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On some general classes of partial linear complex vector functional equations

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Abstract. In this paper four classes of partial linear complex vector functional equations are solved, namely the general derived cyclic functional equations, the generalized paracyclic functional equations, the general semicyclic functional equation and the general special cyclic functional equation, as well a more general class of functional equations.

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§0. Introduction

In [1] the so called *basic cyclic functional equation* and its derived functional equation are solved. Some special generalized cases of these functional equations are considered in [2, 5, 7]. For the general derived cyclic functional equation in [9] one unsolved research problem is given, which is solved in the first section of this paper, and after that some functional equations are also solved. The results obtained generalize those given in [13].

At the end of §1 and in §4, following the referee's suggestion, we have found the general solution of a more general class of functional equations, where the unknown functions may depend on different number of variables.

The paracyclic functional equations of the first kind considered in [8, 14] are solved here in a generalized complex vector form without use of a cyclic operator.

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In the third section of the present paper the general semicyclic complex vector functional equation is solved. For a particular case see the results from [3].

The fourth section includes the solution of the general special cyclic complex vector functional equation.

§1. General Derived Cyclic Complex Vector Functional Equations

First we will introduce the following necessary notations:

Let $\mathcal{V}$ be a finite dimensional complex vector space and let there exist mappings $f_i:\ \mathcal{V}^p \mapsto \mathcal{V}$ $(1 \le i \le k)$. Throughout this section $\mathbf{Z}_i$ $(1 \le i \le n)$ are vectors in $\mathcal{V}$, and $\mathcal{C}_i$ are constant vectors in the same space. We may assume that $\mathbf{Z}_i = (z_{i1}(t), \cdots, z_{in}(t))^T$, where the components $z_{ij}(t)$ $(1 \le i \le n;\ 1 \le j \le n)$ are complex functions, and that $\mathbf{O} = (0, 0, \cdots, 0)^T$ is the zero vector in $\mathcal{V}$.

Next we will give the following results.

Theorem 1.1. *The general solution of the derived cyclic complex vector functional equation*

(1.1)
$$\sum_{i=1}^{k} f_i(\mathbf{Z}_i, \mathbf{Z}_{i+1}, \ldots, \mathbf{Z}_{i+p-1}) = \mathbf{O} \qquad (p < n < 2p-1;\ k \le n; \quad \mathbf{Z}_{n+i} \equiv \mathbf{Z}_i)$$

is given by the formulas

$$(1.2)\quad \begin{aligned} f_r(\mathbf{Z}_1, \mathbf{Z}_2, \ldots, \mathbf{Z}_p) &= \sum_{i=1}^{\min(k-r,n-p)} (-1)^{i-1} F_{ri}(\mathbf{Z}_{i+1}, \mathbf{Z}_{i+2}, \ldots, \mathbf{Z}_p) \\ &\quad + \sum_{i=n-r+1}^{n-p} (-1)^{i-1} F_{ri}(\mathbf{Z}_{i+1}, \mathbf{Z}_{i+2}, \ldots, \mathbf{Z}_p) \\ &+ \sum_{i=n-p+1}^{\min(k-r,p-1)} (-1)^{i-1} F_{ri}(\mathbf{Z}_{i+1}, \mathbf{Z}_{i+2}, \ldots, \mathbf{Z}_p, \mathbf{Z}_1, \mathbf{Z}_2, \ldots, \mathbf{Z}_{p+i}) \\ &+ \sum_{i=\max(n-r+1,n-p+1)}^{p-1} (-1)^{n-i} F_{i+r,n-i}(\mathbf{Z}_1, \mathbf{Z}_2, \ldots, \mathbf{Z}_{p+i}, \mathbf{Z}_{i+1}, \mathbf{Z}_{i+2}, \ldots, \mathbf{Z}_p) \\ &+ \sum_{i=p}^{k-r} (-1)^{n-i} F_{i+r,n-i}(\mathbf{Z}_1, \mathbf{Z}_2, \ldots, \mathbf{Z}_{p+i}) \end{aligned}$$

$$+\sum_{i=\max(n-r+1,p)}^{n-1}(-1)^{n-i}F_{i+r,n-i}(\mathbf{Z}_1,\mathbf{Z}_2,\dots,\mathbf{Z}_{p+i})\quad(1\le r\le k;\ k\le n),$$

where

$$\sum_a^s=\mathbf{O}\ (a>s);\qquad F_{m+n,i}=F_{mi};\qquad F_{i,m+n}=F_{im}$$

and F_{ij} *are arbitrary vector functions from* $\mathcal{V}$.

Proof. The proof of this theorem will be given by induction.

For $k=2$, the equation (1.1) becomes

$$f_1(\mathbf{Z}_1,\mathbf{Z}_2,\dots,\mathbf{Z}_p)+f_2(\mathbf{Z}_2,\mathbf{Z}_3,\dots,\mathbf{Z}_{p+1})=\mathbf{O}. \tag{1.3}$$

Putting $\mathbf{Z}_1=\mathcal{C}_1$ into the equation (1.3), we obtain

$$f_2(\mathbf{Z}_2,\mathbf{Z}_3,\dots,\mathbf{Z}_{p+1})=-F_{11}(\mathbf{Z}_2,\mathbf{Z}_3,\dots,\mathbf{Z}_p), \tag{1.4}$$

where the notation

$$F_{11}(\mathbf{Z}_2,\mathbf{Z}_3,\dots,\mathbf{Z}_p)=f_1(\mathcal{C}_1,\mathbf{Z}_2,\mathbf{Z}_3\dots,\mathbf{Z}_p)$$

is introduced. If we put (1.4) into (1.3), we have

$$f_1(\mathbf{Z}_1,\mathbf{Z}_2,\dots,\mathbf{Z}_p)=F_{11}(\mathbf{Z}_2,\mathbf{Z}_3,\dots,\mathbf{Z}_p). \tag{1.5}$$

Since

$$\begin{array}{ll}\min(1,n-p)=1, & \min(1,p-1)=1,\\ \min(0,n-p)=0, & \min(0,p-1)=0,\\ \max(n,n-p+1)=n, & \max(n,p)=n,\\ \max(n-1,n-p+1)=n-1, & \max(n-1,p)=n-1,\end{array}$$

from (1.2) we obtain (1.5) and (1.4) respectively for $k=2,\ r=1$ and for $k=2,\ r=2$.

Therefore, the theorem holds for $k=2$.

Now we will suppose that the theorem holds for some fixed $k<n$, i.e. let the general solution of the functional equation

$$\sum_{i=1}^{k}g_i(\mathbf{Z}_i,\mathbf{Z}_{i+1},\dots,\mathbf{Z}_{i+p-1})=\mathbf{O}\quad(k<n) \tag{1.6}$$

be given by (1.2), where in the place of f_r we have put g_r $(1\le r\le k)$.

Let us consider the following functional equation

$$\sum_{i=1}^{k+1}f_i(\mathbf{Z}_i,\mathbf{Z}_{i+1},\dots,\mathbf{Z}_{i+p-1})=\mathbf{O}\quad(k+1\le n). \tag{1.7}$$

We will distinguish the following three cases:

1^o Let $2 \le k < n-p+1$. If we put $\mathbf{Z}_i = \mathcal{C}_i$ for $i \not\equiv k+1, k+2, \dots, k+p \pmod n$, then the equation (1.7) becomes

$$f_{k+1}(\mathbf{Z}_{k+1}, \mathbf{Z}_{k+2}, \dots, \mathbf{Z}_{k+p}) = \sum_{i=n-k}^{n-1} (-1)^{n-i} F_{i+k+1,n-i}(\mathbf{Z}_{k+1}, \mathbf{Z}_{k+2}, \dots, \mathbf{Z}_{i+p+k}), \tag{1.8}$$

where we introduced the notation

$$(-1)^{n-i+1} F_{i+k+1,n-i}(\mathbf{Z}_{k+1}, \mathbf{Z}_{k+2}, \dots, \mathbf{Z}_{i+p+k}) = f_{i+k+1-n}(\mathbf{Z}_{i+k+1-n}, \mathbf{Z}_{i+k+2-n}, \dots, \mathbf{Z}_{i+k+p-n})\Big|_{\substack{\mathbf{Z}_i = \mathcal{C}_i \text{ for} \\ i \not\equiv k+1, k+2, \dots, k+p \pmod n}}.$$

From (1.8) it follows that

$$f_{k+1}(\mathbf{Z}_1, \mathbf{Z}_2, \dots, \mathbf{Z}_p) = \sum_{i=n-k}^{n-1} (-1)^{n-i} F_{i+k+1,n-i}(\mathbf{Z}_1, \mathbf{Z}_2, \dots, \mathbf{Z}_{i+p}). \tag{1.9}$$

If we substitute (1.8) into (1.7) and introduce new notations by the equalities

$$\begin{aligned} g_i(\mathbf{Z}_i, \mathbf{Z}_{i+1}, \dots, \mathbf{Z}_{i+p-1}) &= f_i(\mathbf{Z}_i, \mathbf{Z}_{i+1}, \dots, \mathbf{Z}_{i+p-1}) \\ &+ (-1)^{1+k-i} F_{i,1+k-i}(\mathbf{Z}_{k+1}, \mathbf{Z}_{k+2}, \dots, \mathbf{Z}_{i+p-1}) \quad (i = 1, \dots, k), \end{aligned} \tag{1.10}$$

we obtain the equation (1.6).

For $2 \le k < n-p$, since

$$\min(k-r, n-p) = \min(k-r, p-1) = k-r, \qquad n-p+1 > k-r,$$

on the basis of the expressions (1.2) the general solution of the equation (1.6) is

$$\begin{aligned} g_r(\mathbf{Z}_1, \mathbf{Z}_2, \dots, \mathbf{Z}_p) &= \sum_{i=1}^{k-r} (-1)^{i-1} F_{ri}(\mathbf{Z}_{i+1}, \mathbf{Z}_{i+2}, \dots, \mathbf{Z}_p) \\ &+ \sum_{i=n-r+1}^{n-p} (-1)^{i-1} F_{ri}(\mathbf{Z}_{i+1}, \mathbf{Z}_{i+2}, \dots, \mathbf{Z}_p) \\ &+ \sum_{i=\max(n-r+1,n-p+1)}^{p-1} (-1)^{n-i} F_{i+r,n-i}(\mathbf{Z}_1, \mathbf{Z}_2, \dots, \mathbf{Z}_{p+i}, \mathbf{Z}_{i+1}, \mathbf{Z}_{i+2}, \dots, \mathbf{Z}_p) \\ &+ \sum_{i=\max(n-r+1,p)}^{n-1} (-1)^{n-i} F_{i+r,n-i}(\mathbf{Z}_1, \mathbf{Z}_2, \dots, \mathbf{Z}_{p+i}) \qquad (1 \le r \le k). \end{aligned} \tag{1.11}$$

On the basis of the expressions (1.9), (1.10) and (1.11), we obtain

$$
\begin{aligned}
(1.12)\quad f_r(\mathbf{Z}_1,\mathbf{Z}_2,\dots,\mathbf{Z}_p) &= \sum_{i=1}^{k+1-r}(-1)^{i-1}F_{ri}(\mathbf{Z}_{i+1},\mathbf{Z}_{i+2},\dots,\mathbf{Z}_p)\\
&\quad+\sum_{i=n-r+1}^{n-p}(-1)^{i-1}F_{ri}(\mathbf{Z}_{i+1},\mathbf{Z}_{i+2},\dots,\mathbf{Z}_p)\\
&\quad+\sum_{i=\max(n-r+1,n-p+1)}^{p-1}(-1)^{n-i}F_{i+r,n-i}(\mathbf{Z}_1,\mathbf{Z}_2,\dots,\mathbf{Z}_{p+i},\mathbf{Z}_{i+1},\mathbf{Z}_{i+2},\dots,\mathbf{Z}_p)\\
&\quad+\sum_{i=\max(n-r+1,p)}^{n-1}(-1)^{n-i}F_{i+r,n-i}(\mathbf{Z}_1,\mathbf{Z}_2,\dots,\mathbf{Z}_{p+i})\qquad(1\le r\le k+1).
\end{aligned}
$$

Therefore, for $2\le k<n-p+1$ the theorem holds for $k+1$ if it is true for k. This means that the theorem holds for all such k, and also for $k=n-p+1$.

