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ON THE CONTINUOUS AND THE COMPACT PSEUDO-OPEN MAPPINGS

Tamezo MASUDA and Yoshitomo MATSUO

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§1. Introduction.

Let $f : X \longrightarrow Y$ be a continuous mapping onto Y. If $\mathcal{D}$ is a discrete collection of Y, then $\{f^{-1}(d) : d \in \mathcal{D}\}$ is the discrete collection of X. (Theorem 1) As a mutually exclusive open collection of a separable space is a countable one, a normal space that can be expressed as a continuous image of a separable collectionwise normal space is the collectionwise normal space. (Theorem 2) Ernest, Michael [3] has prouved that each open covering G of a pointwise paracompact collectionwise normal space X has a locally finite open covering of X which refines G. With this results a pointwise paracompact normal space that can be expressed as a compact pseudo-open image of a separable collecttionwise normal T_1 space is a paracompact space. (Theorem 3) A regular screenable space that can be expressed as a continuous image of a separable space is the pointwise paracompact normal space. (Theorem 4) A pointwise paracompact normal screenable T_1 space is paracompact. [4] With this results a normal screenable space that can be expressed as a compact pseudo-open image of a Hausdorff separable space is a paracompact space.

§2. Definition

1. A mapping $f : X \longrightarrow Y$ is continuous, if the inverse of each open set is open.
2. A continuous mapping $f : X \longrightarrow Y$ is said to be pseudo-open, if for any point y and any neighbourhood U of its pre-image $f^{-1}(y)$, Int $f(U)$ (the interior of $f(U)$) contains y.
3. A mapping $f : X \longrightarrow Y$ is said to be compact, if for any point y, $f^{-1}(y)$ is compact.
4. A collection D of X is said to be discrete, if for any subcollection D_1 of D, $\{\bar{d} : d \in D_1\}$ is a mutually exclusive collection and $\cup\{\bar{d} : d \in D_1\}$ is closed.
5. A space X is said to be pointwise paracompact, if each open covering G of X has a point finite open covering of X which refines G.

6. A space X is said to be screenable, if for any open covering H of X, there exists a countable mutually exclusive open collections H_1, H_2, ······, such that $\sum_{n\in N} H_n$ refines H and covers X.
7. The open covering G_1, G_2,······, of X are said to be developments of X, if for any integer n, $G_n > G_{n+1}$ and for any point p and any neighboarhood U of p, there exists an integer m such as $G_m(p) \subset U$. This space X is said to be a developable space.
8. A space X is said to be collectionwise normal, if for any discrete collection $\{d_\alpha\}_{\alpha\in\Lambda}$, there exists a mutually exclusive open collection $\{U_\alpha\}_{\alpha\in\Lambda}$, such that for any α of Λ, $d_\alpha \subset U_\alpha$.
9. A space X is said to be F_σ-screenable, if for any open covering G, there exists a countable discrete collections D_1, D_2,······, such that $\sum_{n\in N} D_n$ covers X and $\sum_{n\in N} \{\overline{d} : d \in D_n\} < G$.
10. Let X be a developable space. A compact pseudo-open mapping $f : X \longrightarrow Y$ is said to be a developable mapping (metrizable mapping), if for any point y of Y and any open set V containing y, there exists an integer n such that if p is a point of Y and $G_n(f^{-1}(p)) \cap f^{-1}(y) \neq \phi$, then Int $f(G_n(f^{-1}(p))) \subset V$ $(f(G_n(f^{-1}(p))) \subset V)$.

