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Multiple Solutions of Impulsive Boundary Value Problems on the Half-line in Banach Spaces *

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Abstract. Existence results of multiple solutions are obtained under suitable conditions for impulsive boundary value problems on the half-line in Banach spaces which may be singular at the boundary.

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§1. INTRODUCTION

Boundary value problems on the half line arise quite naturally in the study of radially symmetric solutions of nonlinear elliptic equations, see [1]-[4] for example. In general, problems of this kind are singular. Up to now, most known results in this area concern only boundary value problems with sub-linear nonlinearity, see [1][5]. In [7], the authors discussed multiple solutions of boundary value problems with finite impulses on finite intervals in Banach spaces. The purpose of the present paper is to study the existence of multiple solutions for semi-linear impulsive boundary value on the half line in Banach spaces. Moreover, the problems have singular nature at the boundary. Our main technique is a new fixed point index theory established in Section 2 for cone mappings which are not strict set contractions. Finally, we give an example in Section 4.

Let E be a Banach Space, θ be its zero element, and P be a solid cone in E. We introduce in E an order relation by defining $x \leq y$ if $y - x \in P$, for

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$x, y \in E$. Further we suppose P is a normal cone, i.e. there exists a constant (called normal constant) $N > 0$ such that $\|x\| \leq N\|y\|$ for all $\theta \leq x \leq y$.

Fix $0 < t_1 < t_2 < \cdots < t_m < +\infty$, and put

$$\begin{aligned} PC([0,+\infty),E) = \{x|\ & x = x(t) \text{ is a function from } [0,+\infty) \text{ into } E, \\ & \text{continuous at } t \neq t_k, \text{ left continuous at } t = t_k, \\ & \text{and } x(t_k^+) = \lim_{t\to t_k+0} x(t) \text{ exist } (1 \leq k \leq m)\}. \end{aligned}$$

Take $p \in PC([0,+\infty),R^1) \cap C^1(0,+\infty)$, $p(t) > 0$ for $t \in R^1 = (0,+\infty)$, and $f \in C(R^1 \times E,\, P)$.

Now we consider the following impulsive boundary value problem on the half-line $[0,+\infty)$: To find $x \in PC([0,+\infty),E)$ such that

$$\begin{cases} (Lx)(t) + f(t,x(t)) = \theta, \quad t \neq t_k, k = 1,2,\cdots,m; \\ \Delta x|_{t_k} = (I_k x)(t_k), 0 < t_1 < t_2 < \cdots < t_k < \cdots < t_m; \\ \lambda x(0) - \beta \lim\limits_{t\to 0} p(t)x'(t) = a; \\ \gamma x(\infty) + \delta \lim\limits_{t\to\infty} p(t)x'(t) = b; \\ x(t) \text{ is bounded on } [0,+\infty). \end{cases} \tag{1.1}$$

in which $(Lx)(t) = \frac{1}{p(t)}(p(t)x(t))'$, $a, b \in P$ are given elements,

$$\Delta x|_{t_k} = \lim_{\varepsilon\to 0^+}[x(t_k+\varepsilon) - x(t_k-\varepsilon)]. \tag{1.2}$$

and $I_k \in C(E,P)$. Further λ, β, γ, $\delta \geq 0$ are given constants with $\beta\gamma + \lambda\delta + \lambda\gamma > 0$. For later use, we define for $n \in N$

$$\begin{aligned} PC([\tfrac{1}{n},n],E) = \{x|\ & x = x(t) \text{ is a function from } [\tfrac{1}{n},n] \text{ into } E, \\ & \text{continuous at } t \neq t_k, \text{ left continuous at } t = t_k, \\ & \text{and } x(t_k^+) = \lim_{t\to t_k+0} x(t) \text{ exist } (1 \leq k \leq m)\}. \end{aligned}$$

For $x, y \in PC([0,+\infty),E)$, we define

$$d(x,y) = \sum_{i=1}^{+\infty} \frac{1}{2^i} \frac{\rho_i(x-y)}{1+\rho_i(x-y)},$$

where $\rho_i(x-y) = \sup\limits_{t\in[0,i]}\{\|x(t)-y(t)\|\}$. Clearly $PC([0,+\infty),E)$ is a locally convex space with the topology defined by the distance $d(x,y)$ defined above. The following conditions will be assumed throughout.

$$\int_0^{+\infty} \frac{dt}{p(t)} < +\infty. \tag{1.3}$$

Denote $\tau_0(t) = \int_0^t \frac{1}{p(s)} ds$, $\tau_\infty(t) = \int_t^\infty \frac{1}{p(s)}\, ds$, $\rho^2 = \beta\gamma + \lambda\delta + \lambda\gamma \int_0^\infty \frac{1}{p(t)} dt$, and $\rho > 0$. Define

$$u(t) = \frac{1}{\rho}[\delta + \gamma\tau_\infty(t)], \quad v(t) = \frac{1}{\rho}[\beta + \lambda\tau_0(t)], \tag{1.4}$$

Then $\gamma v + \lambda u \equiv \rho$. Let

$$G(t,s) = \begin{cases} u(t)v(s)p(s), & 0 \le s \le t < \infty \\ v(t)u(s)p(s), & 0 \le t \le s < \infty \end{cases} \tag{1.5}$$

$$e(t) = \frac{1}{\rho^2}[b\lambda\tau_0(t) + a\gamma\tau_\infty(t)] + \frac{1}{\rho^2}(a\delta + b\beta) \tag{1.6}$$

From (1.4), (1.5) and (1.6), there exist $t_m < a^* < b^* < +\infty$ and $1 \ge c^* = c^*(a^*, b^*) > 0$ such that

$$G(t,s) \ge c^* G(r,s) \quad \text{for} \quad t \in [a^*, b^*], r \in [0, +\infty), s \in [0, +\infty), \tag{1.7}$$

$$e(t) \ge c^* e(s) \quad \text{for} \quad t \in [a^*, b^*], s \in [0, +\infty), \tag{1.8}$$

$$\delta + \gamma\tau_\infty(t) \ge c^*[\delta + \gamma\tau_\infty(s)] \quad \text{for} \quad t \in [a^*, b^*], s \in [0, +\infty). \tag{1.9}$$

Write $Q = \{x \in PC([0, +\infty), E), x(t) \ge \theta$ with $x(t) \ge c^* x(s)$ for $t \in [a^*, b^*], s \in [0, +\infty)\}$ $Q_n = \{x \in PC([\frac{1}{n}, n], E), x(t) \ge \theta$ with $x(t) \ge c^* x(s)$ for $t \in [a^*, b^*], s \in [\frac{1}{n}, n]\}$ $(n > \max\{b^*, \frac{1}{t_1}\})$.