2^o. Let $n-p+1\le k<p$. If we put $\mathbf{Z}_i=\mathcal{C}_i$ for $i\not\equiv k+1,k+2,\dots,k+p$ (mod n), then the equation (1.7) becomes

$$
\begin{aligned}
(1.13)\quad & f_{k+1}(\mathbf{Z}_{k+1},\mathbf{Z}_{k+2},\dots,\mathbf{Z}_{k+p})\\
&=\sum_{i=n-k}^{p-1}(-1)^{n-i}\times\\
&\qquad F_{i+k+1,n-i}(\mathbf{Z}_{k+1},\mathbf{Z}_{k+2},\dots,\mathbf{Z}_{p+k+i},\mathbf{Z}_{i+k+1},\mathbf{Z}_{i+k+2},\dots,\mathbf{Z}_{p+k})\\
&\quad+\sum_{i=p}^{n-1}(-1)^{n-i}F_{i+k+1,n-i}(\mathbf{Z}_{k+1},\mathbf{Z}_{k+2},\dots,\mathbf{Z}_{p+k+i}),
\end{aligned}
$$

where we have introduced the notation

$$
\begin{aligned}
& f_{i+k+1-n}(\mathbf{Z}_{i+k+1-n},\mathbf{Z}_{i+k+2-n},\dots,\mathbf{Z}_{i+k+p-n})\Big|_{\substack{\mathbf{Z}_i=\mathcal{C}_i\text{ for}\\ i\not\equiv k+1,k+2,\dots,k+p\pmod n}}\\
&=\begin{cases}(-1)^{n-i+1}F_{i+k+1,n-i}(\mathbf{Z}_{k+1},\mathbf{Z}_{k+2},\dots,\mathbf{Z}_{p+k+i},\mathbf{Z}_{i+k+1},\mathbf{Z}_{i+k+2},\dots,\mathbf{Z}_{p+k})\\ \qquad (i=n-k,n-k+1,\dots,p-1),\\ (-1)^{n-i+1}F_{i+k+1,n-i}(\mathbf{Z}_{k+1},\mathbf{Z}_{k+2},\dots,\mathbf{Z}_{p+k+i})\\ \qquad (i=p,p+1,\dots,n-1).\end{cases}
\end{aligned}
$$

Substituting (1.13) into (1.7) and introducing new functions by

$$
(1.14)\qquad g_i=f_i+(-1)^{1+k-i}F_{i,1+k-i}\quad(i=1,\dots,k),
$$

we obtain the equation (1.6).

On the basis of the inductive hypothesis and the equation (1.14), the general solution of the equation (1.7) is

$$f_r(\mathbf{Z}_1, \mathbf{Z}_2, \dots, \mathbf{Z}_p) = \sum_{i=1}^{n-p} (-1)^{i-1} F_{ri}(\mathbf{Z}_{i+1}, \mathbf{Z}_{i+2}, \dots, \mathbf{Z}_p)$$
$$+ \sum_{i=n-p+1}^{k-r} (-1)^{i-1} F_{ri}(\mathbf{Z}_{i+1}, \mathbf{Z}_{i+2}, \dots, \mathbf{Z}_p, \mathbf{Z}_1, \mathbf{Z}_2, \dots, \mathbf{Z}_{p+i})$$
$$+ (-1)^{k-r} F_{r,k+1-r}(\mathbf{Z}_{k+2-r}, \mathbf{Z}_{k+3-r}, \dots, \mathbf{Z}_p, \mathbf{Z}_1, \mathbf{Z}_2, \dots, \mathbf{Z}_{p+k-r+1})$$
$$+ \sum_{i=n-r+1}^{n-p} (-1)^{i-1} F_{ri}(\mathbf{Z}_{i+1}, \mathbf{Z}_{i+2}, \dots, \mathbf{Z}_p)$$
$$+ \sum_{i=\max(n-r+1,n-p+1)}^{p-1} (-1)^{n-i} F_{i+r,n-i}(\mathbf{Z}_1, \mathbf{Z}_2, \dots, \mathbf{Z}_{p+i}, \mathbf{Z}_{i+1}, \mathbf{Z}_{i+2}, \dots, \mathbf{Z}_p)$$
$$+ \sum_{i=\max(n-r+1,p)}^{n-1} (-1)^{n-i} F_{i+r,n-i}(\mathbf{Z}_1, \mathbf{Z}_2, \dots, \mathbf{Z}_{p+i})$$
$$(r = 1, 2, \dots, p+k-n);$$

$$f_r(\mathbf{Z}_1, \mathbf{Z}_2, \dots, \mathbf{Z}_p) = \sum_{i=1}^{k-r} (-1)^{i-1} F_{ri}(\mathbf{Z}_{i+1}, \mathbf{Z}_{i+2}, \dots, \mathbf{Z}_p) \tag{1.15}$$
$$+ (-1)^{k-r} F_{r,k+1-r}(\mathbf{Z}_{k+2-r}, \mathbf{Z}_{k+3-r}, \dots, \mathbf{Z}_p)$$
$$+ \sum_{i=n-r+1}^{n-p} (-1)^{i-1} F_{ri}(\mathbf{Z}_{i+1}, \mathbf{Z}_{i+2}, \dots, \mathbf{Z}_p)$$
$$+ \sum_{i=\max(n-r+1,n-p+1)}^{p-1} (-1)^{n-i} F_{i+r,n-i}(\mathbf{Z}_1, \mathbf{Z}_2, \dots, \mathbf{Z}_{p+i}, \mathbf{Z}_{i+1}, \mathbf{Z}_{i+2}, \dots, \mathbf{Z}_p)$$
$$+ \sum_{i=\max(n-r+1,p)}^{n-1} (-1)^{n-i} F_{i+r,n-i}(\mathbf{Z}_{i+1}, \mathbf{Z}_{i+2}, \dots, \mathbf{Z}_p)$$
$$(r = p+k-n+1, \dots, k).$$

On the basis of the expressions (1.13) and (1.15), the general solution of the equation (1.7) is determined by (1.2), where k must be replaced by $k+1$.

Therefore, for $n-p+1 \le k < p$ the theorem holds for $k+1$ if it holds for k, i.e. the theorem is true for all such k, and also for $k=p$.

3^o Let $p \le k \le n-1$. For $\mathbf{Z}_i = \mathcal{C}_i$ when $i \not\equiv k+1, k+2, \dots, k+p \pmod n$ the equation (1.7) becomes

(1.16)

$$\begin{aligned} f_{k+1}(\mathbf{Z}_{k+1}, \mathbf{Z}_{k+2}, \dots, \mathbf{Z}_{k+p}) &= \sum_{i=n-k}^{n-p} (-1)^{i-1} F_{k+1,i}(\mathbf{Z}_{i+1}, \mathbf{Z}_{i+2}, \dots, \mathbf{Z}_p) \\ &+ \sum_{i=n-p+1}^{p-1} (-1)^{n-i} \times \\ &\qquad F_{i+k+1,n-i}(\mathbf{Z}_{k+1}, \mathbf{Z}_{k+2}, \dots, \mathbf{Z}_{p+k+i}, \mathbf{Z}_{i+k+1}, \mathbf{Z}_{i+k+2}, \dots, \mathbf{Z}_{p+k}) \\ &+ \sum_{i=p}^{n-1} (-1)^{n-i} F_{i+k+1,n-i}(\mathbf{Z}_{k+1}, \mathbf{Z}_{k+2}, \dots, \mathbf{Z}_{p+k+i}), \end{aligned}$$

where we have introduced the notation

$$\begin{aligned} & f_{i+k+1-n}(\mathbf{Z}_{i+k+1-n}, \mathbf{Z}_{i+k+2-n}, \dots, \mathbf{Z}_{i+k+p-n})\Big|_{\substack{\mathbf{Z}_i = \mathcal{C}_i \text{ for} \\ i \not\equiv k+1, k+2, \dots, k+p \pmod n}} \\ &= \begin{cases} (-1)^i F_{k+1,i}(\mathbf{Z}_{i+1}, \mathbf{Z}_{i+2}, \dots, \mathbf{Z}_p) \quad (i = n-k, \dots, n-p), \\ (-1)^{n-i+1} F_{i+k+1,n-i}(\mathbf{Z}_{k+1}, \mathbf{Z}_{k+2}, \dots, \mathbf{Z}_{p+k+i}, \mathbf{Z}_{i+k+1}, \mathbf{Z}_{i+k+2}, \dots, \mathbf{Z}_{p+k}) \\ \qquad\qquad (i = n-p+1, \dots, p-1), \\ (-1)^{n-i+1} F_{i+k+1,n-i}(\mathbf{Z}_{k+1}, \mathbf{Z}_{k+2}, \dots, \mathbf{Z}_{p+k+i}) \quad (i = p, p+1, \dots, n-1). \end{cases} \end{aligned}$$

Now, we will introduce the following notation

$$g_i = f_i + \begin{cases} (-1)^{i+n-k} F_{k+1,i+n-k-1} & (i = 1, \dots, k+1-p), \\ (-1)^{1+k-i} F_{i,1+k-i} & (i = k+2-p, \dots, k). \end{cases} \tag{1.17}$$

If we substitute the function f_{k+1} determined by (1.16) into (1.7), in view of the notation (1.17) we obtain the functional equation (1.6). On the basis of the expression (1.17) and the general solution of the functional equation (1.6), we obtain that the general solution of the functional equation (1.7) is determined by

$$f_r(\mathbf{Z}_1, \mathbf{Z}_2, \dots, \mathbf{Z}_p) = \sum_{i=1}^{n-p} (-1)^{i-1} F_{ri}(\mathbf{Z}_{i+1}, \mathbf{Z}_{i+2}, \dots, \mathbf{Z}_p)$$

$$+ \sum_{i=n-p+1}^{p-1} (-1)^{i-1} F_{ri}(\mathbf{Z}_{i+1}, \mathbf{Z}_{i+2}, \dots, \mathbf{Z}_p, \mathbf{Z}_1, \mathbf{Z}_2, \dots, \mathbf{Z}_{p+i})$$

$$+ \sum_{i=p}^{k-r} (-1)^{n-i} F_{i+r,n-i}(\mathbf{Z}_1, \mathbf{Z}_2, \dots, \mathbf{Z}_{p+i})$$

$$+ (-1)^{n-k+1-r} F_{k,n-k+1-r}(\mathbf{Z}_1, \mathbf{Z}_2, \dots, \mathbf{Z}_{p+k-r})$$

$$+ \sum_{i=n-r+1}^{n-p} (-1)^{i-1} F_{ri}(\mathbf{Z}_{i+1}, \mathbf{Z}_{i+2}, \dots, \mathbf{Z}_p)$$

$$+ \sum_{i=\max(n-r+1,n-p+1)}^{p-1} (-1)^{n-i} F_{i+r,n-i}(\mathbf{Z}_1, \mathbf{Z}_2, \dots, \mathbf{Z}_{p+i}, \mathbf{Z}_{i+1}, \mathbf{Z}_{i+2}, \dots, \mathbf{Z}_p)$$

$$+ \sum_{i=\max(n-r+1,p)}^{n-1} (-1)^{n-i} F_{i+r,n-i}(\mathbf{Z}_1, \mathbf{Z}_2, \dots, \mathbf{Z}_{p+i})$$

$$(r = 1, 2, \dots, n-k-p+1);$$

$$f_r(\mathbf{Z}_1, \mathbf{Z}_2, \dots, \mathbf{Z}_p) = \sum_{i=1}^{n-p} (-1)^{i-1} F_{ri}(\mathbf{Z}_{i+1}, \mathbf{Z}_{i+2}, \dots, \mathbf{Z}_p) \tag{1.18}$$

$$+ \sum_{i=n-p+1}^{k-r} (-1)^{i-1} F_{ri}(\mathbf{Z}_{i+1}, \mathbf{Z}_{i+2}, \dots, \mathbf{Z}_p, \mathbf{Z}_1, \mathbf{Z}_2, \dots, \mathbf{Z}_{p+i})$$

$$+ (-1)^{k-r} F_{r,k-r+1}(\mathbf{Z}_{k-r+2}, \mathbf{Z}_{k-r+3}, \dots, \mathbf{Z}_p, \mathbf{Z}_1, \mathbf{Z}_2, \dots, \mathbf{Z}_{p+k-r+1})$$

$$+ \sum_{i=n-r+1}^{n-p} (-1)^{i-1} F_{ri}(\mathbf{Z}_{i+1}, \mathbf{Z}_{i+2}, \dots, \mathbf{Z}_p)$$

$$+ \sum_{i=\max(n-r+1,n-p+1)}^{p-1} (-1)^{n-i} F_{i+r,n-i}(\mathbf{Z}_1, \mathbf{Z}_2, \dots, \mathbf{Z}_{p+i}, \mathbf{Z}_{i+1}, \mathbf{Z}_{i+2}, \dots, \mathbf{Z}_p)$$

$$+ \sum_{i=\max(n-r+1,p)}^{n-1} (-1)^{n-i} F_{i+r,n-i}(\mathbf{Z}_1, \mathbf{Z}_2, \dots, \mathbf{Z}_{p+i})$$

$$(r = n-k-p+2, \dots, p-1);$$

$$f_r(\mathbf{Z}_1, \mathbf{Z}_2, \dots, \mathbf{Z}_p) = \sum_{i=1}^{k-r} (-1)^{i-1} F_{ri}(\mathbf{Z}_{i+1}, \mathbf{Z}_{i+2}, \dots, \mathbf{Z}_p)$$

$$+ (-1)^{k-r} F_{r,k+1-r}(\mathbf{Z}_{k+2-r}, \mathbf{Z}_{k+3-r}, \dots , \mathbf{Z}_p)$$
$$+ \sum_{i=n-r+1}^{n-p} (-1)^{i-1} F_{ri}(\mathbf{Z}_{i+1}, \mathbf{Z}_{i+2}, \dots , \mathbf{Z}_p)$$
$$+ \sum_{i=\max(n-r+1,n-p+1)}^{p-1} (-1)^{n-i} F_{i+r,n-i}(\mathbf{Z}_1, \mathbf{Z}_2, \dots , \mathbf{Z}_{p+i}, \mathbf{Z}_{i+1}, \mathbf{Z}_{i+2}, \dots , \mathbf{Z}_p)$$
$$+ \sum_{i=\max(n-r+1,p)}^{n-1} (-1)^{n-i} F_{i+r,n-i}(\mathbf{Z}_1, \mathbf{Z}_2, \dots , \mathbf{Z}_{p+i}) \quad (r = p, p+1, \dots , k).$$

On the basis of the equalities (1.16) and (1.18) we obtain that the general solution of the functional equation (1.7) in the case $p \le k \le n-1$ is determined by (1.2), where k must be replaced by $k+1$. □

Therefore, the solution of the research problem given in [9] is presented by this theorem. As particular cases see the results given in [5, 7].

