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TESTS FOR n RANKINGS WHEN THE HYPOTHESES NOT BEING LIMITED TO RANDOM RANKING

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1. INTRODUCTION. In the test based on the method of n rankings, the null hypothesis is usually given that the ranks are assigned at random by each of judges, that is, all rankings are equally frequent in the population of rankings. However we are frequently confronted with the case in which the hypothesis of random ranking is unsuitable. For instance, when we wish to test for equality of mean ranks between two sets of rankings (namely, two-sample test), we can take no longer the assumption of random ranking as the null hypothesis.

The author considered in the recent paper (1977) a general population of rankings for m objects, specified by the probabilities for the $m!$ different rankings, and showed that if the ranking population has a mean vector $\underset{\sim}{\mu} = (\mu_1,\ldots,\mu_m)'$, $\sum_i \mu_i = m(m+1)/2$ and dispersion matrix $\Sigma = (\sigma_{ij})$, $i,j = 1,\ldots,m$, then the following properties hold:

(1) $\sum_{j=1}^{m} \sigma_{ij} = 0$, for $i = 1,\ldots,m$

(2) $tr\Sigma \overset{d}{=} \sum_{i=1}^{m} \sigma_{ii} = -2\sum_{i<j} \sigma_{ij} = \sum_{i=1}^{m} (i^2 - \mu_i^2)$.

(3) If we assume equi-variances and equi-covariances, then it holds

$$\Sigma = \frac{tr\Sigma}{m-1}\,(I-\frac{1}{m}E), \tag{1.1}$$

where I is a $m\times m$ identity matrix and E denotes a $m\times m$ matrix with all components 1.

(4) In general, Σ is denoted by using $H = (h_{ij})$ which is $m\times m$ symmetric metrix with properties $\sum_i h_{ij} = 0$ and $\sum_i h_{ii} = 0$, as follows

$$\Sigma = \frac{tr\Sigma}{m-1}\,\{\,(I-\frac{1}{m}E) + H\,\}. \tag{1.2}$$

(5) For some constant $\rho(\geq 1)$, a necessary and sufficient condition for

$$\Sigma^- = \frac{m-1}{\rho\, tr\Sigma}\,(I-\frac{1}{m}E) \tag{1.3}$$

to be g-inverse matrix of Σ, is that the following relation holds

$$\Sigma^2 = [\rho\, tr\Sigma/(m-1)]\Sigma \tag{1.4}$$

These properties play and important part in the problem of n ranking as the hypothesis of random ranking is unsuitable. Let $\underset{\sim}{r}_j = (r_{1j},\ldots,r_{mj})'$, $j = 1, \ldots, n$, be a sample from the ranking population with mean vector $\underset{\sim}{\mu} = (\mu_1,\ldots,\mu_m)'$ and dispersion matrix $\Sigma = (\sigma_{ij})$. Then sample mean vector $\underset{\sim}{\bar{R}} = (\bar{R}_1,\ldots,\bar{R}_m)'$, where $\bar{R}_i = (\sum_j r_{ij})/n$ is asymptotically distributed according to m-variate (degenerate) normal distribution. The author (1977) studied the distributions of quadratic forms related to the sample mean vector $\underset{\sim}{\bar{R}}$ and obtained the following fundamental theorem.

THEOREM 1.1 *Suppose the sample mean vector $\underset{\sim}{\bar{R}}$ has the m-variate (degenerate) normal distribution $N(\underset{\sim}{\mu}, \frac{1}{n}\Sigma)$. Then a necessary and sufficient condition for the quadratic form :*

$$\chi_*^2 = \frac{n(m-1)}{\rho tr\Sigma} \sum_{i=1}^{m} (\bar{R}_i - \mu_i)^2 \tag{1.5}$$

to be distributed according to chi-square distribution with $(m-1)/\rho$ degrees of freedom, is that there exists the value of $\rho(\geqq 1)$ which satisfies the relation in (1.4).

The result can be immediately extended to the case of two samples and can be applied to two-sample test for n rankings as discussed by the author (1977). The above theorem is the fundamental fact in setting up tests concerning $\underset{\sim}{\mu}$. In the present paper, the following problems are studied based on the statistic χ_*^2.

