\eta(x^1, x^2, \cdots, x^n)$ = constant appeared in the Theorem 2.5 by parametric equations

$$x^i = x^i(u^\lambda) \quad ,$$

where Greek index takes values $1, 2, \cdots, n-1$, then we have

$$(2.7) \qquad \eta_a B_\lambda{}^a = 0 \quad ,$$

where $B_\lambda{}^a = \dfrac{\partial x^a}{\partial u^\lambda}$. Induced Riemannian metric $g_{\mu\lambda}$ in the hypersurface is given by

$$g_{\mu\lambda} = B_\mu{}^b B_\lambda{}^a g_{ba} \quad .$$

Remember the following formula

$$(2.8) \qquad \nabla_\mu B_\lambda{}^h = H_{\mu\lambda}\xi^h .$$

The left hand side of this equation is so-called Bortolotti-van der Waerden covariant derivatives and $H_{\mu\lambda}$ is the second fundamental tensor of the hypersurface.

From (2.7) we have

$$B_\mu{}^b B_\lambda{}^a \nabla_b \eta_a + \eta_a \nabla_\mu B_\lambda{}^a = 0.$$

Substituting (1.4) and (2.8) into above equation, we get

$$B_\mu{}^b B_\lambda{}^a \phi_{ba} + H_{\mu\lambda} = 0.$$

Contracting above equation with $g^{\mu\lambda}$ and using (1.1), we obtain

$$\phi + H = 0,$$

where $H = g^{\beta\alpha} H_{\beta\alpha}$, $\frac{1}{n-1}|H|$ is so-called mean curvature of the hypersurface.

Thus from the Lemma 2.4 we have

THEOREM 2.6. *In a P-Sasakian manifold, the mean curvature of a hypersurface $\eta(x^1, x^2, \cdots, x^n)$= constant satisfying (2.6) is a constant.*

We shall prove the following

THEOREM 2.7. *A P-Sasakian manifold is an SP-Sasakian one if and only if the following relation holds good*

$$(2.9) \qquad \phi = -(n-1) \quad .$$

PROOF. In a P-Sasakian manifold we have an identical equation

$$\{\phi_{ba}-(-g_{ba}+\eta_b\eta_a)\}\{\phi^{ba}-(-g^{ba}+\xi^b\xi^a)\}=2\{\phi+(n-1)\}.$$

Hence, since the manifold has a positive definite metric, (2.9) is equivalent to

$$\phi_{ji}=-g_{ji}+\eta_j\eta_i,$$

which is the condition for the manifold to be SP-Sasakian. Q.E.D.

From Theorems 2.6 and 2.7 we have

THEOREM 2.8. *In a P-Sasakian manifold, let m be the mean curvature of a hypersurface η=constant satisfying (2.6), then we have*

$$0 \leqq m \leqq 1 \quad .$$

And therefore, for all hypersurfaces η =constant, when $m = 0$, ξ^h is harmonic, and when m=1, the manifold is SP-Sasakian.

§3. CONFORMALLY FLAT P-SASAKIAN MANIFOLDS

If the Ricci tensor R_{ji} of a P-Sasakian manifold satisfies the relation

$$R_{ji} = ag_{ji} + b\eta_j\eta_i,$$

where a and b are certain scalars which are said the associated functions of R_{ji}, then the manifold is called an η-Einstein one [5].

We shall start the following

Lemma 3.1. *If a P-Sasakian manifold is conformally flat, the manifold is an η-Einstein one.*

PROOF. We assume that a P-Sasakian manifold is conformally flat, then we have

$$R_{kji}{}^h = \frac{1}{n-2}(g_{ji}R_k{}^h-g_{ki}R_j{}^h+R_{ji}\delta_k^h-R_{ki}\delta_j^h)-\frac{R}{(n-1)(n-2)}(g_{ji}\delta_k^h-g_{ki}\delta_j^h). \tag{3.1}$$