Example. For $n = 8$, $p = 5$ and $k = 6$ the complex vector functional equation (1.1) becomes

$$\begin{aligned} &f_1(\mathbf{Z}_1, \mathbf{Z}_2, \mathbf{Z}_3, \mathbf{Z}_4, \mathbf{Z}_5) + f_2(\mathbf{Z}_2, \mathbf{Z}_3, \mathbf{Z}_4, \mathbf{Z}_5, \mathbf{Z}_6) + f_3(\mathbf{Z}_3, \mathbf{Z}_4, \mathbf{Z}_5, \mathbf{Z}_6, \mathbf{Z}_7) \\ &\quad + f_4(\mathbf{Z}_4, \mathbf{Z}_5, \mathbf{Z}_6, \mathbf{Z}_7, \mathbf{Z}_8) + f_5(\mathbf{Z}_5, \mathbf{Z}_6, \mathbf{Z}_7, \mathbf{Z}_8, \mathbf{Z}_1) + f_6(\mathbf{Z}_6, \mathbf{Z}_7, \mathbf{Z}_8, \mathbf{Z}_1, \mathbf{Z}_2) \\ &= \mathbf{O}, \end{aligned}$$

whose general solution is given by

$$\begin{aligned} f_1(\mathbf{Z}_1, \mathbf{Z}_2, \mathbf{Z}_3, \mathbf{Z}_4, \mathbf{Z}_5) &= F_{11}(\mathbf{Z}_2, \mathbf{Z}_3, \mathbf{Z}_4, \mathbf{Z}_5) - F_{12}(\mathbf{Z}_3, \mathbf{Z}_4, \mathbf{Z}_5) \\ &\quad + F_{13}(\mathbf{Z}_4, \mathbf{Z}_5) - F_{14}(\mathbf{Z}_5, \mathbf{Z}_1) - F_{63}(\mathbf{Z}_1, \mathbf{Z}_2), \end{aligned}$$

$$\begin{aligned} f_2(\mathbf{Z}_1, \mathbf{Z}_2, \mathbf{Z}_3, \mathbf{Z}_4, \mathbf{Z}_5) &= F_{21}(\mathbf{Z}_2, \mathbf{Z}_3, \mathbf{Z}_4, \mathbf{Z}_5) - F_{22}(\mathbf{Z}_3, \mathbf{Z}_4, \mathbf{Z}_5) \\ &\quad + F_{23}(\mathbf{Z}_4, \mathbf{Z}_5) - F_{24}(\mathbf{Z}_5, \mathbf{Z}_1) - F_{11}(\mathbf{Z}_1, \mathbf{Z}_2, \mathbf{Z}_3, \mathbf{Z}_4), \end{aligned}$$

$$\begin{aligned} f_3(\mathbf{Z}_1, \mathbf{Z}_2, \mathbf{Z}_3, \mathbf{Z}_4, \mathbf{Z}_5) &= F_{31}(\mathbf{Z}_2, \mathbf{Z}_3, \mathbf{Z}_4, \mathbf{Z}_5) - F_{32}(\mathbf{Z}_3, \mathbf{Z}_4, \mathbf{Z}_5) \\ &\quad + F_{33}(\mathbf{Z}_4, \mathbf{Z}_5) + F_{12}(\mathbf{Z}_1, \mathbf{Z}_2, \mathbf{Z}_3) - F_{21}(\mathbf{Z}_1, \mathbf{Z}_2, \mathbf{Z}_3, \mathbf{Z}_4), \end{aligned}$$

$$\begin{aligned} f_4(\mathbf{Z}_1, \mathbf{Z}_2, \mathbf{Z}_3, \mathbf{Z}_4, \mathbf{Z}_5) &= F_{41}(\mathbf{Z}_2, \mathbf{Z}_3, \mathbf{Z}_4, \mathbf{Z}_5) - F_{42}(\mathbf{Z}_3, \mathbf{Z}_4, \mathbf{Z}_5) \\ &\quad - F_{13}(\mathbf{Z}_1, \mathbf{Z}_2) + F_{22}(\mathbf{Z}_1, \mathbf{Z}_2, \mathbf{Z}_3) - F_{31}(\mathbf{Z}_1, \mathbf{Z}_2, \mathbf{Z}_3, \mathbf{Z}_4), \end{aligned}$$

$$\begin{aligned} f_5(\mathbf{Z}_1, \mathbf{Z}_2, \mathbf{Z}_3, \mathbf{Z}_4, \mathbf{Z}_5) &= F_{51}(\mathbf{Z}_2, \mathbf{Z}_3, \mathbf{Z}_4, \mathbf{Z}_5) + F_{14}(\mathbf{Z}_1, \mathbf{Z}_5) \\ &\quad - F_{23}(\mathbf{Z}_1, \mathbf{Z}_2) + F_{32}(\mathbf{Z}_1, \mathbf{Z}_2, \mathbf{Z}_3) - F_{41}(\mathbf{Z}_1, \mathbf{Z}_2, \mathbf{Z}_3, \mathbf{Z}_4), \end{aligned}$$

$$f_6(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3,\mathbf{Z}_4,\mathbf{Z}_5) = F_{63}(\mathbf{Z}_4,\mathbf{Z}_5) + F_{24}(\mathbf{Z}_1,\mathbf{Z}_5) - F_{33}(\mathbf{Z}_1,\mathbf{Z}_2) + F_{42}(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3) - F_{51}(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3,\mathbf{Z}_4),$$

where F_{ij} are arbitrary complex vector functions from $\mathcal{V}$.

Now we will give two particular cases of the above theorem.

Theorem 1.2. *The general solution of the functional equation*

$$(1.19)\qquad \sum_{i=1}^{n} f_i(\mathbf{Z}_i,\mathbf{Z}_{i+1},\dots,\mathbf{Z}_{i+p-1}) = \mathbf{O} \qquad (p < n < 2p-1;\ \mathbf{Z}_{n+i} \equiv \mathbf{Z}_i)$$

is given by

$$(1.20)\qquad f_r(\mathbf{Z}_1,\mathbf{Z}_2,\dots,\mathbf{Z}_p) = \sum_{i=1}^{n-p}(-1)^{i-1}F_{ri}(\mathbf{Z}_{i+1},\mathbf{Z}_{i+2},\dots,\mathbf{Z}_p)$$
$$+ \sum_{i=n-p+1}^{\min(n-r,p-1)} (-1)^{i-1}F_{ri}(\mathbf{Z}_{i+1},\mathbf{Z}_{i+2},\dots,\mathbf{Z}_p,\mathbf{Z}_1,\mathbf{Z}_2,\dots,\mathbf{Z}_{p+i})$$
$$+ \sum_{i=\max(n-r+1,n-p+1)}^{p-1} (-1)^{n-i}F_{i+r,n-i}(\mathbf{Z}_1,\mathbf{Z}_2,\dots,\mathbf{Z}_{p+i},\mathbf{Z}_{i+1},\mathbf{Z}_{i+2},\dots,\mathbf{Z}_p)$$
$$+ \sum_{i=p}^{n-1}(-1)^{n-i}F_{i+r,n-i}(\mathbf{Z}_1,\mathbf{Z}_2,\dots,\mathbf{Z}_{p+i}) \qquad (1 \le r \le n),$$

where F_{ij} are arbitrary complex vector functions from $\mathcal{V}$.

Proof. The proof of this theorem immediately follows from the previous theorem for $k = n$. □

This theorem generalizes the results given in [2].

Theorem 1.3. *The general solution of the functional equation*

$$(1.21)\qquad \sum_{i=1}^{n} f(\mathbf{Z}_i,\mathbf{Z}_{i+1},\dots,\mathbf{Z}_{i+p-1}) = \mathbf{O} \qquad (p < n < 2p-1;\ \mathbf{Z}_{n+i} \equiv \mathbf{Z}_i)$$

is given by

$$(1.22)\qquad f(\mathbf{Z}_1,\mathbf{Z}_2,\dots,\mathbf{Z}_p) = F_0(\mathbf{Z}_1,\mathbf{Z}_2,\dots,\mathbf{Z}_{p-1}) - F_0(\mathbf{Z}_2,\mathbf{Z}_3,\dots,\mathbf{Z}_p)$$
$$+ \sum_{i=1}^{p-[n/2]} \big[F_i(\mathbf{Z}_1,\mathbf{Z}_2,\dots,\mathbf{Z}_i,\mathbf{Z}_{n-p+i+1},\dots,\mathbf{Z}_p) - F_i(\mathbf{Z}_{p-i+1},\dots,\mathbf{Z}_p,\mathbf{Z}_1,\mathbf{Z}_2,\dots,\mathbf{Z}_{2p-n-i})\big],$$

where F_i $(0 \le i \le p-[n/2])$ are arbitrary complex vector functions from $\mathcal{V}$.

Proof. By summing up the functions f_i $(1 \le i \le n)$ determined by (1.20) and putting $f_1 = f_2 = \cdots = f_n = f$, we obtain (1.22), where we have introduced the following notations

$$F_0(\mathbf{Z}_1, \mathbf{Z}_2, \ldots, \mathbf{Z}_{p-1}) = \frac{1}{n}\sum_{r=1}^{n-p}\sum_{i=1}^{r} G_r(\mathbf{Z}_i, \mathbf{Z}_{i+1}, \ldots, \mathbf{Z}_{p-r-1+i}),$$

$$G_r(\mathbf{Z}_1, \mathbf{Z}_2, \ldots, \mathbf{Z}_{p-r}) = \sum_{i=1}^{n}(-1)^r F_{ir}(\mathbf{Z}_1, \mathbf{Z}_2, \ldots, \mathbf{Z}_{p-r}),$$

$$\begin{aligned} &F_i(\mathbf{Z}_1, \mathbf{Z}_2, \ldots, \mathbf{Z}_i, \mathbf{Z}_{n-p+i+1}, \ldots, \mathbf{Z}_p) \\ &\quad = \frac{(-1)^{i+1}}{n}\Bigg[\sum_{r=1}^{n-p+i} F_{r,p-i}(\mathbf{Z}_1, \mathbf{Z}_2, \ldots, \mathbf{Z}_i, \mathbf{Z}_{n-p+i+1}, \ldots, \mathbf{Z}_p) \\ &\qquad - \sum_{r=1}^{p-i} F_{r,n-p+i}(\mathbf{Z}_{n-p+i+1}, \ldots, \mathbf{Z}_p, \mathbf{Z}_1, \mathbf{Z}_2, \ldots, \mathbf{Z}_i)\Bigg] \\ &\qquad\qquad (1 \le i \le p - [(n+1)/2]). \end{aligned}$$

In particular, if $n = 2m$, we get

$$\begin{aligned} &F_{p-m}(\mathbf{Z}_1, \mathbf{Z}_2, \ldots, \mathbf{Z}_{p-m}, \mathbf{Z}_{m+1}, \ldots, \mathbf{Z}_p) \\ &= \frac{(-1)^{p-m+1}}{n}\sum_{r=1}^{m} F_{rm}(\mathbf{Z}_1, \mathbf{Z}_2, \ldots, \mathbf{Z}_{p-m}, \mathbf{Z}_{m+1}, \ldots, \mathbf{Z}_p), \end{aligned}$$

where

$$\begin{aligned} &F_{r+m,m}(\mathbf{Z}_1, \ldots, \mathbf{Z}_{p-m}, \mathbf{Z}_{m+1}, \ldots, \mathbf{Z}_p) \\ &\qquad = -F_{rm}(\mathbf{Z}_{m+1}, \ldots, \mathbf{Z}_p, \mathbf{Z}_1, \ldots, \mathbf{Z}_{p-m}) \ (1 \le r \le m). \ \square \end{aligned}$$

We have noticed that in [13] special generalized cases of the equations (1.1), (1.19) and (1.21) are considered. They are solved in a complicated manner using a cyclic operator.