(a) Test for equality of mean ranks of m objects,

(b) Test for equality of mean ranks of a certain subset of m objects,

(c) Subdivision of χ_*^2 into components,

(d) A multipul comparison test for n rankings.

2. TEST FOR EQUALITY OF MEAN RANKS OF m OBJECTS.

An important problem is that of testing the hypothesis that the mean ranks of m objects all equal, *i.e.* $H_0: \mu_1 = \ldots = \mu_m = (m+1)/2$. Stuart (1951) considered the test based on the coefficient of concordance:

$$W = \frac{12}{m(m+1)(m-1)} \sum_{i=1}^{m} (\bar{R}_i - \frac{m+1}{2})^2. \tag{2.1}$$

He calculated four moments of W and then used K. Pearson's criterion to decide on the most appropriate type of distribution. However it seems the procedure is very complicated.

Now we can introduce the following corollary from the theorem 1.1 mentioned in section 1.

COROLLARY 2.1 *Under the same condition in theorem 1.1,*

(2.2) $$\chi_*^{\cdot 2} = \frac{n(m-1)}{\rho\, tr\Sigma} \sum_{i=1}^{m} (\bar{R}_i - \frac{m+1}{2})^2$$

is distributed according to a non-central chi-square distribution with (m-1)/ρ d.f. and the non-centrality parameter:

(2.3) $$\lambda = \frac{n(m-1)}{\rho tr\Sigma} \sum_{i=1}^{m} (\mu_i - \frac{m+1}{2})^2.$$

Under the hypothesis H_O: $\mu_1 = \ldots = \mu_m = (m+1)/2$, we have $tr\Sigma = m(m+1)(m-1)/12$ and $\lambda = 0$. Therefore $\chi_*^{\cdot 2}$ becomes

(2.4) $$\chi_*^{\cdot 2} = \frac{12n}{\rho m(m+1)} \sum_{i=1}^{m} (\bar{R}_i - \frac{m+1}{2})^2$$

and under the condition:

(2.5) $$\Sigma^2 = [\rho m(m+1)/12]\Sigma,$$

the statistic is distributed according to central chi-square distribution with $(m-1)/\rho$ d.f. Note that if Σ has equi-variances and equi-covariances, then under H_O it becomes

(2.6) $$\Sigma = \frac{m(m+1)}{12} (I - \frac{1}{m}E),$$

and the condition (2.5) holds for $\rho = 1$. Thus, in this case $\chi_*^{\cdot 2}$ coincides with the Friedman's statistic χ_r^2.

We may use the statistic $\chi_*^{\cdot 2}$ in (2.4) to test the null hypothesis H_O: $\mu_1 = \ldots = \mu_m = (m+1)/2$. However, if the test has to be applied Σ must be estimated in H_O. We propose to estimate Σ by $(tr\Sigma/trS)S$, *i.e.* $[m(m+1)(m-1)/(12trS)]\,S$, where $S = (S_{ij})$ denotes sample dispersion matrix. In this case, the condition in (2.5) becomes

(2.7) $$S^2 = [\rho trS/(m-1)]S$$

and by using least squares method the approximate value of ρ is obtained as follows:

(2.8) $$\rho = (m-1) \sum_{i=1}^{m} \sum_{j=1}^{m} (s_{ij} \sum_{\xi=1}^{m} s_{i\xi}\, s_{\xi j}) / (\sum_{i=1}^{m} s_{ii}) (\sum_{i=1}^{m} \sum_{j=1}^{m} s_{ij}^2)$$

The significance of the observed value of $\chi_*^{\cdot 2}$ is determined by reference to the table of chi-square distribution. However, in the practical case, since the value of $(m-1)/\rho$ is not integer, we must use the table of Incomplete Gamma Function. If the observed value of $\chi_*^{\cdot 2}$ exceeds the selected value in the table, then we may reject H_O.

As an illustration, we take up Stuart's (1951) data. That is, we have a sample of *50* rankings of eight occupations. We wish to test whether this sample could have arisen from a population in which all occupations had the same mean rank. The sample mean vector $\bar{R}$ and the sample dispersion matrix $S = (S_{ij})$ are given in the Stuart's paper.