§3. Theorems

THEOREM 1. *Let $f : X \longrightarrow Y$ be a continuous mapping onto Y. If $\{d_i\}_{i\in\Lambda}$ is a discrete collection of Y, then $\{f^{-1}(d_i)\}_{i\in\Lambda}$ is a discrete collection of X.*

PROOF. As f is continuous, $f^{-1}(\overline{d}_i)$ is a closed set. Hence $\overline{f^{-1}(d_i)} \subset f^{-1}(\overline{d}_i)$. For two members d_i, d_j of $\{d_i\}_{i\in\Lambda}$, $\overline{d_i} \cap \overline{d_j} = \phi$. Hence $f^{-1}(\overline{d}_i) \cap f^{-1}(\overline{d}_j) = \phi$. Hence $\overline{f^{-1}(d_i)} \cap \overline{f^{-1}(d_j)} = \phi$. Suppose $\cup_{i\in\Lambda'}\left\{\overline{f^{-1}(d_i)}\right\}$ $(\Lambda' \subsetneq \Lambda)$ is not closed set. There exists a point $x \in X$ such that $x \notin \cup_{i\in\Lambda'}\left\{\overline{f^{-1}(d_i)}\right\}$ and each open set U containing x intersects $\cup_{i\in\Lambda'}\left\{\overline{f^{-1}(d_i)}\right\}$. Let $f(x) = y$. If $y \notin \cup_{i\in\Lambda'}\{\overline{d_i}\}$, there exists an open set U containing y and $U \cap (\cup_{i\in\Lambda'}\{\overline{d_i}\}) = \phi$. Hence $f^{-1}(U)$ containing x dose not interesect $\cup_{i\in\Lambda'}\left\{\overline{f^{-1}(d_i)}\right\}$. If $y \in \overline{d_j}$ $(j \in \Lambda')$, there exists an open set V such that $y \in V$ and $V \cap (\cup_{i\in\Lambda'-j}\{\overline{d_i}\}) = \phi$. Hence $f^{-1}(V)$ containing x dose not intersect $\cup_{i\in\Lambda'-j}\left\{\overline{f^{-1}(d_i)}\right\}$. Let $W = X \sim \overline{f^{-1}(d_j)}$. $f^{-1}(V)\cap W$ is an open set containing x and dose not intersect $\cup_{i\in\Lambda'}\left\{\overline{f^{-1}(d_i)}\right\}$. By this contreadiction, $\cup_{i\in\Lambda'}\left\{\overline{f^{-1}(d_i)}\right\}$ is a closed set. Hence $\{f^{-1}(d_i)\}_{i\in\Lambda}$ is a discrete collection of X. ■

THEOREM 2. *Let $f: X \longrightarrow Y$ be a continuous mapping from X onto normal space Y. If X is a collectionwise normal separable space, then Y is a collectionwise normal separable space.*

PROOF. It is obvious that Y is separable.

Let $\{d_i\}_{i\in\Lambda}$ is a discrete collection of Y. By theorem 1, $\{f^{-1}(d_i)\}_{i\in\Lambda}$ is a discrete collection of X. As X is a collectionwise normal space, there exists a mutually exclusive open collection $\{U_i\}_{i\in\Lambda}$ such that for each $i\in\Lambda$, $f^{-1}(d_i)\subset U_i$. As X is separable, $\{U_i\}_{i\in\Lambda}$ is a countable collection. Hence, $\{d_i\}_{i\in\Lambda}$ is a countable collection. Then $\{\overline{d_i}\}_{i\in\Lambda}$ is a countable closed discrete collection. As Y is a normal space, there exists mutually exclusive open collection $\{V_i\}_{i\in\Lambda}$ such that for each $i\in\Lambda$, $d_i\subset\overline{d_i}\subset V_i$. Hence Y is a collectionwise normal separable space. ■

Every point-finite open covering G of a collectionwise normal space S has a locally finite open covering of S which refines G. [2]

THEOREM 3. *Let $f: X \longrightarrow Y$ be a compact pseudo-open mapping onto a pointwise paracompact normal space Y. If X is a collectionwise normal separable T_1 space, then Y is a paracompact separable space.*

PROOF. Let p and q are points of Y. As X is a Hausdorff space and $f^{-1}(p)$ and $f^{-1}(q)$ are compact sets, $f^{-1}(p)$ and $f^{-1}(q)$ are closed sets and $f^{-1}(p)\cap f^{-1}(q)=\phi$. Let $U = X\sim f^{-1}(q)$. Then $p\in \operatorname{Int} f(U)$ and $q\notin \operatorname{Int} f(U)$. Hence Y is a T_1 space. By theorem 2, Y is a collectionwise normal separable space.