At the end of the present section we give two more theorems.

Theorem 1.4. *The general solution of the functional equation*

$$\sum_{i=1}^{n} f_i(\mathbf{Z}_i, \mathbf{Z}_{i+1}, \ldots, \mathbf{Z}_{i+p-1}) = \mathbf{O} \qquad (\mathbf{Z}_{n+i} \equiv \mathbf{Z}_i) \tag{1.23}$$

is given by

$$\begin{aligned} & f_r(\mathbf{Z}_r, \mathbf{Z}_{r+1}, \ldots, \mathbf{Z}_{r+p-1}) \\ & \quad = \sum_{j=1}^{r-1} (-1)^{r+1} F_{jr}(\{\mathbf{Z}_r, \mathbf{Z}_{r+1}, \ldots, \mathbf{Z}_{r+p-1}\} \cap \{\mathbf{Z}_j, \mathbf{Z}_{j+1}, \ldots, \mathbf{Z}_{j+p-1}\}) \\ & \quad + \sum_{j=r+1}^{n} (-1)^{j} F_{rj}(\{\mathbf{Z}_r, \mathbf{Z}_{r+1}, \ldots, \mathbf{Z}_{r+p-1}\} \cap \{\mathbf{Z}_j, \mathbf{Z}_{j+1}, \ldots, \mathbf{Z}_{j+p-1}\}) \\ & \qquad\qquad\qquad\qquad (1 \le r \le n), \end{aligned}$$

where F_{rj} $(1 \le r \le n-1,\ r+1 \le j \le n)$ *are arbitrary complex vector functions from* $\mathcal{V}$ *such that*

$$F_{rj}(\{\mathbf{Z}_r, \mathbf{Z}_{r+1}, \ldots, \mathbf{Z}_{r+p-1}\} \cap \{\mathbf{Z}_j, \mathbf{Z}_{j+1}, \ldots, \mathbf{Z}_{j+p-1}\}) = A_{rj}$$

if

$$\{\mathbf{Z}_r, \mathbf{Z}_{r+1}, \ldots, \mathbf{Z}_{r+p-1}\} \cap \{\mathbf{Z}_j, \mathbf{Z}_{j+1}, \ldots, \mathbf{Z}_{j+p-1}\} = \emptyset,$$

where A_{rj} *are constant vectors from* $\mathcal{V}$ *and* $\sum_{\sigma}^{\nu} = \mathbf{O}$ *for* $\sigma > \nu$.

This theorem generalizes the theorem 1.2 where it is assumed that $p < n < 2p - 1$. Some other particular cases of the equation (1.23) were considered in [11, 12] under the hypothesis that the functions and the independent variables are real.

Theorem 1.4 is proved in [10]. The idea of its proof with slight modifications will be used below to prove a more general theorem which was kindly suggested to us by the referee.

Consider the equation

$$\sum_{i=1}^{k} f_i(\mathbf{Z}_{\bar{i}(1)}, \mathbf{Z}_{\bar{i}(2)}, \ldots, \mathbf{Z}_{\bar{i}(s_i)}) = \mathbf{O}, \tag{1.24}$$

where $1 \le s_i \le n$, $f_i : \mathcal{V}^{s_i} \mapsto \mathcal{V}$ $(1 \le i \le k)$, $\bar{i}(\nu) \in \{1, 2, \ldots, n\}$ for $1 \le \nu \le s_i$, $\bar{i}(\mu) \ne \bar{i}(\nu)$ for $\mu \ne \nu$.

For two sets of indices $\{\bar{i}(1), \bar{i}(2), \ldots, \bar{i}(s_i)\}$ and $\{\bar{j}(1), \bar{j}(2), \ldots, \bar{j}(s_j)\}$ we denote their intersection by $\{\overline{ij}(1), \overline{ij}(2), \ldots, \overline{ij}(t_{ij})\}$.

Theorem 1.5. *The general solution of the functional equation* (1.24) *is*

$$\begin{aligned} & f_i(\mathbf{Z}_{\bar{i}(1)}, \mathbf{Z}_{\bar{i}(2)}, \ldots, \mathbf{Z}_{\bar{i}(s_i)}) \\ & \quad = \sum_{j=1}^{i-1} F_{ji}(\mathbf{Z}_{\overline{ji}(1)}, \ldots, \mathbf{Z}_{\overline{ji}(t_{ji})}) - \sum_{j=i+1}^{k} F_{ij}(\mathbf{Z}_{\overline{ij}(1)}, \ldots, \mathbf{Z}_{\overline{ij}(t_{ij})}), \end{aligned} \tag{1.25}$$

where $F_{ij}(\mathbf{Z}_{\overline{ij}(1)},\dots,\mathbf{Z}_{\overline{ij}(t_{ij})})$ *are arbitrary functions from* $\mathcal{V}$ *for* $1 \le i \le k-1$, $i+1 \le j \le k$, *and* $\sum_{\sigma}^{\nu} = \mathbf{O}$ *for* $\sigma > \nu$.

Proof. We will prove the theorem by induction.

For $k=2$, equation (1.24) becomes

$$f_1(\mathbf{Z}_{\overline{1}(1)},\mathbf{Z}_{\overline{1}(2)},\dots,\mathbf{Z}_{\overline{1}(s_1)}) + f_2(\mathbf{Z}_{\overline{2}(1)},\mathbf{Z}_{\overline{2}(2)},\dots,\mathbf{Z}_{\overline{2}(s_2)}) = \mathbf{O}.$$

It is obvious that the functions f_1 and f_2 depend just on the variables $\mathbf{Z}_{\overline{12}(1)}$, $\mathbf{Z}_{\overline{12}(2)}$, …, $\mathbf{Z}_{\overline{12}(t_{12})}$, thus we may write

$$\begin{aligned} f_1(\mathbf{Z}_{\overline{1}(1)},\mathbf{Z}_{\overline{1}(2)},\dots,\mathbf{Z}_{\overline{1}(s_1)}) &= -F_{12}(\mathbf{Z}_{\overline{12}(1)},\mathbf{Z}_{\overline{12}(2)},\dots,\mathbf{Z}_{\overline{12}(t_{12})}),\\ f_2(\mathbf{Z}_{\overline{2}(1)},\mathbf{Z}_{\overline{2}(2)},\dots,\mathbf{Z}_{\overline{2}(s_2)}) &= F_{12}(\mathbf{Z}_{\overline{12}(1)},\mathbf{Z}_{\overline{12}(2)},\dots,\mathbf{Z}_{\overline{12}(t_{12})}) \end{aligned}$$

for an arbitrary function $F_{12}:\mathcal{V}^{t_{12}} \mapsto \mathcal{V}$, i.e., for $k=2$ the general solution of (1.24) is given by (1.25).

For some fixed k suppose that the general solution of the functional equation (1.24) is given by the formula (1.25).

Now, let us consider the equation

$$\sum_{i=1}^{k+1} f_i(\mathbf{Z}_{\overline{i}(1)},\mathbf{Z}_{\overline{i}(2)},\dots,\mathbf{Z}_{\overline{i}(s_i)}) = \mathbf{O}. \tag{1.26}$$

If we put $\mathbf{Z}_i = \mathcal{C}_i$ for $i \notin \{\overline{k+1}(1),\overline{k+1}(2),\dots,\overline{k+1}(s_{k+1})\}$, we obtain the representation

$$\begin{aligned} f_{k+1}(\mathbf{Z}_{\overline{k+1}(1)},\mathbf{Z}_{\overline{k+1}(2)},\dots,\mathbf{Z}_{\overline{k+1}(s_{k+1})}) \\ = \sum_{j=1}^{k} F_{j,k+1}(\mathbf{Z}_{\overline{j,k+1}(1)},\mathbf{Z}_{\overline{j,k+1}(2)},\dots,\mathbf{Z}_{\overline{j,k+1}(t_{j,k+1})}) \end{aligned} \tag{1.27}$$

where

$$\begin{aligned} F_{j,k+1}(\mathbf{Z}_{\overline{j,k+1}(1)},\mathbf{Z}_{\overline{j,k+1}(2)},\dots,\mathbf{Z}_{\overline{j,k+1}(t_{j,k+1})}) \\ = -f_j(\mathbf{Z}_{\overline{j}(1)},\mathbf{Z}_{\overline{j}(2)},\dots,\mathbf{Z}_{\overline{j}(s_j)})\Big|_{\substack{\mathbf{Z}_i=\mathcal{C}_i \text{ for} \\ i\notin\{\overline{k+1}(1),\overline{k+1}(2),\dots,\overline{k+1}(s_{k+1})\}}}. \end{aligned}$$

If we substitute (1.27) into (1.26) and denote

$$\begin{aligned} g_i(\mathbf{Z}_{\overline{i}(1)},\mathbf{Z}_{\overline{i}(2)},\dots,\mathbf{Z}_{\overline{i}(s_i)}) &= f_i(\mathbf{Z}_{\overline{i}(1)},\mathbf{Z}_{\overline{i}(2)},\dots,\mathbf{Z}_{\overline{i}(s_i)}) \\ &+ F_{i,k+1}(\mathbf{Z}_{\overline{i,k+1}(1)},\mathbf{Z}_{\overline{i,k+1}(2)},\dots,\mathbf{Z}_{\overline{i,k+1}(t_{i,k+1})}), \end{aligned}$$

then equation (1.26) becomes

$$\sum_{i=1}^{k} g_i(\mathbf{Z}_{\bar{i}(1)}, \mathbf{Z}_{\bar{i}(2)}, \dots, \mathbf{Z}_{\bar{i}(s_i)}) = \mathbf{O}.$$

By assumption its general solution is

$$g_i(\mathbf{Z}_{\bar{i}(1)}, \mathbf{Z}_{\bar{i}(2)}, \dots, \mathbf{Z}_{\bar{i}(s_i)}) = \sum_{j=1}^{i-1} F_{ji}(\mathbf{Z}_{\overline{ji}(1)}, \dots, \mathbf{Z}_{\overline{ji}(t_{ji})}) - \sum_{j=i+1}^{k} F_{ij}(\mathbf{Z}_{\overline{ij}(1)}, \dots, \mathbf{Z}_{\overline{ij}(t_{ij})}),$$

and hence

$$f_i(\mathbf{Z}_{\bar{i}(1)}, \mathbf{Z}_{\bar{i}(2)}, \dots, \mathbf{Z}_{\bar{i}(s_i)}) = \sum_{j=1}^{i-1} F_{ji}(\mathbf{Z}_{\overline{ji}(1)}, \dots, \mathbf{Z}_{\overline{ji}(t_{ji})}) - \sum_{j=i+1}^{k+1} F_{ij}(\mathbf{Z}_{\overline{ij}(1)}, \dots, \mathbf{Z}_{\overline{ij}(t_{ij})})$$

for $1 \le i \le k$. Together with (1.27) this yields (1.25) with $k+1$ instead of k. □

§2. Generalized Paracyclic Complex Vector Functional Equations of the First Kind

Let $\mathcal{V}$ be a complex vector space with complex dimension n, and let the complex vectors $\mathbf{X}_i$, $\mathbf{Y}_j \in \mathcal{V}$ $(1 \le i, j \le n)$ be given as in the previous section. Throughout the section $\mathcal{C}$, $\mathcal{D}$, $\mathcal{C}_i$ and $\mathcal{D}_i$ are constant vectors in $\mathcal{V}$.

Also, let there exist mappings $f_i : \mathcal{V}^{p+q} \mapsto \mathcal{V}$ $(1 \le i \le k)$.

Now we will consider the following generalized paracyclic complex vector functional equation of the first kind

$$\sum_{i=1}^{k} f_i(\mathbf{X}_i, \mathbf{X}_{i+1}, \dots, \mathbf{X}_{i+p-1}, \mathbf{Y}_i, \mathbf{Y}_{i+1}, \dots, \mathbf{Y}_{i+q-1}) = \mathbf{O} \quad (k \le n; \quad \mathbf{X}_{n+i} \equiv \mathbf{X}_i, \quad \mathbf{Y}_{n+i} \equiv \mathbf{Y}_i). \tag{2.1}$$

In order to determine the general solution of the functional equation (2.1), we must distinguish the following six cases:

1^0 $q < 2q-1 \le p = n$, 2^0 $q < p = n < 2q-1$,
3^0 $q < p < n < 2q-1 < 2p-1$,
4^0 $q < p < 2q-1 \le n \le p+q-1 < 2p-1$,
5^0 $q < p < p+q-1 < n < 2p-1$,
6^0 $q < p < 2q-1 < 2p-1 \le n$.