When we apply the test introduced in above, the results are

$$m = 8, \quad n = 50,$$
$$\rho = 1.8352, \qquad \nu = (m-1)/\rho = 3.8143,$$
$$\chi_*'^2 = 152.76$$

For 4 degrees of freedom this value of $\chi_*'^2$ is highly significant. This result agrees with Stuart's findings.

3. TEST FOR EQUALITY OF MEAN RANKS OF A CERTAIN SUBSET OF m OBJECTS. We next consider to test whether the mean ranks of a certain subset of m objects are identical. In this case the hypothesis to be tested is that H_0^*: $\mu_1^* = \ldots = \mu_k^*$, $(m \geq 3,\ 2 \leq k \leq m)$, where $(\mu_1^*, \ldots, \mu_k^*)$ denotes a certain subset of $(\mu_1, \ldots, \mu_m)$. Suppose sampling data is $n$rankings of m objects and the sample mean vector $\bar{\underset{\sim}{R}}$ has the normal distribution $N(\underset{\sim}{\mu}, \frac{1}{n}\Sigma)$. Let $\bar{\underset{\sim}{R}}^* = (\bar{R}_1^*, \ldots, \bar{R}_k^*)'$ denote the sample mean vector corresponding to $\underset{\sim}{\mu}^* = (\mu_1^*, \ldots, \mu_k^*)'$. Then the distribution of $\bar{\underset{\sim}{R}}^*$ is k-variate normal distribution $N(\underset{\sim}{\mu}^*, \frac{1}{n}\Sigma^*)$, where Σ^* is a k-dimensional sub-matrix of Σ. In the following, we denote by I_k the $k\chi k$ identity matrix and by E_k the $k\chi k$ matrix with all components 1. By the similar way used to theorem 1.1, we can introduce the following result.

THEOREM 3.1 *If there exists the value of* ρ^* *which satisfies the relation*

$$\Sigma^{*2} = [\rho^* tr(I_k - \frac{1}{k}E_k)\Sigma^*/(k-1)]\Sigma^*, \tag{3.1}$$

then the quadratic form:

$$\chi_{\#}^2 = \bar{\underset{\sim}{R}}^{*\prime} Q \bar{\underset{\sim}{R}}^* = \frac{n(k-1)}{\rho^*(\sum_i \sigma_{ii}^* - \frac{1}{k}\sum_i \sum_j \sigma_{ij}^*)} \sum_{i=1}^{k} (\bar{R}_i^* - \frac{1}{k}\sum_{i=1}^{k} \bar{R}_i^*)^2, \tag{3.2}$$

where

$$Q = \frac{n(k-1)}{\rho^* tr(I_k - \frac{1}{k}E_k)\Sigma^*}(I_k - \frac{1}{k}E_k), \tag{3.3}$$

is distributed according to non-central chi-square distribution with $(k-1)/\rho^*$ *d.f. and non-centrality parameter*

$$\lambda^* = \frac{n(k-1)}{\rho^*(\sum_i \sigma_{ii}^* - \frac{1}{k}\sum_i \sum_j \sigma_{ij}^*)} \sum_{i=1}^{k} (\mu_i^* - \frac{1}{k}\sum_{i=1}^{k}\mu_i^*)^2. \tag{3.4}$$

when we especially put $k = m$, it becomes that $\Sigma^* = \Sigma$,

$$tr\ (I_k - \frac{1}{k}E_k)\Sigma^* = tr\Sigma, \text{ and } \frac{1}{k}\sum_{i=1}^{k} \bar{R}_i^* = \frac{m+1}{2}.$$