Let us list the following conditions for later use:

(H_1) (a) $f \in C(R^+ \times E, P), I_k \in C(E, P)(k = 1, 2, \cdots, m)$ and for any $n > 0, r > 0$, $I_k(\overline{B}_r)(k = 1, 2, \cdots, m)$ is bounded and $f(t,x)$ is uniformly continuous on $[\frac{1}{n}, n] \times \overline{B}_r$, where $\overline{B}_r = \{x \in E, \|x\| \le r\}$;

(b) there exist $\phi \in C(R^+, [0, +\infty))$ and $\{M_k\}_{k=1}^\infty$ such that for any bounded $D \subseteq E$ and $t \in R^+$

$$\alpha(f(t, D) \le \phi(t)\alpha(D),$$

$$\alpha(I_k(D)) \le M_k \alpha(D), k = 1, 2, \cdots, m$$

where $\alpha(D)$ denotes the Kuratowski measure of noncompactness of bounded D in Banach space E[10, p.41] and

$$q = \sup_{t\in[0,+\infty)} [\int_0^{+\infty} G(t,s)\phi(s) ds + (\delta + \gamma\tau_\infty(t)) \sum_{k=1}^m \frac{M_k}{\delta + \gamma\tau_\infty(t_k)}] < 1;$$

(c) there exist $R > 0, \psi \in C(R^+, [0, +\infty)), \Phi \in C(E, E)$ which is uniformly continuous on any bounded set with $\|f(t,x)\| \le \psi(t)\|\Phi(x)\|$ for $(t,x) \in R^+ \times E$. Moreover,

$$\sup_{t\in[0,+\infty)} \|e(t)\| + [\sup_{t\in[0,+\infty)} \int_0^{+\infty} G(t,s)\psi(s) ds] \cdot \sup\{\|\Phi(x)\|, x \in \overline{B}_R\}$$

$$+\sup_{t\in[0,+\infty)}[\delta+\gamma\tau_\infty(t)]\sup_{x\in B_R}\sum_{k=1}^{m}\frac{\|(I_k(x)\|}{\delta+\gamma\tau_\infty(t_k)}<\frac{R}{N};$$

where $N \geq 1$ is the normal-constant of P;

(H_2) there exists a $g \in P^{o*}$ such that

$$\lim_{\|x\|\to+\infty}\frac{g(f(t,x))}{g(x)}=+\infty$$

uniformly for $t \in [a^*, b^*]$, where $P^{o*} = \{g \in E^* : g(x) > 0 \quad \text{for} \quad x > \theta\}$.

(H_3) there exist a $g \in P^{o*}$ such that

$$\lim_{\|x\|\to 0}\frac{g(f(t,x))}{g(x)}=+\infty$$

uniformly for $t \in [a^*, b^*]$, where $P^{o*} = \{g \in E^* : g(x) > 0 \quad \text{for} \quad x > \theta\}$.

Remark 1. Condition (H_1) is widely applied in [7][11] and can easily be satisfied.

§2. ESTABLISHMENT OF FIXED POINT INDEX THEORY

First we will establish the degree theory for operators which are not strict set contractions. Assume that A is an operator from a bounded set $S \subseteq PC(I,E)$ into $PC(I,E)$, where $I = [t', t''], t_1 > t' > 0, t'' > b^*$. For $x \in PC(I,E)$, define $\|x\|_I = \sup\{\|x(t)\|, t \in I\}$, $Q_I = \{x \in PC(I,E), x(t) \geq \theta \quad \text{with} \quad x(t) \geq c^*x(s) \quad \text{for} \quad t \in [a^*, b^*], s \in I\}$. We now give a new definition.

Definition 2.1. Let $A : S \to PC(I,E)$. We will say that A satisfies the S-r-S condition iff A is a bounded and continuous operator, $A(S)$ is piecewise equicontinuous, and

$$\alpha(A(D)) \leq r\alpha(D)$$

for any bounded and piecewise equicontinuous $D \subseteq S$, where $0 \leq r < 1$ is a constant.

If $A : PC(I,E) \to PC(I,E)$ and satisfy the S-r-S condition for any bounded $S \subseteq PC(I,E)$, we will say that A satisfies S-r-S condition on $PC(I,E)$.

Clearly this definition is different from the definition of the strict set contractions(see[10],[12],[15]).

Let $\Omega \subseteq PC(I,E)$ be open and bounded, and $A : \overline{\Omega} \to PC(I,E), h = id - A$, where id denotes the *indentity* operator.

Lemma 2.1. *Let $A : \overline{\Omega} \to PC(I, E)$ satisfy the S-r-S condition, then*

1) *h is proper, i.e., $h^{-1}(D)$ is compact for any compact set $D \subseteq PC(I, E)$;*
2) *h is closed, i.e., $h(S)$ is closed for any closed set $S \subseteq \overline{\Omega}$.*

The proof of Lemma 2.1 is similar to those of strict set contractions in [10](Proposition 9.1, p.70).

Lemma 2.2. *If $D \subseteq PC(I, E)$ is bounded and piecewise equicontinuous, then $\overline{co}(D)$ is bounded and piecewise equicontinuous.*

The proof of Lemma 2.2 is easy and can be omitted.

Lemma 2.3. *Let $\{S_i\} \subseteq E$ be bounded, closed and $S_1 \supseteq S_2 \supseteq S_3 \supseteq \cdots \supseteq S_n \supseteq \cdots$, $S_n \neq \emptyset, n = 1, 2, 3, \cdots$. If $\alpha(S_n) \to 0$, then $S = \bigcap\limits_{i=1}^{\infty} S_i$ is a nonempty and compact set.*

The proof is similar to the proof of Theorem 9.1 in [10](P.71).