Theorem 2.1. *If $q < 2q-1 \le p = n$, then the general solution of the functional equation* (2.1) *is given by*

$$
\begin{aligned}
(2.2)\quad & f_r(\mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_n, \mathbf{Y}_1, \mathbf{Y}_2, \dots, \mathbf{Y}_q) \\
= & \sum_{i=1}^{\min(k-r,\, q-1)} (-1)^{i-1} F_{ri}(\mathbf{X}_{i+1}, \mathbf{X}_{i+2}, \dots, \mathbf{X}_i, \mathbf{Y}_{i+1}, \mathbf{Y}_{i+2}, \dots, \mathbf{Y}_q) \\
+ & \sum_{i=n-r+1}^{q-1} (-1)^{i-1} F_{ri}(\mathbf{X}_{i+1}, \mathbf{X}_{i+2}, \dots, \mathbf{X}_i, \mathbf{Y}_{i+1}, \mathbf{Y}_{i+2}, \dots, \mathbf{Y}_q) \\
+ & \sum_{i=q}^{\min(k-r, n-q)} (-1)^{i-1} F_{ri}(\mathbf{X}_{i+1}, \mathbf{X}_{i+2}, \dots, \mathbf{X}_i) \\
+ & \sum_{i=\max(n-r+1,\, q)}^{n-q} (-1)^{n-i} F_{i+r,n-i}(\mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_n) \\
+ & \sum_{i=n-q+1}^{k-r} (-1)^{n-i} F_{i+r,n-i}(\mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_n, \mathbf{Y}_1, \mathbf{Y}_2, \dots, \mathbf{Y}_{q+i}) \\
+ & \sum_{i=\max(n-r+1, n-q+1)}^{n-1} (-1)^{n-i} F_{i+r,n-i}(\mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_n, \mathbf{Y}_1, \mathbf{Y}_2, \dots, \mathbf{Y}_{q+i}) \\
& \qquad (1 \le r \le k),
\end{aligned}
$$

where F_{ij} are arbitrary complex vector functions from $\mathcal{V}$.

Proof. The proof of this theorem is completely analogous to the proof of the theorem 1.1 given in the previous section. □

Theorem 2.2. *If $q < p = n < 2q-1$, then the general solution of the functional equation* (2.1) *will be*

$$
\begin{aligned}
(2.3)\quad & f_r(\mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_n, \mathbf{Y}_1, \mathbf{Y}_2, \dots, \mathbf{Y}_q) \\
= & \sum_{i=1}^{\min(k-r, n-q)} (-1)^{i-1} F_{ri}(\mathbf{X}_{i+1}, \mathbf{X}_{i+2}, \dots, \mathbf{X}_i, \mathbf{Y}_{i+1}, \mathbf{Y}_{i+2}, \dots, \mathbf{Y}_q) \\
+ & \sum_{i=n-r+1}^{n-q} (-1)^{i-1} F_{ri}(\mathbf{X}_{i+1}, \mathbf{X}_{i+2}, \dots, \mathbf{X}_i, \mathbf{Y}_{i+1}, \mathbf{Y}_{i+2}, \dots, \mathbf{Y}_q)
\end{aligned}
$$

$$+ \sum_{i=n-q+1}^{\min(k-r,q-1)} (-1)^{i-1} \times$$
$$F_{ri}(\mathbf{X}_{i+1}, \mathbf{X}_{i+2}, \dots, \mathbf{X}_i, \mathbf{Y}_{i+1}, \mathbf{Y}_{i+2}, \dots, \mathbf{Y}_q, \mathbf{Y}_1, \mathbf{Y}_2, \dots, \mathbf{Y}_{q+i})$$
$$+ \sum_{i=\max(n-r+1,n-q+1)}^{q-1} (-1)^{n-i} \times$$
$$F_{i+r,n-i}(\mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_n, \mathbf{Y}_1, \mathbf{Y}_2, \dots, \mathbf{Y}_{q+i}, \mathbf{Y}_{i+1}, \mathbf{Y}_{i+2}, \dots, \mathbf{Y}_q)$$
$$+ \sum_{i=q}^{k-r} (-1)^{n-i} F_{i+r,n-i}(\mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_n, \mathbf{Y}_1, \mathbf{Y}_2, \dots, \mathbf{Y}_{q+i})$$
$$+ \sum_{i=\max(n-r+1,\,q)}^{n-1} (-1)^{n-i} F_{i+r,n-i}(\mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_n, \mathbf{Y}_1, \mathbf{Y}_2, \dots, \mathbf{Y}_{q+i})$$
$$(1 \le r \le k),$$

where F_{ij} are arbitrary complex vector functions from $\mathcal{V}$.

Proof. The proof of this theorem is analogous to that of the previous theorem 2.1. □

Theorem 2.3. *If $q < p < n < 2q - 1 < 2p - 1$, then the general solution of the functional equation* (2.1) *is*

$$(2.4) \quad f_r(\mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_p, \mathbf{Y}_1, \mathbf{Y}_2, \dots, \mathbf{Y}_q)$$
$$= \sum_{i=1}^{\min(n-p,k-r)} (-1)^{i-1} F_{ri}(\mathbf{X}_{i+1}, \mathbf{X}_{i+2}, \dots, \mathbf{X}_p, \mathbf{Y}_{i+1}, \mathbf{Y}_{i+2}, \dots, \mathbf{Y}_q)$$
$$+ \sum_{i=n-r+1}^{n-p} (-1)^{i-1} F_{ri}(\mathbf{X}_{i+1}, \mathbf{X}_{i+2}, \dots, \mathbf{X}_p, \mathbf{Y}_{i+1}, \mathbf{Y}_{i+2}, \dots, \mathbf{Y}_q)$$
$$+ \sum_{i=n-p+1}^{\min(k-r,n-q)} (-1)^{i-1} \times$$
$$F_{ri}(\mathbf{X}_{i+1}, \mathbf{X}_{i+2}, \dots, \mathbf{X}_p, \mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_{i+p}, \mathbf{Y}_{i+1}, \mathbf{Y}_{i+2}, \dots, \mathbf{Y}_q)$$
$$+ \sum_{i=\max(n-p+1,n-r+1)}^{n-q} (-1)^{i-1} \times$$
$$F_{ri}(\mathbf{X}_{i+1}, \mathbf{X}_{i+2}, \dots, \mathbf{X}_p, \mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_{i+p}, \mathbf{Y}_{i+1}, \mathbf{Y}_{i+2}, \dots, \mathbf{Y}_q)$$

$$+ \sum_{i=n-q+1}^{\min(k-r,q-1)} (-1)^{i-1} F_{ri}(\mathbf{X}_{i+1}, \mathbf{X}_{i+2}, \ldots, \mathbf{X}_p, \mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_{i+p}, \mathbf{Y}_{i+1}, \mathbf{Y}_{i+2}, \ldots, \mathbf{Y}_q, \mathbf{Y}_1, \mathbf{Y}_2, \ldots, \mathbf{Y}_{i+q})$$

$$+ \sum_{i=\max(n-q+1,n-r+1)}^{q-1} (-1)^{n-i} F_{i+r,n-i}(\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_{i+p}, \mathbf{X}_{i+1}, \mathbf{X}_{i+2}, \ldots, \mathbf{X}_p, \mathbf{Y}_1, \mathbf{Y}_2, \ldots, \mathbf{Y}_{i+q}, \mathbf{Y}_{i+1}, \mathbf{Y}_{i+2}, \ldots, \mathbf{Y}_q)$$

$$+ \sum_{i=q}^{\min(k-r,p-1)} (-1)^{n-i} \times F_{i+r,n-i}(\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_{i+p}, \mathbf{X}_{i+1}, \mathbf{X}_{i+2}, \ldots, \mathbf{X}_p, \mathbf{Y}_1, \mathbf{Y}_2, \ldots, \mathbf{Y}_{i+q})$$

$$+ \sum_{i=\max(q,n-r+1)}^{p-1} (-1)^{n-i} \times F_{i+r,n-i}(\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_{i+p}, \mathbf{X}_{i+1}, \mathbf{X}_{i+2}, \ldots, \mathbf{X}_p, \mathbf{Y}_1, \mathbf{Y}_2, \ldots, \mathbf{Y}_{i+q})$$

$$+ \sum_{i=p}^{k-r} (-1)^{n-i} F_{i+r,n-i}(\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_{i+p}, \mathbf{Y}_1, \mathbf{Y}_2, \ldots, \mathbf{Y}_{i+q})$$

$$+ \sum_{i=\max(p,n-r+1)}^{n-1} (-1)^{n-i} F_{i+r,n-i}(\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_{i+p}, \mathbf{Y}_1, \mathbf{Y}_2, \ldots, \mathbf{Y}_{i+q}) \quad (1 \le r \le k),$$

where F_{ij} are arbitrary complex vector functions from $\mathcal{V}$.

Theorem 2.4. *If $q < p < 2q - 1 \le n \le p + q - 1 < 2p - 1$, then the general solution of the functional equation (2.1) is given by*

$$f_r(\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_p, \mathbf{Y}_1, \mathbf{Y}_2, \ldots, \mathbf{Y}_q) \tag{2.5}$$

$$= \sum_{i=1}^{\min(n-p,k-r)} (-1)^{i-1} F_{ri}(\mathbf{X}_{i+1}, \mathbf{X}_{i+2}, \ldots, \mathbf{X}_p, \mathbf{Y}_{i+1}, \mathbf{Y}_{i+2}, \ldots, \mathbf{Y}_q)$$

$$+ \sum_{i=n-r+1}^{n-p} (-1)^{i-1} F_{ri}(\mathbf{X}_{i+1}, \mathbf{X}_{i+2}, \ldots, \mathbf{X}_p, \mathbf{Y}_{i+1}, \mathbf{Y}_{i+2}, \ldots, \mathbf{Y}_q)$$

$$+ \sum_{i=n-p+1}^{\min(k-r,q-1)} (-1)^{i-1} \times F_{ri}(\mathbf{X}_{i+1}, \mathbf{X}_{i+2}, \ldots, \mathbf{X}_p, \mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_{i+p}, \mathbf{Y}_{i+1}, \mathbf{Y}_{i+2}, \ldots, \mathbf{Y}_q)$$

$$+ \sum_{i=\max(n-p+1,n-r+1)}^{q-1} (-1)^{i-1} \times$$

$$F_{ri}(\mathbf{X}_{i+1}, \mathbf{X}_{i+2}, \dots, \mathbf{X}_p, \mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_{i+p}, \mathbf{Y}_{i+1}, \mathbf{Y}_{i+2}, \dots, \mathbf{Y}_q)$$

$$+ \sum_{i=q}^{\min(k-r,n-q)} (-1)^{i-1} F_{ri}(\mathbf{X}_{i+1}, \mathbf{X}_{i+2}, \dots, \mathbf{X}_p, \mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_{i+p})$$

$$+ \sum_{i=\max(n-i+1,\, q)}^{n-q} (-1)^{n-i} F_{i+r,n-i}(\mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_{i+p}, \mathbf{X}_{i+1}, \mathbf{X}_{i+2}, \dots, \mathbf{X}_p)$$

$$+ \sum_{i=n-q+1}^{\min(k-r,p-1)} (-1)^{n-i} \times$$

$$F_{i+r,n-i}(\mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_{i+p}, \mathbf{X}_{i+1}, \mathbf{X}_{i+2}, \dots, \mathbf{X}_p, \mathbf{Y}_1, \mathbf{Y}_2, \dots, \mathbf{Y}_{i+q})$$

$$+ \sum_{i=\max(n-q+1,n-r+1)}^{p-1} (-1)^{n-i} \times$$

$$F_{i+r,n-i}(\mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_{i+p}, \mathbf{X}_{i+1}, \mathbf{X}_{i+2}, \dots, \mathbf{X}_p, \mathbf{Y}_1, \mathbf{Y}_2, \dots, \mathbf{Y}_{i+q})$$

$$+ \sum_{i=p}^{k-r} (-1)^{n-i} F_{i+r,n-i}(\mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_{i+p}, \mathbf{Y}_1, \mathbf{Y}_2, \dots, \mathbf{Y}_{i+q})$$

$$+ \sum_{i=\max(p,n-r+1)}^{n-1} (-1)^{n-i} F_{i+r,n-i}(\mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_{i+p}, \mathbf{Y}_1, \mathbf{Y}_2, \dots, \mathbf{Y}_{i+q})$$

$$(1 \le r \le k),$$

where F_{ij} are arbitrary complex vector functions from $\mathcal{V}$.