Thus, in this case $\chi_{\#}^2$ coincides with $\chi_*'^2$ in (2.2). Note that if $k \neq m$ and Σ has equi-variances and equi-covariances, then the value of ρ^* satisfied (3.1) does not exist. We shall discuss for such a case the last part of this section. Under the hypothesis $H_0^*: \mu_1^* = \ldots = \mu_k^*$, it finds $\lambda^* = 0$, and hence $\chi_{\#}^2$

has central chi-square distribution with $(k-1)/\rho^*$ d.f. The approximate value of ρ^* satisfied (3.1) can be found by least squares method as follows:

$$\rho^* = (k-1)\sum_{i=1}^{k}\sum_{j=1}^{k}(\sigma^*_{ii}-\sum_{\xi=1}^{k}\sigma^*_{i\xi}\sigma^*_{\xi j})/(\sum_{i=1}^{k}\sigma^*_{ii}-\frac{1}{k}\sum_{i=1}^{k}\sum_{j=1}^{k}\sigma^*_{ij})(\sum_{i=1}^{k}\sum_{j=1}^{k}\sigma^{*2}_{ij}) \tag{3.5}$$

For the same problem, Linhart (1960) has given an approximate test. Namely it is proposed to use a chi-square distribution with ν d.f. for the statistic t/α, where

$$\begin{aligned} t &= \sum_{i=1}^{k}\sum_{j=1}^{k}(\bar{R}^*_i - \bar{R}^*_j)^2, \\ \alpha &= Var(t)/\{2E(t)\}, \\ \nu &= 2E^2(t)/Var(t) = E(t)/\alpha \end{aligned} \tag{3.6}$$

However, in his paper there is no reason to conclude that t/α is distributed as chi-square distribution. Now we can give an interpretation to the distribution of the statistic t/α based on the result mentioned above. If we put $\alpha = E(t)/\nu$, where let ν be a parameter, and noting that under H^*_o: $\mu^*_1 = \ldots = \mu^*_k$, the followings are obtained

$$\begin{aligned} t &= \sum_{i=1}^{k}\sum_{j=1}^{k}(\bar{R}^*_i - \bar{R}^*_j)^2 = 2k\sum_{i=1}^{k}(\bar{R}^*_i - \frac{1}{k}\sum_{i=1}^{k}\bar{R}^*_i)^2, \\ E(t) &= \frac{1}{n}\sum_{i=1}^{k}\sum_{j=1}^{k}(\sigma^*_{ii}+\sigma^*_{jj}-2\sigma^*_{ij}) = \frac{2k}{n}(\sum_{i=1}^{k}\sigma^*_{ii} - \frac{1}{k}\sum_{i=1}^{k}\sum_{j=1}^{k}\sigma^*_{ij}), \end{aligned} \tag{3.7}$$

then we can see that t/α is denoted as follows

$$t/\alpha = \frac{n\nu}{\sum_i\sigma^*_{ii} - \frac{1}{k}\sum_i\sum_j\sigma^*_{ij}}\sum_{i=1}^{k}(\bar{R}^*_i - \frac{1}{k}\sum_{i=1}^{k}\bar{R}^*_i)^2. \tag{3.8}$$

Hence, putting $\nu = (k-1)/\rho^*$, it can be seen that under the hypothesis H^*_o, the statistic t/α agrees with $\chi^2_{\#}$ defined in (3.2) and therefore under the condition in (3.1) it has chi-square distribution with $\nu = (k-1)/\rho^*$ d.f.

As an illustration, we will take up stuart's (1951) data. The following results are obtained in a test for equality of mean ranks of jobs B,D, and G.

$$m = 8,\quad n = 50,\quad k = 3,$$
$$\rho^* = 1.2689,\quad \nu = (k-1)/\rho^* = 1.5762,$$
$$\chi^2_{\#} = 42.2168$$

This value is highly significant for 2 d.f. For reference, we show the results of Linhart's (1960) approximate test in the following.

$$t = 8.51,\quad E(t) = 0.318,\quad Var(t) = 0.112,$$
$$\alpha = 0.177,\quad \nu = 1.80,\quad \chi^2 = t/\alpha = 48.1$$

Finally, let us consider the special case in which Σ has equi-variances and equi-covariances, i.e. Σ is denoted as (1.1).
The statistic $\bar{R}^* = (\bar{R}^*_1,\ldots,\bar{R}^*_k)$ has the distribution $N(\mu^*, \frac{1}{n}\Sigma^*)$, where Σ^* is especially denoted as