Now we give the definition of the degree for our operators.

Definition 2.2. Let $\Omega \subseteq PC(I, E)$ be open and bounded, $A : \overline{\Omega} \to PC(I, E)$ satisfies S-r-S condition, $0 \leq r < 1, h = id - A$,

(1) Assume that $\theta \notin h(\partial\Omega)$. Let $D_1 = \overline{co}(A(\overline{\Omega}))$ and $D_n = \overline{co}(A(D_{n-1} \cap \overline{\Omega})), n = 2, 3, \cdots$.

1) If there exists n_0 such that $D_{n_0} = \emptyset$, then we define $deg(h, \Omega, \theta) = 0$.

2) If $D_n \neq \emptyset$ for $n = 1, 2, \cdots$, then $D_n \cap \overline{\Omega}$ is bounded and closed($n = 1, 2, \cdots$.). Let $D = \bigcap\limits_{i=1}^{\infty} D_n$. Then D is bounded, convex, closed and nonempty. Obviously $D_1 \supseteq D_2$. If $D_{n-1} \supseteq D_n$, then $D_n = \overline{co}(A(D_{n-1} \cap \overline{\Omega})) \supseteq \overline{co}(A(D_n \cap \overline{\Omega})) = D_{n+1}$. So $D_{n-1} \supseteq D_n, n = 2, 3, \cdots$. By Lemma 2.2, D is piecewise equicontinuous and

$$\alpha(D_n) = \alpha(A(D_{n-1} \cap \overline{\Omega})) \leq r\alpha(D_{n-1} \cap \overline{\Omega}) \leq r\alpha(D_{n-1}),$$

So $\alpha(D_n) \leq r^{n-1}\alpha(D_1)$. By $r < 1$ and Lemma 2.3, we know D is nonempty and compact. Because of $D_{n-1} \cap \overline{\Omega} \supseteq D_n \cap \overline{\Omega}$ and $\alpha(D_n \cap \overline{\Omega}) \to 0$, we know $D \cap \overline{\Omega} = (\bigcap\limits_{n=1}^{\infty} D_n) \cap \overline{\Omega}$ is nonempty and compact. On the other hand, from

$$A(D_n \cap \overline{\Omega}) \subseteq \overline{co}(A(D_{n-1} \cap \overline{\Omega})) = D_n.$$

we have

$$A(D \cap \overline{\Omega}) \subseteq \bigcap_{n=1}^{\infty} A(D_n \cap \overline{\Omega}) \subseteq \bigcap_{n=1}^{\infty} D_n = D. \tag{2.1}$$

Since D is compact, $A : D \cap \overline{\Omega} \to D$ is completely continuous. So by the extention theorem of completely continuous operator (see propsition 8.3, p.56 in [10]), there exists a completely continuous operator $A_1 : \overline{\Omega} \to D$ such that $A_1x = Ax$ for all $x \in D \cap \overline{\Omega}$. Let $h_1 = id - A_1$. It is easy to see that $\theta \notin h_1(\partial\Omega)$(In fact, if there exists a $x \in \partial\Omega$ such that $x - A_1x = \theta$, then $x = A_1x$, which implies that $x \in D$. Therefore, $x = A_1x = Ax$, which contradict $\theta \notin (id - A)(\partial\Omega)$. So $deg_{\mathrm{LS}}(h_1, \Omega, \theta)$ is well be defined. Define

$$deg(h, \Omega, \theta) = deg_{\mathrm{LS}}(h_1, \Omega, \theta). \tag{2.2}$$

It is easy to see that the above definition is independent on h_1. In fact, let $A_2 : \overline{\Omega} \to D$ be another extension of A, and $h_2 = id - A_2$. Suppose that $H(t, x) = x - tA_1x - (1 - t)A_2x, x \in \overline{\Omega}, 0 \le t \le 1$. We will prove $H(t, x) \ne \theta$ for $t \in [0, 1]$ and $x \in \partial\Omega$. On the other hand, if there exist $t_0, 0 \le t_0 \le 1$, and $x_0 \in \partial\Omega$ such that $H(t_0, x_0) = \theta$, i.e., $x_0 = t_0A_1x_0 + (1 - t_0)A_2x_0$. Since $A_1x_0 \in D, A_2x_0 \in D$, and D is convex, we know $x_0 \in D$. So $x_0 = t_0A_1x_0 + (1 - t_0)A_2x_0 = A_1x_0$. This contradicts to $\theta \notin h(\partial\Omega)$. Thus we have

$$deg_{\mathrm{LS}}(h_1, \Omega, \theta) = deg_{\mathrm{LS}}(h_2, \Omega, \theta). \tag{2.3}$$

So the definition is not related to the choice of h_1.

(2) Suppose $p \notin h(\partial\Omega)$. It is easy to see $\theta \notin (h-p)(\partial\Omega)$ and thus we define

$$deg(h, \Omega, p) = deg(h - p, \Omega, \theta). \tag{2.4}$$

Remark 2. If A has a fixed point $x' \in \overline{\Omega}$, we have $D_n \cap \Omega \ne \emptyset, n = 1, 2, \cdots$. So the fixed points set $F \subseteq D \cap \overline{\Omega}$. And the above degree theory has the similar properties and fixed theorems as those for strict set contractions.

Remark 3. Similar to Definition 2.2, we can define the fixed point index for cone mappings which satisfy the S-r-S conditions and can obtain similar properties such as the normaliztion, the additivity, the homotopy invariance, and the permanence property in fixed point index theory for strict set contractions(see[10], p.238). And moreover, the following theorem is true.

