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SOME EQUIVALENTS OF THE BROUWER'S FIXED POINT THEOREM

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INTRODUCTION

The most usual way to prove the Brouwer's fixed point theorem is completed according to following order; Theorem on the degree of continuous mapping, Retraction Theorem and the Fixed Point Theorem. The theorem on the degree of continuous mapping is a generalization of the Intermediate Value Theorem on continuous function.

We will show another generalization of the intermediate value theorem which is equivalent to the fixed point theorem, in this paper.

1. THE THREE THEOREMS

First of all, we state three theorems that we are going to prove their equivalence. We use k-dimensional cube I^k where I is the interval [-1, 1], instead of k-dimensional sphere used usually. We denote the i-th coordinate of a point p in k-dimensional space by p_i and we use so-called maximum norm M(p):

$$M(p) = (\text{Max of absolute value of } P_i,\ i = 1, 2, \cdots, k)$$

as the metric of our k-dimensional space. The k-dimensional cube I^k is nothing else the unit sphere with respect to maximum norm. $b(I^k)$ means the boundary of the cube I^k.

RETRACTION THEOREM. *Let* f *be a mapping from* I^k *onto* $b(I^k)$. *If each point on the boundary* $b(I^k)$ *is fixed by the mapping* f, *that is, the restriction of the mapping* f *on the boundary is identity, then the mapping* f *is not continuous.*

FIXED POINT THEOREM. *Let* f *be a continuous mapping from* I^k *to* I^k. *Then there exists at least one fixed point* p, *that is* $f(p) = p$.

Here, we state our generalization of the intermediate value theorem.

NULL POINT THEOREM. *Let* f *be a continuous mapping from* I^k *to k-dimensional space* R^k. *If the mapping* f *satisfies following boundary condition, then there exists at least one null point* p, *that is,* $f(p) = 0$.

Boundary condtion. On the i-th coordinate of each boundary point p *and that of its image* f(p), $(i=1,2,\cdots,k)$,

$$p_i = 1 \text{ implies } f(p)_i > 0$$
$$p_i = -1 \text{ implies } f(p)_i < 0.$$

2. PROOF OF EQUIVALENCE OF THREE THEOREMS

Firstly, it is well known that Retraction Theorem implies Fixed Point Theorem.

Secondly, the mapping mentioned in Retraction Theorem satisfies the boundary condition of Null Point Theorem. Then Null Point Theorem implies Retraction Theorem.

At last, we will show that Fixed Point Theorem implies Null point Theorem.

Left f be a continuous mapping from I^k to R^k satisfying the boundary condition.

There exists a positive number K such that,

$$M(f(p)) \leq K \text{ for any point p of the cube } I^k,$$

since I^k is compact.

We can strengthen our boundary codition according to the compactness of the boundary $b(I^k)$, the continuity of the mapping f and the boundary condition, as follows:

There exists a positive number c $(c < 1/2)$ such that:

$$1-c < p_i \leq 1 \quad \text{implies} \quad f(p)_i > 0$$
$$-1 \leq p_i < -1+c \quad \text{implies} \quad f(p)_i < 0$$

Now we define a new mapping g defined on I^k by

$$g(p) = p - (c/K)f(p)$$

where the subtraction is the one as vector operation in k-dimensional space. This mapping g is continuous, and we will show the mapping g transforms I^k to I^k, and then we can apply Fixed Point Theorem for the mapping g. The fixed point of the mapping g is nothing else the null point of the mapping f.

The definitions of constants K and c imply

(i) a point $(c/K)f(p)$ belongs to the cube $[-c,c]^k$, that is,

$$M((c/K)f(p)) \leq c.$$

(ii) $M(g(p)) \leq M(p) + c$.

Then, if $M(p) \leq 1 - c$, we get $M(g(p)) \leq 1$.