(3.9) $$\Sigma^* = \frac{tr\Sigma}{m-1}\left(I_k - \frac{1}{m}E_k\right).$$

Then, for the quadratic form:

(3.10) $$\chi^2_{\#} = \bar{R}^{*\prime} A \bar{R}^* = \frac{n(m-1)}{tr\Sigma}\sum_{i=1}^{k}\left(\bar{R}_i^* - \frac{1}{k}\sum_{i=1}^{k}\bar{R}_i^*\right)^2,$$

where $$A = \frac{n(m-1)}{tr\Sigma}\left(I_k - \frac{1}{k}E_k\right),$$

we can see that (i) $A(\frac{1}{n}\Sigma^*)$ is idempotent matrix, (ii) $trA(\frac{1}{n}\Sigma^*) = k-1$. Thus we have the following result.

THEOREM 3.2 *If Σ has equi-variances and equi-covariances, then $\chi^2_{\#}$ defined in (3.10) is distributed as non-central chi-square distribution with $k-1$ d.f. and non-centrality parameter*

(3.11) $$\frac{n(m-1)}{tr\Sigma}\sum_{i=1}^{k}\left(\mu_i^* - \frac{1}{k}\sum_{i=1}^{k}\mu_i^*\right)^2$$

NOTE: We can show that the quadratic form $\chi^2_{\#}$ is a component of the $\chi^{\prime 2}_{*}$ when Σ has equi-variances and equi-covariances.

4. SUBDIVISION OF χ^2_* INTO COMPONENTS.

4.1 SUBDIVISION OF χ^2_* INTO SINGLE COMPONENTS. We consider a comparison among the $\bar{R}_1,\dots,\bar{R}_m$, that is,

(4.1) $$Z_j = \sum_{i=1}^{m} \ell_{ji}\bar{R}_i, \quad \text{where} \quad \sum_{i=1}^{m}\ell_{ji} = 0.$$

Suppose that $\bar{R} = (\bar{R}_1,\dots,\bar{R}_m)'$ has a normal distribution $N(\mu, \frac{1}{n}\Sigma)$, where we especially assume that Σ has equi-variances and equi-covariances, *i.e.* Σ is denoted as (1.1). Then it is easily seen that Z_j is distributed as normal distribution with mean θ_j and variance D_j, where

(4.2) $$\theta_j = \sum_{i=1}^{m}\ell_{ji}\mu_i, \qquad D_j = \frac{tr\Sigma}{n(m-1)}\sum_{i=1}^{m}\ell_{ji}^2,$$

and further the statistic:

(4.3) $$(Z_j - \theta_j)^2/D_j$$

is distributed according to chi-square distribution with 1 d.f.

Now we consider two comparisons $Z_\alpha = \sum_i \ell_{\alpha i}\bar{R}_i$ and $Z_\beta = \sum_i \ell_{\beta i}\bar{R}_i$, where $\sum_i \ell_{\alpha i} = 0$ and $\sum_i \ell_{\beta i} = 0$. The covariance of Z_α and Z_β is denoted by using two vectors $\ell_\alpha = (\ell_{\alpha 1},\dots,\ell_{\alpha m})'$ and $\ell_\beta = (\ell_{\beta 1},\dots,\ell_{\beta m})'$ as follows

$$(4.4)\qquad \begin{aligned} Cor(Z_\alpha, Z_\beta) &= \frac{1}{n}\ell'_\alpha \Sigma \ell_\beta \\ &= \frac{tr\Sigma}{n(m-1)}\ell'_\alpha (I - \frac{1}{m}E)\ell_\beta \\ &= \frac{tr\Sigma}{n(m-1)}\ell'_\alpha \ell_\beta \\ &= \frac{tr\Sigma}{n(m-1)}\sum_i \ell_{\alpha i}\ell_{\beta i} \end{aligned}$$

Therefore, if $\sum_i \ell_{\alpha i}\ell_{\beta i} = 0$, *i.e.* Z_α and Z_β are orthogonal, then it becomes that $cov(Z_\alpha, Z_\beta) = 0$. Since the joint distribution of Z_α and Z_β is a bivariate normal distribution, it means Z_α and Z_β are independent of each other.