We can prove the previous two theorems in the same way as the following theorem.

Theorem 2.5. *If $q < p < p + q - 1 < n < 2p - 1$, then the general solution of the functional equation (2.1) is given by the formulas*

(2.6) $f_r(\mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_p, \mathbf{Y}_1, \mathbf{Y}_2, \dots, \mathbf{Y}_q)$

$$= \sum_{i=1}^{\min(q-1,k-r)} (-1)^{i-1} F_{ri}(\mathbf{X}_{i+1}, \mathbf{X}_{i+2}, \dots, \mathbf{X}_p, \mathbf{Y}_{i+1}, \mathbf{Y}_{i+2}, \dots, \mathbf{Y}_q)$$

$$+ \sum_{i=n-r+1}^{q-1} (-1)^{i-1} F_{ri}(\mathbf{X}_{i+1}, \mathbf{X}_{i+2}, \dots, \mathbf{X}_p, \mathbf{Y}_{i+1}, \mathbf{Y}_{i+2}, \dots, \mathbf{Y}_q)$$

$$+ \sum_{i=q}^{\min(k-r,n-p)} (-1)^{i-1} F_{ri}(\mathbf{X}_{i+1}, \mathbf{X}_{i+2}, \dots, \mathbf{X}_p)$$

$$+ \sum_{i=\max(q,n-r+1)}^{n-p} (-1)^{i-1} F_{ri}(\mathbf{X}_{i+1}, \mathbf{X}_{i+2}, \dots, \mathbf{X}_p)$$

$$+ \sum_{i=n-p+1}^{\min(k-r,p-1)} (-1)^{i-1} F_{ri}(\mathbf{X}_{i+1}, \mathbf{X}_{i+2}, \dots, \mathbf{X}_p, \mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_{i+p})$$

$$+ \sum_{i=\max(n-p+1,n-r+1)}^{p-1} (-1)^{n-i} \times$$

$$F_{i+r,n-i}(\mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_{i+p}, \mathbf{X}_{i+1}, \mathbf{X}_{i+2}, \dots, \mathbf{X}_p)$$

$$+ \sum_{i=p}^{\min(k-r,n-q)} (-1)^{n-i} F_{i+r,n-i}(\mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_{i+p})$$

$$+ \sum_{i=\max(p,n-r+1)}^{n-q} (-1)^{n-i} F_{i+r,n-i}(\mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_{i+p})$$

$$+ \sum_{i=n-q+1}^{k-r} (-1)^{n-i} F_{i+r,n-i}(\mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_{i+p}, \mathbf{Y}_1, \mathbf{Y}_2, \dots, \mathbf{Y}_{i+q})$$

$$+ \sum_{i=\max(n-q+1,n-r+1)}^{n-1} (-1)^{n-i} F_{i+r,n-i}(\mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_{i+p}, \mathbf{Y}_1, \mathbf{Y}_2, \dots, \mathbf{Y}_{i+q})$$

$$(1 \le r \le k),$$

where F_{ij} are arbitrary complex vector functions from $\mathcal{V}$.

Proof. The proof of this theorem is based on mathematical induction.

For $k = 2$ the functional equation (2.1) has the form

$$f_1(\mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_p, \mathbf{Y}_1, \mathbf{Y}_2, \dots, \mathbf{Y}_q) + f_2(\mathbf{X}_2, \mathbf{X}_3, \dots, \mathbf{X}_{p+1}, \mathbf{Y}_2, \mathbf{Y}_3, \dots, \mathbf{Y}_{q+1}) = \mathbf{O}. \tag{2.7}$$

Putting $\mathbf{X}_{p+1} = \mathcal{C},\ \mathbf{Y}_{q+1} = \mathcal{D}$ into the equation (2.7), we obtain

$$f_1(\mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_p, \mathbf{Y}_1, \mathbf{Y}_2, \dots, \mathbf{Y}_q) = F_{11}(\mathbf{X}_2, \mathbf{X}_3, \dots, \mathbf{X}_p, \mathbf{Y}_2, \mathbf{Y}_3, \dots, \mathbf{Y}_q). \tag{2.8}$$

If we put (2.8) into (2.7), we have

$$f_2(\mathbf{X}_2,\mathbf{X}_3,\dots,\mathbf{X}_{p+1},\mathbf{Y}_2,\mathbf{Y}_3,\dots,\mathbf{Y}_{q+1}) = -F_{11}(\mathbf{X}_2,\mathbf{X}_3,\dots,\mathbf{X}_p,\mathbf{Y}_2,\mathbf{Y}_3,\dots,\mathbf{Y}_q),$$

i.e.

$$(2.9)\quad f_2(\mathbf{X}_1,\mathbf{X}_2,\dots,\mathbf{X}_p,\mathbf{Y}_1,\mathbf{Y}_2,\dots,\mathbf{Y}_q) = -F_{11}(\mathbf{X}_1,\mathbf{X}_2,\dots,\mathbf{X}_{p-1},\mathbf{Y}_1,\mathbf{Y}_2,\dots,\mathbf{Y}_{q-1}).$$

For $k=2$ and $r=1$ and $r=2$, from (2.6) we deduce (2.8) and (2.9), which means that the theorem holds for $k=2$.

Now we will suppose that the general solution of the functional equation

$$(2.10)\qquad \sum_{i=1}^{k} g_i(\mathbf{X}_i,\mathbf{X}_{i+1},\dots,\mathbf{X}_{i+p-1},\mathbf{Y}_i,\mathbf{Y}_{i+1},\dots,\mathbf{Y}_{i+q-1}) = \mathbf{O}$$

is given by (2.6) with f_r replaced by g_r.

Let us consider the following functional equation

$$(2.11)\qquad \sum_{i=1}^{k+1} f_i(\mathbf{X}_i,\mathbf{X}_{i+1},\dots,\mathbf{X}_{i+p-1},\mathbf{Y}_i,\mathbf{Y}_{i+1},\dots,\mathbf{Y}_{i+q-1}) = \mathbf{O}.$$

We will distinguish the following five cases:

1^o Let $1<k<q$. The substitutions

$$(2.12)\qquad \begin{array}{llll} \mathbf{X}_i = \mathcal{C}_i & \text{for} & i \not\equiv k+1,k+2,\dots,k+p & (\mathrm{mod}\ n), \\ \mathbf{Y}_i = \mathcal{D}_i & \text{for} & i \not\equiv k+1,k+2,\dots,k+q & (\mathrm{mod}\ n) \end{array}$$

transform the equation (2.11) into

$$(2.13)\quad f_{k+1}(\mathbf{X}_{k+1},\mathbf{X}_{k+2},\dots,\mathbf{X}_{p+k},\mathbf{Y}_{k+1},\mathbf{Y}_{k+2},\dots,\mathbf{Y}_{q+k}) = \sum_{i=n-k}^{n-1} (-1)^{n-i} \times F_{i+k+1,n-i}(\mathbf{X}_{k+1},\mathbf{X}_{k+2},\dots,\mathbf{X}_{i+p+k},\mathbf{Y}_{k+1},\mathbf{Y}_{k+2},\dots,\mathbf{Y}_{i+q+k}).$$

Putting (2.13) into (2.11) and introducing new functions by

$$(2.14)\qquad g_i = f_i + (-1)^{k+1-i} F_{i,k+1-i} \qquad (1 \le i \le k),$$

we obtain the equation (2.10). According to (2.6) and (2.14), the general solution of the equation (2.11) is

$$(2.15)\quad f_r(\mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_p, \mathbf{Y}_1, \mathbf{Y}_2, \dots, \mathbf{Y}_q)$$

$$= \sum_{i=1}^{k-r} (-1)^{i-1} F_{ri}(\mathbf{X}_{i+1}, \mathbf{X}_{i+2}, \dots, \mathbf{X}_p, \mathbf{Y}_{i+1}, \mathbf{Y}_{i+2}, \dots, \mathbf{Y}_q)$$

$$+ \sum_{i=n-r+1}^{q-1} (-1)^{i-1} F_{ri}(\mathbf{X}_{i+1}, \mathbf{X}_{i+2}, \dots, \mathbf{X}_p, \mathbf{Y}_{i+1}, \mathbf{Y}_{i+2}, \dots, \mathbf{Y}_q)$$

$$+ \sum_{i=\max(q, n-r+1)}^{n-p} (-1)^{i-1} F_{ri}(\mathbf{X}_{i+1}, \mathbf{X}_{i+2}, \dots, \mathbf{X}_p)$$

$$+ \sum_{i=\max(n-p+1, n-r+1)}^{p-1} (-1)^{n-i} \times F_{i+r,n-i}(\mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_{i+p}, \mathbf{X}_{i+1}, \mathbf{X}_{i+2}, \dots, \mathbf{X}_p)$$

$$+ \sum_{i=\max(p, n-r+1)}^{n-q} (-1)^{n-i} F_{i+r,n-i}(\mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_{i+p})$$

$$+ \sum_{i=\max(n-q+1, n-r+1)}^{n-1} (-1)^{n-i} F_{i+r,n-i}(\mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_{i+p}, \mathbf{Y}_1, \mathbf{Y}_2, \dots, \mathbf{Y}_{i+q})$$

$$+ (-1)^{k-r} F_{r,k+1-r}(\mathbf{X}_{k-r+2}, \mathbf{X}_{k-r+3}, \dots, \mathbf{X}_p, \mathbf{Y}_{k-r+2}, \mathbf{Y}_{k-r+3}, \dots, \mathbf{Y}_q)$$

$$= \sum_{i=1}^{k+1-r} (-1)^{i-1} F_{ri}(\mathbf{X}_{i+1}, \mathbf{X}_{i+2}, \dots, \mathbf{X}_p, \mathbf{Y}_{i+1}, \mathbf{Y}_{i+2}, \dots, \mathbf{Y}_q)$$

$$+ \sum_{i=n-r+1}^{q-1} (-1)^{i-1} F_{ri}(\mathbf{X}_{i+1}, \mathbf{X}_{i+2}, \dots, \mathbf{X}_p, \mathbf{Y}_{i+1}, \mathbf{Y}_{i+2}, \dots, \mathbf{Y}_q)$$

$$+ \sum_{i=\max(q, n-r+1)}^{n-p} (-1)^{i-1} F_{ri}(\mathbf{X}_{i+1}, \mathbf{X}_{i+2}, \dots, \mathbf{X}_p)$$

$$+ \sum_{i=\max(n-p+1, n-r+1)}^{p-1} (-1)^{n-i} \times F_{i+r,n-i}(\mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_{i+p}, \mathbf{X}_{i+1}, \mathbf{X}_{i+2}, \dots, \mathbf{X}_p)$$

$$+ \sum_{i=\max(p, n-r+1)}^{n-q} (-1)^{n-i} F_{i+r,n-i}(\mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_{i+p})$$

$$+ \sum_{i=\max(n-q+1,n-r+1)}^{n-1} (-1)^{n-i} F_{i+r,n-i}(\mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_{i+p}, \mathbf{Y}_1, \mathbf{Y}_2, \dots, \mathbf{Y}_{i+q})$$

$$(1 \le r \le k).$$

On the basis of the expressions (2.13) and (2.15), if $1 < k < q$, the theorem holds for $k+1$. Thus it holds for all such k, and also for $k = q$.