In the case of $1 - c < M(p) \leq 1$, we estimate the i-th coordinate of the point g(p):

$$g(p)_i = p_i - (c/K)f(p)_i$$

We put $q_i = (c/K)f(p)_i$ then, $-c \le q_i \le c$, $g(p)_i = p_i - q_i$. On account of the strengthened boundary condition, we get;

if $1 - c < p_i \le 1$, then $q_i > 0$ and so, $0 < g(p)_i < p_i \le 1$,

if $-1 \le p_i \le -1 + c$, then $q_i < 0$ and so, $-1 \le p_i < g(p)_i \le 0$,

otherwise, $-1 \le g(p)_i \le 1$, as obtained already.

Therefore $M(g(p)) \le 1$, in any case, that is, $g(p)$ belongs to the cube I^k.

Thus three theorems are mutually equivalent each other.

3. ONE MORE EQUIVALENT THEOREM

We can modify the boundary condition of our theorem as weaker form. So, we get one more generalization of the intermediate value theorem which is also equivalent to three theorems.

NULL POINT THEOREM (revised). *Let* f *be a continuous mapping from* I^k *to k-dimensional space* R^k *satisfying;*

Boundary condition (revised). On the i-th coordinate of each boundary point p *and that of its image* $f(p)$, $(i=1,2,\cdots,k)$,

$p_i = 1$ implies $f(p)_i \ge 0$,

$p_i = -1$ implies $f(p)_i \le 0$.

Then there exists at least one null point p, *that is,* $f(p) = 0$.

We revised the boundary condition concerning to i-th coordinate of $f(p)$,

from positive to non-negative

from negative to non-positive.

Obviously, the revised null point theorem implies the previous theorem.

The fixed point theorem can be proved useing the revised null point theorem, easily. For a given continuous mapping f from I^k to I^k, we put $g(p) = f(p) - p$ and then we can apply the revised null point theorem to the mapping g.

Now, we will show that the null point theorem implies the revised one.

Let J be the interval $[-2,2]$ and we consider k-dimensional cube J^k which is $2I^k$ as vector operation (the number 2 is not essential, we may use any number greater than 1). We are going to construct a mapping g from J^k to R^k to whom we apply the null point theorem.

Let's define $g(p)$ for each point p in J^k:

$$g(p) = f(p) \quad \text{for a point } p \text{ in } I^k,$$

and for a point p outside point of I^k, let $t(p)$ be the point which is the intersection point of the ray through the point p originated the point 0 with the boundary $b(I^k)$, we put

$$g(p) = f(t(p)) + p - t(p)$$

Then the mapping g is a continuous one from J^k to R^k and

$$\text{if } p_i = 2, \text{ then } t(p)_i = 1 \text{ and so } g(p)_i = f(t(p))_i + 1 > 0,$$
$$\text{if } p_i = -2, \text{ then } t(p)_i = -1 \text{ and so } g(p)_i = f(t(p))_i - 1 < 0.$$

Therefore, the mapping g defined on J^k satisfies the boundary condition of the null point the theorem for the cube J^k. Then there exists a null point p of the mapping g, $g(p) = 0$.

This null point can not be a outside point of the cube I^k. Because, if the null point p of the mapping g is outside of I^k, then it must satisfies,

$$f(t(p)) = t(p) - p. \qquad (*)$$

Consider the i-th coordinate of above point, and

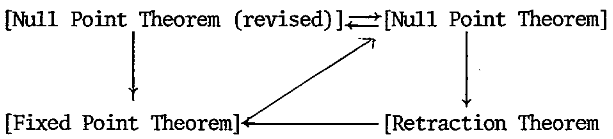
$$\text{if } t(p)_i = 1, \text{ then } f(t(p))_i \geq 0 \text{ and } 1 < P_i \leq 2$$
$$\text{if } t(p)_i = -1, \text{ then } f(t(p))_i \leq 0 \text{ and } -2 \leq p_i < -1,$$

then the equality (*) is false in any cases. (the point t(p) is on $b(I^k)$, so its i-th coordinate is +1 or -1 for some i.)

Therefore, the null point p should belongs to I^k, that is, the point p is a null point of the mapping f, $f(p) = 0$.

Thus the revised null point theorem is equivalent to the other three theorems.

At last, we show the implication diagramm of four theorems;



We shall show the other equivalent theorem equivalent to the fixed point theorem in coming paper.

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