THEOREM 4.1 *If the comparisions $Z_1,\ldots,Z_{m-1}$ are mutually orthogonal, i.e. $\sum_i \ell_{\alpha i}\ell_{\beta i} = 0$, for $\alpha \neq \beta = 1,\ldots,m-1$, then the following subdivision holds.*

$$(4.5)\qquad \chi^2_* = \frac{n(m-1)}{tr\Sigma}\sum_{i=1}^{m}(\bar{R}_i - \mu_i)^2 = \sum_{j=1}^{m-1}(Z_j - \theta_j)^2/D_j$$

where χ^2_* has the chi-square distribution with m-1 d.f. and $(Z_j - \theta_j)^2/D_j$ $j=1,\ldots,m-1$ are mutually independently distributed as chi-square distribution with 1 d.f., respectively.

Similarly, the subdivision of χ'^2_* into single components is also considered. The result is the following.

COROLLARY 4.1 *Under the same condition in theorem 4.1, the following subdivision holds.*

$$(4.6)\qquad \chi'^2_* = \frac{n(m-1)}{tr\Sigma}\sum_{i=1}^{m}(\bar{R}_i - \frac{m+1}{2})^2 = \sum_{j=1}^{m-1}Z_j^2/D_j$$

where χ'^2_* has the non-central chi-square distribution with m-1 d.f. and non-centrality parameter

$$(4.7)\qquad \lambda = \frac{n(m-1)}{tr\Sigma}\sum_{i=1}^{m}(\mu_i - \frac{m+1}{2})^2,$$

and in the right hand side Z_j^2/D_j, $j=1,\ldots,m-1$, are mutually independently distributed according to non-central chi-square distribution with 1 d.f. and non-centrality parameter $\lambda_j = \theta_j^2/D_j$, respectively, and it holds $\lambda = \sum_j \lambda_j$.

4.2 A CERTAIN INCOMPLETE SUBDIVISION. We here consider an important incomplete subdivision. In this case, the results given by Hogg and Craig (1958), and Ogawa (1949) may be usefull. As mentioned in section 3, when Σ has equi-variances and equi-covariances, the statistic $\chi^2_\#$ defined in (3.10) has a non-central chi-square distribution and becomes a component of the χ'^2_*. Based on the result, we can make a certain incomplete subdivision of χ'^2_* under the assumption that Σ has equi-variances and equi-covariances.

Let m objects be classified into g groups and let $(\bar{R}_1^{(j)},\dots,\bar{R}_{m_j}^{(j)})$, $j=1,\dots,g$, be denoted the sample means corresponding to the j-th group. Now we put

$$(4.8)\qquad \begin{aligned} Q_j &= \frac{n(m-1)}{tr\Sigma}\sum_{i=1}^{m_j}(\bar{R}_i^{(j)}-\bar{\bar{R}}^{(j)}), \quad j=1,\dots,g, \\ Q_{g+1} &= \frac{n(m-1)}{tr\Sigma}\sum_{j=1}^{g} m_j(\bar{\bar{R}}^{(j)}-\frac{m+1}{2})^2, \end{aligned}$$

where $\bar{\bar{R}}^{(j)} = \frac{1}{m_j}\sum_{i=1}^{m_j}\bar{R}_i^{(j)}$.

Note that Q_j represents the sum of squares of deviations among the mean ranks in the j-th group and is distributed as non-central chi-square distribution with m_j-1 d.f. and non-centrality parameter

$$(4.9)\qquad \lambda_j = \frac{n(m-1)}{tr\Sigma}\sum_i(\mu_i^{(j)}-\frac{1}{m_j}\sum_i\mu_i^{(j)})^2, \qquad j=1,\dots,g$$

While Q_{g+1} is the sum of squares of deviations among the group totals and is distributed according to non-central chi-square distribution with $g-1$ d.f. and non-centrality parameter

$$(4.10)\qquad \lambda_{g+1} = \frac{n(m-1)}{tr\Sigma}\sum_{j=1}^{g} m_j(\frac{1}{m_j}\sum_i\mu_i^{(j)}-\frac{m+1}{2})^2.$$

These Q_j and Q_{g+1} are components of the statistic $\chi_*^{\prime 2}$. Thus we can obtain the following result.