Theorem 2.1. *Assume $\Omega \subseteq PC(I, E)$ is bounded and open and $A : \Omega \to Q_I$ satifies the S-r-S condition.*

(a) *If $Ax \not\le x$ for $x \in \partial(\Omega \cap Q_I)$, then*

$$i(A, \Omega \cap Q_I, Q_I) = 0;$$

(b) *if $Ax \not\ge x$ for $x \in \partial(\Omega \cap Q_I)$, then*

$$i(A, \Omega \cap Q_I, Q_I) = 1.$$

Proof. (a) Choose $\mu_0 \in (Q_I - \Omega \cap Q_I)$. If there exist $x \in \partial(\Omega \cap Q_I)$ and $0 \le t_0 \le 1$ such that $Ax + t_0\mu_0 = x$, then $Ax \le x$. This is a contradiction. Then $i(A+\mu_0, \Omega \cap Q_I, Q_I) = i(A, \Omega \cap Q_I, Q_I)$. On the other hand, $A+\mu_0$ has not a fixed point in $\overline{\Omega \cap Q_I}$. So $i(A+\mu_0, \Omega \cap Q_I, Q_I) = 0$, i.e. $i(A, \Omega \cap Q_I, Q_I) = 0$.

(b) Similarly, if there exist a $x \in \partial(\Omega \cap Q_I)$ and a $0 \le t_0 \le 1$ such that $t_0 Ax = x$, then $Ax \ge x$. This is a contradiction. Then $i(A, \Omega \cap Q_I, Q_I) = i(\theta, \Omega \cap Q_I, Q_I) = 1$. □

§3. EXISTENCE OF MULTIPLE SOLUTIONS

In this section, we will give two existence theorems. Similar to [7][9], we can prove that for $x \in PC([0,+\infty), E)$ is bounded and satisfies

$$x(t) = e(t) + (Ax)(t) + (Bx)(t), \tag{3.1}$$

then x is a solution of equation (1.1), where

$$(Ax)(t) = \int_0^{+\infty} G(t,s) f(s, x(s)) ds,$$

$$(Bx)(t) = [\delta + \gamma\tau_\infty(t)] \sum_{0<t_k<t} \frac{(I_k x)(t_k)}{\delta + \gamma\tau_\infty(t_k)}.$$

For $x \in Q$, let

$$(Jx)(t) = e(t) + (Ax)(t) + (Bx)(t), \tag{3.2}$$

$$(A_n x)(t) = \int_{\frac{1}{n}}^{n} G(t,s) f(s, x(s)) ds,$$

$$(J_n x)(t) = e(t) + (A_n x)(t) + (Bx)(t), n > \max\{b^*, \frac{1}{t_1}\}. \tag{3.3}$$

Let I be a is bounded closed interval. First we need the following lemmas.

Lemma 3.1(see [14]). *If $S \subseteq PC(I, E)$ is bounded and equicontinuous, then*

$$\alpha(\{\int_I x(t)dt, x \in S\}) \le \int_I \alpha(S(t))dt. \tag{3.4}$$

Lemma 3.2(see [13]). *If $S \subseteq PC(I, E)$ is bounded and piecewise equicontinuous, then*

$$\alpha(S) = \sup\{\alpha(S(t)), t \in I\}.$$

Lemma 3.3. *Assume* (H_1) *is true and* $D = \{x \in PC([0,+\infty),E), \sup_{t\in[0,+\infty)} \|x(t)\| \le R\}$. *Then*

(1) $J : D \to Q$ *is continuous.*

(2) $J_n : Q_n \to Q_n (n > \max\{b^*, \frac{1}{t_1}\})$ *are continuous and the S-q-S condition for* $J_n : Q_n \to Q_n$ *is satisfied for all* $n > \max\{b^*, \frac{1}{t_1}\}$.

Proof. For $x \in Q$, we have $(Jx)(t) \ge 0$. And from (1.7), (1.8) and (1.9), for $t \in [a^*, b^*]$ we have

$$\begin{aligned}
& (Jx)(t) \\
= \;& e(t) + \int_0^{+\infty} G(t,s)f(s,x(s))ds + [\delta + \gamma\tau_\infty(t)] \sum_{k=1}^{m} \frac{(I_k x)(t_k)}{\delta + \gamma\tau_\infty(t_k)} \\
\ge \;& c^*[e(u) + \int_0^{+\infty} G(u,s)f(s,x(s))ds + [\delta + \gamma\tau_\infty(u)] \sum_{k=1}^{m} \frac{(I_k x)(t_k)}{\delta + \gamma\tau_\infty(t_k)}] \\
= \;& c^*(Jx)(u)
\end{aligned}$$

for any $u \in [0,+\infty)$. Therefore $JQ \subseteq Q$. If $x_n \to x_0, x_n, x_0 \in D$, by virtue of the domimated convergence theorem, we have

$$\lim_{n\to+\infty} \int_0^{+\infty} G(t,s)f(s,x_n(s))ds = \int_0^{+\infty} G(t,s)f(s,x_0(s))ds.$$

So $J : D \to Q$ is continuous and bounded. Similarly, we have $J_n : Q_n \to Q_n$ continuous and bounded for all $n > \max\{b^*, \frac{1}{t_1}\}$.

For any bounded $D \subseteq PC([\frac{1}{n}, n], E)$, if D is piecewise equicontinuous, $\{y, y(t) = f(t, x(t)), x \in D\}$ is piecewise equicontinuous on $[\frac{1}{n}, n]$. By Lemma 3.1, we have

$$\begin{aligned}
& \alpha(J_n D(t)) \\
= \;& \alpha(\{e(t) + (A_n x)(t) + (Bx)(t), x \in D\}) \\
\le \;& \alpha(\{(A_n x)(t), x \in D\}) + \alpha(\{(Bx)(t), x \in D\}) \\
\le \;& \int_{\frac{1}{n}}^{n} G(t,s)\alpha(f(s,D(s))ds + (\delta + \gamma\tau_\infty(t)) \sum_{k=1}^{m} \frac{\alpha(I_k(D(t_k))}{\delta + \gamma\tau_\infty(t_k)} \\
\le \;& \int_{\frac{1}{n}}^{n} G(t,s)\phi(s)\alpha(D(s))ds + (\delta + \gamma\tau_\infty(t)) \sum_{k=1}^{m} \frac{M_k\alpha(D)}{\delta + \gamma\tau_\infty(t_k)} \\
\le \;& \int_{\frac{1}{n}}^{n} G(t,s)\phi(s)ds\alpha(D) + (\delta + \gamma\tau_\infty(t)) \sum_{k=1}^{m} \frac{M_k}{\delta + \gamma\tau_\infty(t_k)}\alpha(D) \\
\le \;& [\int_0^{+\infty} G(t,s)\phi(s)ds + (\delta + \gamma\tau_\infty(t)) \sum_{k=1}^{m} \frac{M_k}{\delta + \gamma\tau_\infty(t_k)}]\alpha(D).
\end{aligned}$$