Hence Y is a pointwise paracompact collectionwise normal separable T_1 space. Hence Y is a paracompact separable space. ■

THEOREM 4. *Let $f: X \longrightarrow Y$ be a continuous mapping onto a regular screenable space Y. If X is a sparable space, then Y is a pointwise paracompact normal screenable space.*

PROOF. Let G is an open covering of Y. As Y is a regular space, for each point y of Y there exists an open set U_y containing y and there is an open set V of G and $\overline{U}_y \subset V$. Let $H = \{U_y\}_{y \in Y}$.

As Y is a screenable space there exist mutually exclusive open collections K_i $(i = 1, 2, \cdots\cdots)$ each of which refines H and $\sum_{i \in N} K_i$ covers Y. For each integer i and each $k_1 \in K_i$, $k_2 \in K_i$, $f^{-1}(k_1) \cap f^{-1}(k_2) = \phi$. As X is a separable space, for each integer i $\{f^{-1}(k) : k \in K_i\}$ is a countable open collection. Hence $\sum_{i \in N} K_i$ is a countable open collection which covers Y. Y is a regular space and each open covering G of Y has a countable open covering of Y which refines G. Hence Y is a normal space.

Let $\sum_{i \in N} K_i = \{W_1, W_2, \cdots\cdots\}$. For each W_i, there is an open set $G_i \in G$ such as $\overline{W}_i \in G_i$.

Let $M_1 = G_1$, $M_i = G_i \sim \cup_{j=1}^{i-1} \overline{W}_j$ $(2 \leqq i)$. $\{M_i\}_{i \in N}$ is an open covering of Y and each point p of Y is contained only finite member of $\{M_i\}_{i \in N}$. Hence Y is a pointwise paracompact normal screenable space. ■

A space X is a paracompact space if and only if X is a pointwise paracompact normal screenable T_1 space. [3]

THEOREM 5. *Let $X \longrightarrow Y$ be a compact pseudo-open mapping onto a regular screenable space Y. If X is a Hausdorff separable space, then Y is a paracompact space.*

PROOF. By theorem 3, Y is a T_1 space. By theorem 4, Y is a normal pointwise paracompact space. Hence Y is a normal screenable pointwise paracompact T_1 space. By [3], Y is a paracompact space. ■

Let X be a Hausdorff developable space and Y is a Topological space. If $f : X \longrightarrow Y$ is a developable mapping onto Y, then Y is a developable T_1 space. [4] If f is a metrizable mapping onto Y, then Y is metrizable. [4]

THEOREM 6. *Let $f : X \longrightarrow Y$ be a developable mapping onto a regular screenable space Y. If X is a Hausdorff separable developable space, then Y is metrizable.*

PROOF. By theorem [4], Y is a developable T_1 space. By theorem 4, Y is a normal space. Hence Y is a normal screenable developable T_1 space. Then Y is metriable. [1] ■

COROLLARY 1. *Let $f: X \longrightarrow Y$ be a compact pseudo-open mapping onto a normal F_σ-screenable space Y. If X is a separable collectionwise normal T_1 space, then Y is a paracompact space.*

COROLLARY 2. *Let $f: X \longrightarrow Y$ be a continuous mapping onto a normal Moore space Y. If X is a separable collectionwise normal space, then Y is metrizable.*

COROLLARY 3. *Let $f: X \longrightarrow Y$ be a continuous mapping onto a screenable Moore space Y. If X is a separable space, then Y is metrizable.*

COROLLARY 4. *Let $f: X \longrightarrow Y$ be a developable mapping onto a screenable space Y. If X is a separable Hausdorff developable space, then Y is metrizable.*

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SCIENCE UNIVERSITY OF TOKYO