2^o Let $q \le k < n - p + 1$. For the values (2.12) the functional equation (2.11) becomes

$$f_{k+1}(\mathbf{X}_{k+1}, \mathbf{X}_{k+2}, \dots, \mathbf{X}_{p+k}, \mathbf{Y}_{k+1}, \mathbf{Y}_{k+2}, \dots, \mathbf{Y}_{q+k}) \tag{2.16}$$
$$= \sum_{i=n-k}^{n-q} (-1)^{n-i} F_{i+k+1,n-i}(\mathbf{X}_{k+1}, \mathbf{X}_{k+2}, \dots, \mathbf{X}_{i+p+k})$$
$$+ \sum_{i=n-q+1}^{n-1} (-1)^{n-i} \times$$
$$F_{i+k+1,n-i}(\mathbf{X}_{k+1}, \mathbf{X}_{k+2}, \dots, \mathbf{X}_{i+p+k}, \mathbf{Y}_{k+1}, \mathbf{Y}_{k+2}, \dots, \mathbf{Y}_{i+q+k}).$$

Now we will introduce the notations (2.14). On the basis of the expressions (2.16) and (2.14), the equation (2.11) becomes (2.10). By using the inductive hypothesis and by virtue of the expression (2.14), we can conclude that the general solution of the equation (2.11) is given by the following equality

$$f_r(\mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_p, \mathbf{Y}_1, \mathbf{Y}_2, \dots, \mathbf{Y}_q) \tag{2.17}$$
$$= \sum_{i=1}^{\min(q-1,k+1-r)} (-1)^{i-1} F_{ri}(\mathbf{X}_{i+1}, \mathbf{X}_{i+2}, \dots, \mathbf{X}_p, \mathbf{Y}_{i+1}, \mathbf{Y}_{i+2}, \dots, \mathbf{Y}_q)$$
$$+ \sum_{i=q}^{k+1-r} (-1)^{i-1} F_{ri}(\mathbf{X}_{i+1}, \mathbf{X}_{i+2}, \dots, \mathbf{X}_p)$$
$$+ \sum_{i=n-r+1}^{q-1} (-1)^{i-1} F_{ri}(\mathbf{X}_{i+1}, \mathbf{X}_{i+2}, \dots, \mathbf{X}_p, \mathbf{Y}_{i+1}, \mathbf{Y}_{i+2}, \dots, \mathbf{Y}_q)$$
$$+ \sum_{i=\max(q,n-r+1)}^{n-p} (-1)^{i-1} F_{ri}(\mathbf{X}_{i+1}, \mathbf{X}_{i+2}, \dots, \mathbf{X}_p)$$
$$+ \sum_{i=\max(n-p+1,n-r+1)}^{p-1} (-1)^{n-i} \times$$
$$F_{i+r,n-i}(\mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_{i+p}, \mathbf{X}_{i+1}, \mathbf{X}_{i+2}, \dots, \mathbf{X}_p)$$

$$+ \sum_{i=\max(p,n-r+1)}^{n-q} (-1)^{n-i} F_{i+r,n-i}(\mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_{i+p})$$

$$+ \sum_{i=\max(n-q+1,n-r+1)}^{n-1} (-1)^{n-i} F_{i+r,n-i}(\mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_{i+p}, \mathbf{Y}_1, \mathbf{Y}_2, \dots, \mathbf{Y}_{i+q})$$

$$(1 \le r \le k).$$

Therefore, on the basis of the expressions (2.16) and (2.17), we can conclude that the theorem holds in this case too, and also for $k = n - p + 1$.

3^o Let $n - p + 1 \le k < p$. If we substitute (2.12) into (2.11), we find

$$\begin{aligned}(2.18)\quad & f_{k+1}(\mathbf{X}_{k+1}, \mathbf{X}_{k+2}, \dots, \mathbf{X}_{p+k}, \mathbf{Y}_{k+1}, \mathbf{Y}_{k+2}, \dots, \mathbf{Y}_{q+k}) \\ &= \sum_{i=n-k}^{p-1} (-1)^{n-i} \times \\ &\qquad F_{i+k+1,n-i}(\mathbf{X}_{k+1}, \mathbf{X}_{k+2}, \dots, \mathbf{X}_{i+p+k}, \mathbf{X}_{k+1+i}, \mathbf{X}_{k+2+i}, \dots, \mathbf{X}_{k+p}) \\ &+ \sum_{i=p}^{n-q} (-1)^{n-i} F_{i+k+1,n-i}(\mathbf{X}_{k+1}, \mathbf{X}_{k+2}, \dots, \mathbf{X}_{i+p+k}) \\ &+ \sum_{i=n-q+1}^{n-1} (-1)^{n-i} \times \\ &\qquad F_{i+k+1,n-i}(\mathbf{X}_{k+1}, \mathbf{X}_{k+2}, \dots, \mathbf{X}_{i+p+k}, \mathbf{Y}_{k+1}, \mathbf{Y}_{k+2}, \dots, \mathbf{Y}_{i+q+k}).\end{aligned}$$

If we substitute (2.18) into (2.11) and if we take into account the transformation (2.14), then we obtain the equation (2.10). On the basis of the expression (2.14) and the inductive hypothesis, we find that the general solution of the functional equation (2.11) is given by the following formula

$$\begin{aligned}(2.19)\quad & f_r(\mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_p, \mathbf{Y}_1, \mathbf{Y}_2, \dots, \mathbf{Y}_q) \\ &= \sum_{i=1}^{\min(q-1,k+1-r)} (-1)^{i-1} F_{ri}(\mathbf{X}_{i+1}, \mathbf{X}_{i+2}, \dots, \mathbf{X}_p, \mathbf{Y}_{i+1}, \mathbf{Y}_{i+2}, \dots, \mathbf{Y}_q) \\ &+ \sum_{i=q}^{\min(n-p,k+1-r)} (-1)^{i-1} F_{ri}(\mathbf{X}_{i+1}, \mathbf{X}_{i+2}, \dots, \mathbf{X}_p) \\ &+ \sum_{i=n-p+1}^{k+1-r} (-1)^{i-1} F_{ri}(\mathbf{X}_{i+1}, \mathbf{X}_{i+2}, \dots, \mathbf{X}_p, \mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_{i+p})\end{aligned}$$

$$+\sum_{i=n-r+1}^{q-1}(-1)^{i-1}F_{ri}(\mathbf{X}_{i+1},\mathbf{X}_{i+2},\dots,\mathbf{X}_p,\mathbf{Y}_{i+1},\mathbf{Y}_{i+2},\dots,\mathbf{Y}_q)$$

$$+\sum_{i=\max(q,n-r+1)}^{n-p}(-1)^{i-1}F_{ri}(\mathbf{X}_{i+1},\mathbf{X}_{i+2},\dots,\mathbf{X}_p)$$

$$+\sum_{i=\max(n-p+1,n-r+1)}^{p-1}(-1)^{n-i}\times$$

$$F_{i+r,n-i}(\mathbf{X}_1,\mathbf{X}_2,\dots,\mathbf{X}_{i+p},\mathbf{X}_{i+1},\mathbf{X}_{i+2},\dots,\mathbf{X}_p)$$

$$+\sum_{i=\max(p,n-r+1)}^{n-q}(-1)^{n-i}F_{i+r,n-i}(\mathbf{X}_1,\mathbf{X}_2,\dots,\mathbf{X}_{i+p})$$

$$+\sum_{i=\max(n-q+1,n-r+1)}^{n-1}(-1)^{n-i}F_{i+r,n-i}(\mathbf{X}_1,\mathbf{X}_2,\dots,\mathbf{X}_{i+p},\mathbf{Y}_1,\mathbf{Y}_2,\dots,\mathbf{Y}_{i+q})$$

$$(1\le r\le k).$$

On the basis of the equalities (2.18) and (2.19), the theorem holds for this case as well, and also for $k=p$.

4^o Let $p\le k<n-q+1$. Putting (2.12) into (2.11), we obtain

$$f_{k+1}(\mathbf{X}_{k+1},\mathbf{X}_{k+2},\dots,\mathbf{X}_{p+k},\mathbf{Y}_{k+1},\mathbf{Y}_{k+2},\dots,\mathbf{Y}_{q+k}) \tag{2.20}$$

$$=\sum_{i=n-k}^{n-p}(-1)^{i-1}F_{k+1,i}(\mathbf{X}_{i+k+1},\mathbf{X}_{i+k+2},\dots,\mathbf{X}_{p+k})$$

$$+\sum_{i=n-p+1}^{p-1}(-1)^{n-i}\times$$

$$F_{i+k+1,n-i}(\mathbf{X}_{k+1},\mathbf{X}_{k+2},\dots,\mathbf{X}_{i+k+p},\mathbf{X}_{i+k+1},\mathbf{X}_{i+k+2},\dots,\mathbf{X}_{k+p})$$

$$+\sum_{i=p}^{n-q}(-1)^{n-i}F_{i+k+1,n-i}(\mathbf{X}_{k+1},\mathbf{X}_{k+2},\dots,\mathbf{X}_{i+p+k})$$

$$+\sum_{i=n-q+1}^{n-1}(-1)^{n-i}\times$$

$$F_{i+k+1,n-i}(\mathbf{X}_{k+1},\mathbf{X}_{k+2},\dots,\mathbf{X}_{i+p+k},\mathbf{Y}_{k+1},\mathbf{Y}_{k+2},\dots,\mathbf{Y}_{i+q+k}).$$

If we substitute f_{k+1} given by (2.20) into (2.11), by the substitutions

$$g_i = \begin{cases} f_i + (-1)^{n-k+i} F_{k+1,n-k+i-1} & (1 \le i \le k-p+1), \\ f_i + (-1)^{k+1-i} F_{i,k+1-i} & (i = k-p+2, \dots, k), \end{cases} \tag{2.21}$$

we obtain the equation (2.10). According to (2.21), the general solution of the functional equation (2.11) is given by

$$\begin{aligned}
&f_r(\mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_p, \mathbf{Y}_1, \mathbf{Y}_2, \dots, \mathbf{Y}_q) \\
&= \sum_{i=1}^{\min(q-1,k+1-r)} (-1)^{i-1} F_{ri}(\mathbf{X}_{i+1}, \mathbf{X}_{i+2}, \dots, \mathbf{X}_p, \mathbf{Y}_{i+1}, \mathbf{Y}_{i+2}, \dots, \mathbf{Y}_q) \\
&+ \sum_{i=q}^{\min(n-p,k+1-r)} (-1)^{i-1} F_{ri}(\mathbf{X}_{i+1}, \mathbf{X}_{i+2}, \dots, \mathbf{X}_p) \\
&+ \sum_{i=n-p+1}^{\min(p-1,k+1-r)} (-1)^{i-1} F_{ri}(\mathbf{X}_{i+1}, \mathbf{X}_{i+2}, \dots, \mathbf{X}_p, \mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_{i+p}) \\
&+ \sum_{i=p}^{k+1-r} (-1)^{n-i} F_{i+r,n-i}(\mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_{i+p}) \\
&+ \sum_{i=n-r+1}^{q-1} (-1)^{i-1} F_{ri}(\mathbf{X}_{i+1}, \mathbf{X}_{i+2}, \dots, \mathbf{X}_p, \mathbf{Y}_{i+1}, \mathbf{Y}_{i+2}, \dots, \mathbf{Y}_q) \\
&+ \sum_{i=\max(q,n-r+1)}^{n-p} (-1)^{i-1} F_{ri}(\mathbf{X}_{i+1}, \mathbf{X}_{i+2}, \dots, \mathbf{X}_p) \\
&+ \sum_{i=\max(n-p+1,n-r+1)}^{p-1} (-1)^{n-i} \times \\
&\qquad F_{i+r,n-i}(\mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_{i+p}, \mathbf{X}_{i+1}, \mathbf{X}_{i+2}, \dots, \mathbf{X}_p) \\
&+ \sum_{i=\max(p,n-r+1)}^{n-q} (-1)^{n-i} F_{i+r,n-i}(\mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_{i+p}) \\
&+ \sum_{i=\max(n-q+1,n-r+1)}^{n-1} (-1)^{n-i} F_{i+r,n-i}(\mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_{i+p}, \mathbf{Y}_1, \mathbf{Y}_2, \dots, \mathbf{Y}_{i+q}) \\
&\qquad (1 \le r \le k).
\end{aligned} \tag{2.22}$$

Therefore, the theorem is proved for $p \le k \le n-q+1$.