So by Lemma 3.2 and (H_1)

$$\alpha(J_n D) \leq q\alpha(D).$$

Thus $J_n : Q_n \to Q_n$ satisfies S-q-S condition for all $n > \max\{b^*, \frac{1}{t_1}\}$. The proof is complete.□

Theorem 3.1. *Let* $D = \{x \in PC([0,+\infty), E), \sup_{t\in[0,+\infty)} \|x(t)\| \leq R\}$ *and* $D_n = \{x|_{[\frac{1}{n},n]}, x \in D\}$. *Assume the condition* (H_1) *holds and* $x_n \in D_n$ *and*

$$([0,+\infty)_n x_n)(t) = x_n(t), t \in [\frac{1}{n}, n], n > \max\{b^*, \frac{1}{t_1}\}.$$

Then there exists $x^* \in D$ *such that* $(Jx^*)(t) = e(t) + (Ax^*)(t) + (Bx^*)(t)$.

Proof. Let

$$x_n^*(t) = \begin{cases} x_n(\frac{1}{n}), & t \in [0, \frac{1}{n}]; \\ x_n(t), & t \in (\frac{1}{n}, n] \\ x_n(n), & t \in (n, +\infty). \end{cases}$$

Then $\{x_n^*\} \subseteq D$. And from Lemma 3.1, for any $t \in (0,+\infty)$, there exists a $k > 0$ such that $t \in [\frac{1}{k}, k]$. Hence

$$\begin{aligned} & \alpha(\{x_n^*(t)\}) \\ = \ & \alpha(\{x_n^*(t)\}_{n\geq k}) \\ = \ & \alpha(\{e(t) + \int_{\frac{1}{n}}^{n} G(t,s)f(s,x_n^*(s))ds + (\delta+\gamma\tau_\infty(t))\sum_{k=1}^{m}\frac{I_k(x_n^*(t_k))}{\delta+\gamma\tau_\infty(t_k)}\}_{n\geq k}) \\ \leq \ & \alpha(\{\int_{\frac{1}{n}}^{n} G(t,s)f(s,x_n^*(s))ds\}_{n\geq k}) + (\delta+\gamma\tau_\infty(t))\sum_{k=1}^{m}\frac{\alpha(I_k(\{x_n^*(t_k)\}_{n\geq k})}{\delta+\gamma\tau_\infty(t_k)} \\ \leq \ & \alpha(\{\int_{\frac{1}{n}}^{n} G(t,s)f(s,x_n^*(s))ds\}) + (\delta+\gamma\tau_\infty(t))\sum_{k=1}^{m}\frac{\alpha(I_k(\{x_n^*(t_k)\})}{\delta+\gamma\tau_\infty(t_k)}. \end{aligned}$$

Now for any $T > 0$

$$\begin{aligned} & \alpha(\{\int_{\frac{1}{n}}^{n} G(t,s)f(s,x_n^*(s))ds\}) \\ = \ & \alpha(\{\int_{\frac{1}{n}}^{n} G(t,s)f(s,x_n^*(s))ds\}_{n\geq T}) \\ = \ & \alpha(\{\int_{\frac{1}{n}}^{\frac{1}{T}} G(t,s)f(s,x_n^*(s))ds + \int_{\frac{1}{T}}^{T} G(t,s)f(s,x_n^*(s))ds \end{aligned}$$

$$
\begin{aligned}
&+\int_T^n G(t,s)f(s,x_n^*(s))ds\}_{n\geq T})\\
\leq\;& \alpha(\{\int_{\frac{1}{n}}^{\frac{1}{T}} G(t,s)f(s,x_n^*(s))ds\}_{n\geq T}) + \alpha(\{\int_{\frac{1}{T}}^{T} G(t,s)f(s,x_n^*(s))ds\}_{n\geq T})\\
&+\alpha(\{\int_T^n G(t,s)f(s,x_n^*(s))ds\}_{n\geq T})\\
\leq\;& \alpha(\{\int_{\frac{1}{n}}^{\frac{1}{T}} G(t,s)f(s,x_n^*(s))ds\}) + \int_{\frac{1}{T}}^{T} G(t,s)\alpha(\{f(s,x_n^*(s))\})ds\\
&+\alpha(\{\int_T^n G(t,s)f(s,x_n^*(s))ds\})
\end{aligned}
$$

By virtue of condition (c) in (H_1), we have

$$
\begin{aligned}
&\|\int_{\frac{1}{n}}^{\frac{1}{T}} G(t.s)f(s,x_n^*(s))ds\| \leq \int_{\frac{1}{n}}^{\frac{1}{T}} G(t.s)\|f(s,x_n^*(s))\|ds\\
\leq\;& \int_{\frac{1}{n}}^{\frac{1}{T}} G(t,s)\psi(s)\Phi(\|x_n^*(s))\|ds \leq \int_{\frac{1}{n}}^{\frac{1}{T}} G(t,s)\psi(s)ds\sup\{\Phi(x),|x|\leq R\}\\
\leq\;& \int_0^{\frac{1}{T}} G(t,s)\psi(s)ds\sup\{\Phi(x),|x|\leq R\}
\end{aligned}
$$

and

$$
\begin{aligned}
&\|\int_T^n G(t.s)f(s,x_n^*(s))ds\| \leq \int_T^n G(t.s)\|f(s,x_n^*(s))\|ds\\
\leq\;& \int_T^n G(t,s)\psi(s)\Phi(\|x_n^*(s))\|ds \leq \int_T^n G(t,s)\psi(s)ds\sup\{\Phi(x),|x|\leq R\}\\
\leq\;& \int_T^{+\infty} G(t,s)\psi(s)ds\sup\{\Phi(x),|x|\leq R\}.
\end{aligned}
$$