THEOREM 4.2 *For the quadratic forms $Q_1,\dots,Q_g$, and Q_{g+1} defined in (4.8), it holds that*

$$(4.11)\qquad \begin{aligned} \chi_*^{\prime 2} &= \sum_{j=1}^{g} Q_j + Q_{g+1}, \\ \lambda &= \sum_{j=1}^{g}\lambda_j + \lambda_{g+1}. \end{aligned}$$

NOTE: Under the hypothesis H_0: $\mu_1=\dots=\mu_m=(m+1)/2$, since we assumed Σ has equi-variances and equi-covariances, it becomes $tr\Sigma = m(m+1)(m-1)/12$, and the non-centrality parameters are all equal to zero. Thus, in this case Q_j and Q_{g+1} are especially denoted as

$$(4.12)\qquad \begin{aligned} Q_j &= \frac{12n}{m(m+1)}\sum_{i=1}^{m_j}(\bar{R}_i^{(j)}-\bar{\bar{R}}^{(j)})^2, \qquad j=1,\dots,g \\ Q_{g+1} &= \frac{12n}{m(m+1)}\sum_{j=1}^{g} m_j(\bar{\bar{R}}^{(j)}-\frac{m+1}{2})^2, \end{aligned}$$

and have the central chi-square distributions with m_j-1 and $g-1$ d.f., respectively. For example, if $g=2$, the followings are obtained:

$$(4.13)\qquad \begin{aligned} Q_1 &= \frac{12n}{m(m+1)}\sum_{i=1}^{K}(\bar{R}_i^{(1)}-\bar{\bar{R}}^{(1)})^2, \\ Q_2 &= \frac{12n}{m(m+1)}\sum_{i=1}^{m}(\bar{R}_i^{(2)}-\bar{\bar{R}}^{(2)})^2, \\ Q_3 &= \frac{12n}{m(m+1)}\cdot\frac{k(m-k)}{m}(\bar{\bar{R}}^{(1)}-\bar{\bar{R}}^{(2)})^2, \end{aligned}$$

and it holds that $\chi_*^{\prime 2} = Q_1+Q_2+Q_3$. This result has been already shown in the author's (1974) paper and has been applied to test whether there is a difference between two sets of ranks assigned to k objects in G_1 and $(m-k)$ objects in G_2.

5. A MULTIPLE COMPARISON TEST FOR n RANKINGS. Suppose the sample mean vector $\underset{\sim}{\bar{R}} = (\bar{R}_1,\ldots,\bar{R}_m)'$ is distributed as $N(\underset{\sim}{\mu}, \frac{1}{n}\Sigma)$ and especially assume that there exists the value of ρ which satisfies the relation $\Sigma^2 = [\rho tr\Sigma/(m-1)]\Sigma$. Then as mentioned in theorem 1.1, the statistic χ^2_* has the chi-square distribution with $(m-1)/\rho$ d.f. Hence, if $\chi^2_{(m-1)/\rho}(\alpha)$ is the $100(1-\alpha)\%$ point of this distribution, we have

$$P(\sum_{i=1}^{m}(\bar{R}_i - \mu_i)^2 \leq \delta^2) = 1-\alpha, \tag{5.1}$$

where

$$\delta^2 = \frac{\rho tr\Sigma}{n(m-1)}\chi^2_{(m-1)/\rho}(\alpha). \tag{5.2}$$

If we interpret by use of m-dimensional geometric concepts and terminology, the set of points in the m-dimensional space for the inequality:

$$\sum_{i=1}^{m}(\bar{R}_i - \mu_i)^2 \leq \delta^2, \tag{5.3}$$

is the interior of a $100(1-\alpha)\%$ confiedence sphere for the true parameter point $(\mu_1,\ldots,\mu_m)$. For any particular choise of $(\ell_1,\ldots,\ell_m)$, not all zero, the two parallel hyperplanes having equations

$$\sum_i \ell_i\mu_i = \sum_i \ell_i\bar{R}_i \pm \delta(\sum_i \ell_i^2)^{1/2} \tag{5.4}$$