Transvecting (3.1) with η_h and making use of (1.5), we have

$$g_{ki}\eta_j-g_{ji}\eta_k= -\frac{n-1}{n-2}(g_{ji}\eta_k-g_{ki}\eta_j)+\frac{1}{n-2}(R_{ji}\eta_k-R_{ki}\eta_j)$$
$$-\frac{R}{(n-1)(n-2)}(g_{ji}\eta_k-g_{ki}\eta_j),$$

that is,

$$R_{ji}\eta_k-R_{ki}\eta_j= (\frac{R}{n-1} + 1)(g_{ji}\eta_k-g_{ki}\eta_j).$$

Furthermore, transvecting above equation with ξ^k and using (1.1) and (1.5), we find

$$R_{ji}= (\frac{R}{n-1} + 1)g_{ji}-(\frac{R}{n-1} + n)\eta_j\eta_i. \tag{3.2}$$

Q.E.D.

Now, we shall prove the next

THEOREM 3.1. *If a P-Sasakian manifold is conformally flat, then the manifold becomes an SP-Sasakian one and the curvature tensor of the manifold is given by*

$$(3.3)\quad R_{kjih} = \frac{1}{n-2}\left(\frac{R}{n-1} + 2\right)(g_{ji}g_{kh} - g_{ki}g_{jh})$$
$$-\frac{1}{n-2}\left(\frac{R}{n-1} + n\right)(g_{kh}\eta_j\eta_i - g_{jh}\eta_k\eta_i + g_{ji}\eta_k\eta_h - g_{ki}\eta_j\eta_h).$$

Especially, when $R = -n(n-1)$, *the manifold is of constant curvature* -1.

PROOF. We assume that a P-Sasakian manifold is conformally flat, then from the Lemma 3.1 the manifold is an η-Einstein one. So, substituting (3.2) into (3.1), we obtain (3.3).

Differentiating (3.2) covariantly and making use of (1.4), we have

$$\nabla_k R_{ji} = \frac{1}{n-1} g_{ji}\nabla_k R - \frac{1}{n-1}\eta_j\eta_i\nabla_k R - \left(\frac{R}{n-1} + n\right)(\phi_{kj}\eta_i + \phi_{ki}\eta_j),$$
$$\nabla_j R_{ki} = \frac{1}{n-1} g_{ki}\nabla_j R - \frac{1}{n-1}\eta_k\eta_i\nabla_j R - \left(\frac{R}{n-1} + n\right)(\phi_{jk}\eta_i + \phi_{ji}\eta_k),$$

from which follows

$$(3.4)\quad \nabla_k R_{ji} - \nabla_j R_{ki} = \frac{1}{n-1}(g_{ji}\nabla_k R - g_{ki}\nabla_j R) - \frac{1}{n-1}(\eta_j\eta_i\nabla_k R - \eta_k\eta_i\nabla_j R)$$
$$-\left(\frac{R}{n-1} + n\right)(\phi_{ki}\eta_j - \phi_{ji}\eta_k).$$

On the other hand, since the conformal curvature tensor $C_{kji}{}^h$ vanishes we have

$$\nabla_l C_{kji}{}^h = 0 \quad ,$$

from which follows

$$\nabla_a C_{kji}{}^a = 0 \quad ,$$

that is, when $n > 3$,

$$(3.5)\quad \nabla_k R_{ji} - \nabla_j R_{ki} = \frac{1}{2(n-1)}(g_{ji}\nabla_k R - g_{ki}\nabla_j R).$$

When $n=3$, the equation (3.5) is the condition for the manifold to be conformally flat.