So

$$
\begin{aligned}
&\alpha(\{\int_{\frac{1}{n}}^{n} G(t,s)f(s,x_n^*(s))ds\})\\
\leq\;& 2\int_0^{\frac{1}{T}} G(t,s)\psi(s)ds\sup\{\Phi(x),|x|\leq R\} + \int_{\frac{1}{T}}^{T} G(t,s)\alpha(\{f(s,x_n^*(s))\})ds\\
&+2\int_T^{+\infty} G(t,s)\psi(s)ds\sup\{\Phi(x),|x|\leq R\}\\
=\;& 2\int_0^{\frac{1}{T}} G(t,s)\psi(s)ds\sup\{\Phi(x),|x|\leq R\} + \int_{\frac{1}{T}}^{T} G(t,s)\phi(s)\alpha(\{x_n^*(s)\})ds\\
&+2\int_T^{+\infty} G(t,s)\psi(s)ds\sup\{\Phi(x),|x|\leq R\}.
\end{aligned}
$$

Letting $T \to +\infty$, we get

$$\begin{aligned} &\alpha(\{x_n^*(t)\}) \\ \leq\ & [\int_0^{+\infty} G(t,s)\phi(s)ds + (\delta + \gamma\tau_\infty(t)) \sum_{k=1}^{m} \frac{M_k}{\delta + \gamma\tau_\infty(t_k)}] \sup_{t\in[0,+\infty)} \alpha(\{x_n^*(t)\}). \end{aligned}$$

Thus

$$\sup_{t\in[0,+\infty)} \alpha(\{x_n^*(t)\}) \leq q \sup_{t\in[0,+\infty)} \alpha(\{x_n^*(t)\}).$$

Consequently $\sup_{t\in[0,+\infty)} \alpha(\{x_n^*(t)\}) = 0$. Since $\{x_n^*\}$ is piecewise equicontinuous, for any $T > 0$, $\alpha(\{x_n^*|_{[0,T]}\}) = 0$. Thus $\{x_n^*\}$ is relatively compact.

So there exists a subsequence $\{{x_{n_j}}^*\}$ such that

$$ {x_{n_j}}^* \to x^*, j \to +\infty. \tag{3.5}$$

By virtue of (H_1), we get

$$B{x_{n_j}}^* \to Bx^*, j \to +\infty. \tag{3.6}$$

And the domimated covergence theorem implies

$$\lim_{j\to+\infty} \int_0^{+\infty} G(t,s)\|f(s,{x_{n_j}}^*(s)) - f(s,x^*(s))\|ds = 0. \tag{3.7}$$

From (3.5), (3.7) and the domimated convergence theorem, we have

$$\begin{aligned} & \lim_{j\to+\infty}[(A_{n_j}x_{n_j})(t) - (Ax^*)(t)] \\ =\ & \lim_{j\to+\infty}[(A_{n_j}{x_{n_j}}^*)(t) - (Ax^*)(t)] \\ =\ & \lim_{j\to+\infty}[\int_0^{+\infty} G(t,s)f(s,{x_{n_j}}^*(s))ds - \int_{n_j}^{+\infty} G(t,s)f(s,{x_{n_j}}^*(s))ds \\ & - \int_0^{\frac{1}{n_j}} G(t,s)f(s,{x_{n_j}}^*(s))ds - \int_0^{+\infty} G(t,s)f(s,x^*(s))ds] \\ =\ & \lim_{j\to+\infty}[\int_0^{+\infty} G(t,s)(f(s,{x_{n_j}}^*(s)) - f(s,x^*(s))ds] \\ & - \lim_{j\to+\infty}[\int_{n_j}^{+\infty} G(t,s)f(s,{x_{n_j}}^*(s))ds + \int_0^{\frac{1}{n_j}} G(t,s)f(s,{x_{n_j}}^*(s))ds] \\ =\ & 0. \end{aligned} \tag{3.8}$$

In virtue of the continuity of J, we have

$$\lim_{j\to+\infty}(J{x_{n_j}}^*)(t) = (Jx^*)(t), t \in (0,+\infty). \tag{3.9}$$

From (3.6) and (3.8), we have

$$\lim_{j\to+\infty}[(J_{n_j}x_{n_j})(t)-(Jx_{n_j}{}^*)(t)] = \lim_{j\to+\infty}[(A_{n_j}x_{n_j})(t)-(Ax_{n_j}{}^*)(t)] = 0. \quad (3.10)$$

Consequently

$$\lim_{j\to+\infty}(J_{n_j}x_{n_j})(t) = \lim_{j\to+\infty}(Jx_{n_j}{}^*)(t) = \lim_{j\to+\infty}(Jx^*)(t). \quad (3.11)$$

On the other hand, since

$$\lim_{j\to+\infty}(J_{n_j}x_{n_j})(t) = \lim_{j\to+\infty}x_{n_j}(t) = \lim_{j\to+\infty}x_{n_j}{}^*(t) = x^*(t). \quad (3.12)$$

Thus by (3.11) and (3.12), we have

$$x^*(t) = (Jx^*)(t) = e(t) + (Ax^*)(t) + (Bx^*)(t).$$

The proof is complete.□

The following theorem is based on Theorem 3.1 and Theorem 2.1.