are tangent to the sphere having equation $\sum_i(\bar{R}_i - \mu_i)^2 = \delta^2$. Thus if we consider the set of points contained between all possible pairs of parallel hyperplanes tangent to this sphere, this set of points constitutes the interior of the sphere and the probability associated with this set is $1-\alpha$. That is, the probability becomes $1-\alpha$ that all possible inequalities:

$$\sum_i \ell_i\bar{R}_i - \delta(\sum_i \ell_i^2)^{1/2} \leq \sum_i \ell_i\mu_i \leq \sum_i \ell_i\bar{R}_i + \delta(\sum_i \ell_i^2)^{1/2}, \tag{5.5}$$

for all $\ell_1,\ldots,\ell_m$, not all zero, are simultaneously fulfilled.

THEOREM 5.1 *Suppose* χ^2_* *defined in (1.5) has the chi-square distribution with* $(m-1)/\rho$ *d.f. and let* $\chi^2_{(m-1)/\rho}(\alpha)$ *be the* $100(1-\alpha)\%$ *point of the chi-square distribution. If L is the set of all real vectors* $\underset{\sim}{\ell} = (\ell_1,\ldots,\ell_m)'$, *where* $\sum_i \ell_i = 0$, *not all zero, then the probability is* $1-\alpha$ *that the inequalities (5.5) hold simultaneously for all* $\underset{\sim}{\ell}$ *in L.*

NOTE: If we take only a finite number N of vector $\underset{\sim}{\ell}$, the set of points which lie between all N pairs of hyperplanes corresponding to these N vectors is a m-dimensional polyhedron which circumscribes the sphere having equation

$\sum_i (\bar{R}_i - \mu_i)^2 = \delta^2$. Since the polyhedron contains the sphere, the probability exceeds $1-\alpha$ that any finite number N of the inequalities corresponding to N vectors are simultaneously fulfilled.

The above result may be applied to the simultaneous test for multiple comparisons. The hypothesis to be tested is that

$$H_0: \theta = \sum_i \ell_i \mu_i = 0, \text{ for all } \underset{\sim}{\ell} \text{ in } L.$$

From the theorem 5.1, we can determine a critical value $Z_0 = \delta(\sum_i \ell_i^2)^{1/2}$ for the level of significance α such that, under H_0, the probability is α that the inequalities $|Z| > Z_0$, where $Z = \sum_i \ell_i \bar{R}_i$, hold simultaneously for all $\underset{\sim}{\ell}$ in L. Thus we have the following procedure:

(i) *if* $-Z_0 \leq Z \leq Z_0$, *then* H_0 *is not rejected,*

(ii) *if* $Z > Z_0$, *then we reject* H_0 *and conclude* $\theta > 0$,

(iii) *if* $Z < -Z_0$, *then we reject* H_0 *and conclude* $\theta < 0$.

The above method was introduced in a maner analogous to that of Scheffé's (1953) procedure. On the other hand, Mcdonald and Thompson (1967) has presented a multiple comparison method based on the range of rank sums.

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REFERENCES

[1] HOGG, R. V. and CRAIG, A. T. (1958): On the decomposition of certain χ^2 variables. Ann. Math. Statist., 29, 608-610.

[2] KANNO, R. (1974): A rank sum test in the analysis of variance. TRU Mathematics, 10, 55-64.

[3] ________ (1977): The distributions of quadratic forms related to the problem of n rankings. TRU Mathematics, 13-1, 89-110.

[4] LINHART, H. (1960): Approximate tests for m rankings. Biometrika, 47, 476-480.

[5] McDONALD, B. J. and THOMPSON, W. A. (1967): Rank sum multiple comparisons in one- and two-way classifications. Biometrika, 54, 487-497.

[6] OGAWA, J. (1949): On the independence of biliner and quadratic forms of a random sample from a normal population. Ann. Inst. Math. Stat., 1, 83-108.

[7] SCHEFFÉ, H. (1953): A method for judgeing all contrasts in the analysis of variance. Biometrika, 40, 87-104.

[8] STUART, A. (1951): An application of the distribution of the ranking concordance coefficient. Biometrika, 38, 33-42.