It follows from (3.4) and (3.5) that

$$(3.6)\quad \frac{1}{2}(g_{ji}\nabla_k R - g_{ki}\nabla_j R) = (\eta_j\eta_i\nabla_k R - \eta_k\eta_i\nabla_j R) + \{R+n(n-1)\}(\phi_{ki}\eta_j - \phi_{ji}\eta_k).$$

Transvecting (3.6) with $\xi^k\xi^i$ and using (1.1), we have

$$(3.7)\quad \nabla_j R = \eta_j\xi^a\nabla_a R.$$

Contraction (3.6) with g^{ji} gives by virtue of (1.1) and (3.7)

$$(3.8)\quad \nabla_k R = -2\left(\frac{R}{n-1} + n\right)\phi\eta_k \ .$$

Substituting (3.8) into (3.6), we get

$$-\left(\frac{R}{n-1} + n\right)\phi(g_{ji}\eta_k - g_{ki}\eta_j) = \{R + n(n-1)\}(\phi_{ki}\eta_j - \phi_{ji}\eta_k),$$

from which follows by contraction with ξ^j

$$\{R + n(n-1)\}\{\phi_{ki} - \frac{\phi}{n-1}(g_{ki} - \eta_k\eta_i)\} = 0.$$

Thus, we find either

(3.9) $R = -n(n-1)$

or

(3.10) $\phi_{ki} = \frac{\phi}{n-1}(g_{ki} - \eta_k\eta_i)$.

First, we consider the case of (3.9). Substituting (3.9) into (3.3), we get

(3.11) $R_{kjih} = -(g_{ji}g_{kh} - g_{ki}g_{jh})$,

that is, the manifold is of constant curvature −1. And therefore, we find from the Theorem 2.4 that the manifold is an *SP*-Sasakian one.

Second, we consider the case of (3.10). Contracting (3.10) with $\phi_j^{\ k}$ and making use of (1.1) and (1.2), we get

(3.12) $g_{ji} - \eta_j\eta_i = \frac{\phi}{n-1}\phi_{ji}$.

It follows from (3.10) and (3.12) that

$$\{\phi^2 - (n-1)^2\}(g_{ji} - \eta_j\eta_i) = 0,$$

from which

$$\phi = -(n-1).$$

Thus we find

$$\phi_{ji} = -g_{ji} + \eta_j\eta_i,$$

which is the condition for the manifold to be *SP*-Sasakian. Q.E.D.

REMARK. If we make use of (1.6) instead of (3.5) in the proof of the Theorem 3.1, after all we find either

(3.13) $R = (n-1)(n-4)$

or the manifold is an *SP*-Sasakian manifold. However, we can show that the case of (3.13) does not occur as follows:

Substituting (3.13) into (3.6), we have

$$\phi_{ki}\eta_j - \phi_{ji}\eta_k = 0 \quad .$$

Contracting above equation with ϕ^{ki} and using (1.1), we get

$$(n-1)\eta_j = 0 \quad ,$$

which is inconsistent with our assumption that the vector η_j is a unit vector. This gives another proof of the Theorem 3.1.

In a conformally flat *P*-Sasakian manifold, we find from (3.8) that the scalar curvature R is a function of η alone where $\eta_i = \frac{\partial\eta}{\partial x^i}$. Therefore, the scalar curvature R is a constant along the hypersurface $\eta(x)$ =constant. Concerning the case when the scalar curvature is a constant in the whole manifold, we have the following

THEOREM 3.2. *If a conformally flat P-Sasakian manifold has a non-zero constant scalar curvature, the manifold is of constant curvature.*

PROOF. We assume that a conformally flat P-Sasakian manifold has a non-zero constant scalar curvature. Then from (3.6) we have

$$\{R + n(n - 1)\}(\phi_{ki}\eta_j - \phi_{ji}\eta_k) = 0,$$

from which follows

$$R = - n(n - 1).$$

Thus, from the Theorem 3.1 the manifold is of constant curvature. Q.E.D.

In an SP-Sasakian manifold, from the definition we have

$$\nabla_i\xi^h = - \delta_i^h + \eta_i\xi^h,$$

from which we find that the ξ^h is a concircular vector field.

On the other hand, one of the present authors studied subprojective manifolds [1]. Making use of his theorem and Theorem 3.1, we have the following

THEOREM 3.3. *A conformally flat P-Sasakian manifold is a subprojective one* $(n > 3)$.

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