Theorem 3.2. *Assume conditions* $(H_1), (H_2)$ *are satisfied and* $a(\gamma+\delta)+b(\alpha+\beta) > \theta$*, then equation* (1.1) *has at least two positive solutions.*

Proof. Since $a(\gamma+\delta)+b(\alpha+\beta) > \theta$, we have $e(t) > \theta$ all $t > 0$. It is easy to see that $\inf_{t\in[\frac{1}{n},n]}\|e(t)\| > 0$ for any $n > \max\{b^*, \frac{1}{t_1}\}$. Let $r_n = \frac{1}{2N}\inf_{t\in[\frac{1}{n},n]}\|e(t)\|$ and $r = \sup_{t\in J}\|e(t)\|$. In virtue of (H_2), we can choose $R' > \max\{\frac{2R}{c^*}, r\} > 0$ such that

$$g(f(t,x)) \geq N^*g(x) \quad (3.13)$$

for $\|x\| \geq R'$, where

$$N^* > 2(\inf_{t\in[a^*,b^*]}\int_{a^*}^{b^*}G(t,s)ds)^{-1}$$

and c^*, a^*, b^*, R are defined in Section 1. Write $B_{1,n} = \{x \in PC([\frac{1}{n},n],E),\ \|x\|_{[\frac{1}{n},n]} < r_n\}, B_{2,n} = \{x \in PC([\frac{1}{n},n],E), \|x\|_{[\frac{1}{n},n]} < R\}$, and $B_{3,n} = \{x \in PC([\frac{1}{n},n],E), \|x\|_{[\frac{1}{n},n]} < \frac{NR'}{c^*}\}$. For $x \in \partial(Q_n \cap B_{1,n})$, in virtue of $(J_nx)(t) \geq e(t)$, we have $\inf_{t\in[\frac{1}{n},n]}\|(J_nx)(t)\| \geq \frac{1}{N}\inf_{t\in[\frac{1}{n},n]}\|e(t)\| > \sup_{t\in[\frac{1}{n},n]}\|x(t)\|$. Hence

$$J_nx \not\leq x. \quad (3.14)$$

Now we will prove that

$$J_n x \not\geq x, x \in \partial(Q_n \cap B_{2,n}). \tag{3.15}$$

In fact, if $J_n x \geq x$, then from (H_1) we have

$$\begin{aligned} & x(t) \leq (J_n x)(t) \\ = \ & e(t) + (A_n x)(t) + (B_n x)(t) \\ = \ & e(t) + \int_{\frac{1}{n}}^{n} G(t,s) f(s, x(s)) ds + (Bx)(t). \end{aligned}$$

So

$$\|x(t)\| \leq N[\|e(t)\| + \int_{\frac{1}{n}}^{n} G(t,s)\|f(s,x(s))\|ds + [\delta + \gamma\tau_\infty(t)] \sum_{k=1}^{m} \frac{\|(I_k x)(t_k)\|}{\delta + \gamma\tau_\infty(t_k)}].$$

Moreover, (H_1) yields

$$\|x\|_{[\frac{1}{n},n]} \leq N[\sup_{t\in J}\|e(t)\| + [\sup_{t\in J}\int_0^{+\infty} G(t,s)\psi(s)ds] \sup_{x\in[0,R]} \Phi(x)$$

$$+ \sup_{t\in J}(\delta + \gamma\tau_\infty(t)) \sup_{x\in(Q_n\cap\overline{B_2})} \sum_{k=1}^{m} \frac{\|(I_k x)(t_k)\|}{\delta + \gamma\tau_\infty(t_k)}] < R.$$

This contradicts $x \in \partial(Q_n \cap B_{2,n})$ because $\|x\| = R$ for all $x \in \partial(Q_n \cap B_{2,n})$. Therefore (3.15) is true.

Next we will show that

$$J_n x \not\leq x, x \in \partial(Q_n \cap B_{3,n}). \tag{3.16}$$

In fact, if there exists a $x \in \partial(B_{3,n} \cap Q_n)$ with $J_n x \leq x$, then for $t \in [a^*, b^*]$, $x(t) \geq c^* x(s)$ for all $s \in [\dfrac{1}{n}, n]$, which implies that $\inf\limits_{t\in[a^*,b^*]}\|x(t)\| \geq \dfrac{c^*}{N}\dfrac{NR'}{c^*} = R'$. Because of $g \in P^{\mathrm{o}*}$, we get $\inf_{t\in[a^*,b^*]} g(x(t)) > 0$. Now for $t \in [a^*, b^*]$, if $J_n x \leq x$, then from (3.13) we have

$$(J_n x)(t) \geq (A_n x)(t) = \int_{\frac{1}{n}}^{n} G(t,s) f(s, x(s)) ds.$$

So for $t \in [a^*, b^*]$ we get

$$g(x(t)) \geq g((J_n x)(t)) \geq \int_{a^*}^{b^*} G(t,s) g(f(s, x(s))) ds$$

$$\geq N^* \int_{a^*}^{b^*} G(t,s) g(x(s)) ds \geq N^* \int_{a^*}^{b^*} G(t,s) ds \inf_{t\in[a^*,b^*]} g(x(t)).$$

Hence

$$\inf_{t\in[a^*,b^*]} g(x(t)) \geq 2 \inf_{t\in[a^*,b^*]} g(x(t)),$$

which implies $\inf_{t\in[a^*,b^*]} g(x(t)) = 0$. This is a contradiction. So (3.16) is true.

By (3.14),(3.15),(3.16) and Theorem 2.1, we have

$$i(J_n, Q_n \cap B_{1,n}, Q_n) = 0,$$

$$i(J_n, Q_n \cap B_{2,n}, Q_n) = 1,$$

$$i(J_n, Q_n \cap B_{3,n}, Q_n) = 0.$$

Therefore

$$i(J_n, Q_n \cap (B_{2,n} - \overline{B_{1,n}}), Q_n) = 1,$$

$$i(J_n, Q_n \cap (B_{3,n} - \overline{B_{2,n}}), Q_n) = -1.$$

Consequently there exist $x'_n \in (B_{2,n} - \overline{B_{1,n}}) \cap Q_n$ and $x''_n \in (B_{3,n} - \overline{B_2}) \cap Q_n$ such that $J_n x'_n = x'_n, J_n x''_n = x''_n$ for all $n > \max\{b^*, \frac{1}{t_1}\}$. From Theorem 3.1, we have $x^* \in Q, x^{**} \in Q$ such that

$$Jx^* = x^*, Jx^{**} = x^{**}.$$

And moreover $R \geq \|x^*(t)\| \geq \frac{1}{N}\|e(t)\|$ for $t \in [a^*, b^*]$ and $\frac{NR'}{c^*} \geq \sup_{t\in J}\|x^{**}(t)\| \geq \min\{\|x^{**}(t)\|, t \in [a^*, b^*]\} \geq 2R$. The proof is complete. □

Theorem 3.3. *Assume conditions* $(H_1), (H_2), (H_3)$ *are satisfied and* $a(\gamma + \delta) + b(\alpha + \beta) = \theta$, *then equation* (1.1) *has at least two positive solutions.*

Proof. By (H_3) we can choose $r' > 0$ such that

$$g(f(t,x)) \geq N' g(x) \tag{3.17}$$

for $\|x\| \leq r'$, where

$$N' > 2(\inf_{t\in[a^*,b^*]} \int_{a^*}^{b^*} G(t,s)ds)^{-1}.$$

Let $B_{1,n} = \{x \in PC([\frac{1}{n}, n], E), \|x\|_{[\frac{1}{n},n]} < r'\}$. Now we have

$$J_n x \not\leq x, x \in \partial(B_{1,n} \cap Q_n). \tag{3.18}$$

In fact, if there exists a $x \in \partial(B_{1,n} \cap Q_n)$ with $J_n x \leq x$, then for $t \in [a^*, b^*]$, $x(t) \geq c^* x(s)$ for all $s \in [\frac{1}{n}, n]$, which implies that $\inf_{t\in[a^*,b^*]} \|x(t)\| \geq$

$\frac{c^*}{N} \sup_{s\in[\frac{1}{n},n]} \|x(s)\| = \frac{c^*}{N} r'$. From $g \in P^{\circ *}$, we get $\inf_{t\in[a^*,b^*]} g(x(t)) > 0$. Now for $t \in [a^*, b^*]$, we have

$$(J_n x)(t) \geq (A_n x)(t) = \int_{\frac{1}{n}}^{n} G(t,s) f(s, x(s)) ds \geq \int_{a^*}^{b^*} G(t,s) f(s, x(s)) ds.$$

So (3.17) implies

$$\begin{aligned} g(x(t)) &\geq \int_{a^*}^{b^*} G(t,s) g(f(s, x(s))) ds \geq \int_{a^*}^{b^*} G(t,s) N' g(x(s)) ds \\ &\geq \int_{a^*}^{b^*} G(t,s) ds N' \inf_{t\in[a^*,b^*]} g(x(t)). \end{aligned}$$

Hence

$$\inf_{t\in[a^*,b^*]} g(x(t)) \geq 2 \inf_{t\in[a^*,b^*]} g(x(t))$$

which implies that $\inf_{t\in[a^*,b^*]} g(x(t)) = 0$. This is a contradiction. So

$$i(J_n, B_{1,n} \cap Q_n, Q_n) = 0.$$

And we can choose $B_{2,n}$ and $B_{3,n}$ as those in the proof in Theorem 3.2. The proof is complete.□

§4. AN EXAMPLE

In this section, we will give an example illustrating Theorem 3.3.

Example 4.1. *Consider the following problem*

$$\begin{cases} x_n''(t) + \dfrac{it^{i-1}}{1+t^i} x_n'(t) + \phi(t)(|x_n(t)|^p + |x_{n+1}(t)|^s) = 0, \\ t \in (0,\infty), \quad t \neq t_k, k = 1, 2, \cdots, \overline{m} \\ \Delta x_n|_{t=t_k} = \dfrac{1}{2^{k+1}} x_n(t_k), \quad k = 1, 2, \cdots, \overline{m} \\ x_n(0) = 0, \lim\limits_{t\to+\infty} (1+t^i) x_n'(t) = 0, \\ (n = 1, 2, \cdots, m) \end{cases} \tag{4.1}$$

where $x_{m+n} = x_n (n = 1, 2, \cdots, m)$ *and*

$$\phi(t) = \begin{cases} \dfrac{1}{200} t^{-\frac{1}{2}}, & t \in (0, 1]; \\ \dfrac{1}{50(1+t^i)^2}, & t \in [1, +\infty). \end{cases}$$

$1 < i, 0 < p < 1 < s$, $0 < t_1 < t_2 < t_3 < \cdots < t_k < \cdots < t_{\overline{m}}$, $\overline{m}$ *is the number of impulse points.*

Conclusion: problem (4.1) has at least two positive solutions.

Proof. Let E be the Euclidean space $R^m = \{x = (x_1, x_2, \cdots, x_m)\}$ and $P = \{x = (x_1, x_2, \cdots, x_m) : x_n \geq 0 \text{ for } n \geq 1\}$. Then P is a normal cone in E, $P^* = P$ and problem (4.1) can be regarded as equation (1.1) with $x = (x_1, x_2, \cdots, x_m), f(t,x) = (f_1(t,x), f_2(t,x), \cdots, f_m(t,x))$ and

$$f_n(t,x) = \phi(t)(|x_n(t)|^p + |x_{n+1}(t)|^s),$$

$\lambda = 1, \beta = 0, \gamma = 0, \delta = 1, a = b = (0, 0, \cdots, 0)$. Then (a)(b) of condition (H_1) in Theorem 3.3 are satisfied automaticaly. Let $R = 1$, we can see (c) of condition (H_1) is true. Let $g = (1, 1, \cdots, 1)$, then $g(x) = \sum_{n=1}^{m} x_n > 0$ for $x = (x_1, x_2, \cdots, x_m) > 0$ and

$$\frac{g(f(t,x))}{g(x)} = \frac{\sum\limits_{n=1}^{m} f_n(t,x)}{\sum\limits_{n=1}^{m} x_n} = \frac{\phi(t)(\sum\limits_{n=1}^{m} |x_n|^p + \sum\limits_{n=1}^{m} |x_{n+1}|^s)}{\sum\limits_{n=1}^{m} x_n}.$$

It is easy to see

$$\lim_{\|x\| \to +\infty} \frac{g(f(t,x))}{g(x)} = +\infty \quad \text{and}$$

$$\lim_{\|x\| \to +0} \frac{g(f(t,x))}{g(x)} = +\infty$$

uniformly for any $t \in [t', t'']$, where $t' > 0, t'' < +\infty$. So $(H_2), (H_3)$ are satisfied. By virtue of Theorem 3.3 equation (4.1) has at least two positive solutions. The proof is complete.□

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