5^o Let $n-q+1 \leq k < n$. If we put (2.12) into (2.11), we get

$$\begin{aligned}
(2.23)\quad & f_{k+1}(\mathbf{X}_{k+1}, \mathbf{X}_{k+2}, \ldots, \mathbf{X}_{p+k}, \mathbf{Y}_{k+1}, \mathbf{Y}_{k+2}, \ldots, \mathbf{Y}_{q+k}) \\
&= \sum_{i=n-k}^{q-1} (-1)^{i-1} \times \\
&\qquad F_{k+1,i}(\mathbf{X}_{i+k+1}, \mathbf{X}_{i+k+2}, \ldots, \mathbf{X}_{p+k}, \mathbf{Y}_{i+k+1}, \mathbf{Y}_{i+k+2}, \ldots, \mathbf{Y}_{q+k}) \\
&+ \sum_{i=q}^{n-p} (-1)^{i-1} F_{k+1,i}(\mathbf{X}_{i+k+1}, \mathbf{X}_{i+k+2}, \ldots, \mathbf{X}_{p+k}) \\
&+ \sum_{i=n-p+1}^{p-1} (-1)^{n-i} \times \\
&\qquad F_{i+k+1,n-i}(\mathbf{X}_{k+1}, \mathbf{X}_{k+2}, \ldots, \mathbf{X}_{i+k+p}, \mathbf{X}_{i+k+1}, \mathbf{X}_{i+k+2}, \ldots, \mathbf{X}_{k+p}) \\
&+ \sum_{i=p}^{n-q} (-1)^{n-i} F_{i+k+1,n-i}(\mathbf{X}_{k+1}, \mathbf{X}_{k+2}, \ldots, \mathbf{X}_{i+p+k}) \\
&+ \sum_{i=n-q+1}^{n-1} (-1)^{n-i} \times \\
&\qquad F_{i+k+1,n-i}(\mathbf{X}_{k+1}, \mathbf{X}_{k+2}, \ldots, \mathbf{X}_{i+p+k}, \mathbf{Y}_{k+1}, \mathbf{Y}_{k+2}, \ldots, \mathbf{Y}_{i+q+k}).
\end{aligned}$$

Substituting the function f_{k+1} determined by (2.23) into (2.11) and using the substitutions (2.21), we obtain the equation (2.10). Therefore, the general solution of the functional equation (2.11) in the case considered can be written in the following general form

$$\begin{aligned}
(2.24)\quad & f_r(\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_p, \mathbf{Y}_1, \mathbf{Y}_2, \ldots, \mathbf{Y}_q) \\
&= \sum_{i=1}^{\min(q-1,k+1-r)} (-1)^{i-1} F_{ri}(\mathbf{X}_{i+1}, \mathbf{X}_{i+2}, \ldots, \mathbf{X}_p, \mathbf{Y}_{i+1}, \mathbf{Y}_{i+2}, \ldots, \mathbf{Y}_q) \\
&+ \sum_{i=q}^{\min(k+1-r,n-p)} (-1)^{i-1} F_{ri}(\mathbf{X}_{i+1}, \mathbf{X}_{i+2}, \ldots, \mathbf{X}_p) \\
&+ \sum_{i=n-p+1}^{\min(k+1-r,p-1)} (-1)^{i-1} F_{ri}(\mathbf{X}_{i+1}, \mathbf{X}_{i+2}, \ldots, \mathbf{X}_p, \mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_{i+p}) \\
&+ \sum_{i=p}^{\min(k+1-r,n-q)} (-1)^{n-i} F_{i+r,n-i}(\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_{i+p})
\end{aligned}$$

$$+\sum_{i=n-q+1}^{k+1-r}(-1)^{n-i}F_{i+r,n-i}(\mathbf{X}_1,\mathbf{X}_2,\dots,\mathbf{X}_{i+p},\mathbf{Y}_1,\mathbf{Y}_2,\dots,\mathbf{Y}_{i+q})$$

$$+\sum_{i=n-r+1}^{q-1}(-1)^{i-1}F_{ri}(\mathbf{X}_{i+1},\mathbf{X}_{i+2},\dots,\mathbf{X}_p,\mathbf{Y}_{i+1},\mathbf{Y}_{i+2},\dots,\mathbf{Y}_q)$$

$$+\sum_{i=\max(q,n-r+1)}^{n-p}(-1)^{i-1}F_{ri}(\mathbf{X}_{i+1},\mathbf{X}_{i+2},\dots,\mathbf{X}_p)$$

$$+\sum_{i=\max(n-p+1,n-r+1)}^{p-1}(-1)^{n-i}\times$$

$$F_{i+r,n-i}(\mathbf{X}_1,\mathbf{X}_2,\dots,\mathbf{X}_{i+p},\mathbf{X}_{i+1},\mathbf{X}_{i+2},\dots,\mathbf{X}_p)$$

$$+\sum_{i=\max(p,n-r+1)}^{n-q}(-1)^{n-i}F_{i+r,n-i}(\mathbf{X}_1,\mathbf{X}_2,\dots,\mathbf{X}_{i+p})$$

$$+\sum_{i=\max(n-q+1,n-r+1)}^{n-1}(-1)^{n-i}F_{i+r,n-i}(\mathbf{X}_1,\mathbf{X}_2,\dots,\mathbf{X}_{i+p},\mathbf{Y}_1,\mathbf{Y}_2,\dots,\mathbf{Y}_{i+q})$$

$$(1\le r\le k).$$

Therefore, the theorem holds for $n-q+1\le k\le n$. □

Now we will solve two particular cases of the equation (2.1).

a) By putting $k=n$ into equation (2.1), we obtain the functional equation

$$\sum_{i=1}^{n}f_i(\mathbf{X}_i,\mathbf{X}_{i+1},\dots,\mathbf{X}_{i+p-1},\mathbf{Y}_i,\mathbf{Y}_{i+1},\dots,\mathbf{Y}_{i+q-1})=\mathbf{O}. \tag{2.25}$$

Therefore, if we put $k=n$ into (2.2), (2.3), (2.4), (2.5) and (2.6), then we obtain the general solution of the functional equation (2.25) in the cases considered. For example, if we put $k=n$ into (2.4), we obtain that the general solution of the functional equation (2.25) for $q<p<n<2q-1<2p-1$ is given by the formulas

$$f_r(\mathbf{X}_1,\mathbf{X}_2,\dots,\mathbf{X}_p,\mathbf{Y}_1,\mathbf{Y}_2,\dots,\mathbf{Y}_q) \tag{2.26}$$

$$=\sum_{i=1}^{n-p}(-1)^{i-1}F_{ri}(\mathbf{X}_{i+1},\mathbf{X}_{i+2},\dots,\mathbf{X}_p,\mathbf{Y}_{i+1},\mathbf{Y}_{i+2},\dots,\mathbf{Y}_q)$$

$$+\sum_{i=n-p+1}^{n-q}(-1)^{i-1}\times$$

$$F_{ri}(\mathbf{X}_{i+1},\mathbf{X}_{i+2},\dots,\mathbf{X}_p,\mathbf{X}_1,\mathbf{X}_2,\dots,\mathbf{X}_{p+i},\mathbf{Y}_{i+1},\mathbf{Y}_{i+2},\dots,\mathbf{Y}_q)$$

$$+\sum_{i=n-q+1}^{\min(n-r,q-1)}(-1)^{i-1}F_{ri}(\mathbf{X}_{i+1},\mathbf{X}_{i+2},\dots,\mathbf{X}_p,\mathbf{X}_1\mathbf{X}_2,\dots,\mathbf{X}_{p+i},\\ \mathbf{Y}_{i+1},\mathbf{Y}_{i+2},\dots,\mathbf{Y}_q,\mathbf{Y}_1,\mathbf{Y}_2,\dots,\mathbf{Y}_{q+i})$$

$$+\sum_{i=\max(n-q+1,n-r+1)}^{q-1}(-1)^{n-i}F_{i+r,n-i}(\mathbf{X}_1,\mathbf{X}_2,\dots,\mathbf{X}_{p+i},\mathbf{X}_{i+1},\mathbf{X}_{i+2},\dots,\mathbf{X}_p,\\ \mathbf{Y}_1,\mathbf{Y}_2,\dots,\mathbf{Y}_{q+i},\mathbf{Y}_{i+1},\mathbf{Y}_{i+2},\dots,\mathbf{Y}_q)$$

$$+\sum_{i=q}^{p-1}(-1)^{n-i}\times\\ F_{i+r,n-i}(\mathbf{X}_1,\mathbf{X}_2,\dots,\mathbf{X}_{p+i},\mathbf{X}_{i+1},\mathbf{X}_{i+2},\dots,\mathbf{X}_p,\mathbf{Y}_1,\mathbf{Y}_2,\dots,\mathbf{Y}_{q+i})$$

$$+\sum_{i=p}^{n-1}(-1)^{n-i}F_{i+r,n-i}(\mathbf{X}_1,\mathbf{X}_2,\dots,\mathbf{X}_{i+p},\mathbf{Y}_1,\mathbf{Y}_2,\dots,\mathbf{Y}_{q+i})$$

$$(1\le r\le n),$$

where F_{ij} are arbitrary complex vector functions from $\mathcal{V}$.

The functional equation (2.1) for $q<p<2q-1<2p-1\le n$ had not been previously investigated, but for this case we must additionally determine the general solution of the equation (2.25). Now we will give the following result which treats a more general case than the previous one.

Theorem 2.6. *The general solution of the functional equation* (2.25) *for* $n+1\ge 2\max(p,q)$ *is given by the formulas*

$$f_r(\mathbf{X}_1,\mathbf{X}_2,\dots,\mathbf{X}_p,\mathbf{Y}_1,\mathbf{Y}_2,\dots,\mathbf{Y}_q)\\ =F_r(\mathbf{X}_1,\mathbf{X}_2,\dots,\mathbf{X}_{p-1},\mathbf{Y}_1,\mathbf{Y}_2,\dots,\mathbf{Y}_{q-1})\\ -F_{r+1}(\mathbf{X}_2,\mathbf{X}_3,\dots,\mathbf{X}_p,\mathbf{Y}_2,\mathbf{Y}_3,\dots,\mathbf{Y}_q)\\ (1\le r\le n;\quad F_{n+1}\equiv F_1),\tag{2.27}$$

where F_r are arbitrary complex vector functions from $\mathcal{V}$.

Proof. Using the conventions $f_r\equiv f_{r+n}$, $\mathbf{X}_r\equiv\mathbf{X}_{r+n}$ and $\mathbf{Y}_r\equiv\mathbf{Y}_{r+n}$, the equation (2.25) in an expanded form can be written in the following way

$$f_r(\mathbf{X}_r,\mathbf{X}_{r+1},\dots,\mathbf{X}_{r+p-1},\mathbf{Y}_r,\mathbf{Y}_{r+1},\dots,\mathbf{Y}_{r+q-1})\\ +f_{r+1}(\mathbf{X}_{r+1},\mathbf{X}_{r+2},\dots,\mathbf{X}_{r+p},\mathbf{Y}_{r+1},\mathbf{Y}_{r+2},\dots,\mathbf{Y}_{r+q})+\cdots\\ +f_{r+n-p}(\mathbf{X}_{r+n-p},\mathbf{X}_{r+n-p+1},\dots,\mathbf{X}_{r+n-1},\\ \mathbf{Y}_{r+n-p},\mathbf{Y}_{r+n-p+1},\dots,\mathbf{Y}_{r+n-p+q-1})\tag{2.28}$$

$$+ f_{r+n-p+1}(\mathbf{X}_{r+n-p+1}, \mathbf{X}_{r+n-p+2}, \dots, \mathbf{X}_{r+n-1}, \mathbf{X}_r, \mathbf{Y}_{r+n-p+1}, \mathbf{Y}_{r+n-p+2}, \dots, \mathbf{Y}_{r+n-p+q})$$
$$+ \cdots + f_{r+n-1}(\mathbf{X}_{r+n-1}, \mathbf{X}_r, \dots, \mathbf{X}_{r+p-2}, \mathbf{Y}_{r+n-1}, \mathbf{Y}_r, \dots, \mathbf{Y}_{r+q-2}) = \mathbf{O}.$$

Assuming that $p > q$ (for $p = q$ there are just slight modifications in the following formulas) and putting $\mathbf{Y}_{r+q} = \mathbf{Y}_{r+q+1} = \cdots = \mathbf{Y}_{r+n-1} = \mathcal{C}$ and $\mathbf{X}_{r+p} = \mathbf{X}_{r+p+1} = \cdots = \mathbf{X}_{r+n-1} = \mathcal{C}$ into (2.28), where $\mathcal{C}$ is a fixed vector from $\mathcal{V}$, we obtain

$$\begin{aligned}(2.29)\quad & f_r(\mathbf{X}_r, \mathbf{X}_{r+1}, \dots, \mathbf{X}_{r+p-1}, \mathbf{Y}_r, \mathbf{Y}_{r+1}, \dots, \mathbf{Y}_{r+q-1}) \\ & + f_{r+1}(\mathbf{X}_{r+1}, \mathbf{X}_{r+2}, \dots, \mathbf{X}_{r+p-1}, \mathcal{C}, \mathbf{Y}_{r+1}, \mathbf{Y}_{r+2}, \dots, \mathbf{Y}_{r+q-1}, \mathcal{C}) \\ & + f_{r+2}(\mathbf{X}_{r+2}, \mathbf{X}_{r+3}, \dots, \mathbf{X}_{r+p-1}, \mathcal{C}, \mathcal{C}, \mathbf{Y}_{r+2}, \mathbf{Y}_{r+3}, \dots, \mathbf{Y}_{r+q-1}, \mathcal{C}, \mathcal{C}) + \cdots \\ & + f_{r+p-1}(\mathbf{X}_{r+p-1}, \mathcal{C}, \dots, \mathcal{C}) + f_{r+p}(\mathcal{C}, \mathcal{C}, \dots, \mathcal{C}) + f_{r+p+1}(\mathcal{C}, \mathcal{C}, \dots, \mathcal{C}) + \cdots \\ & + f_{r+n-p}(\mathcal{C}, \mathcal{C}, \mathcal{C}, \dots, \mathcal{C}) + f_{r+n-p+1}(\mathcal{C}, \mathcal{C}, \dots, \mathcal{C}, \mathbf{X}_r, \mathcal{C}, \mathcal{C}, \dots, \mathcal{C}) \\ & + f_{r+n-p+2}(\mathcal{C}, \mathcal{C}, \dots, \mathcal{C}, \mathbf{X}_r, \mathbf{X}_{r+1}, \mathcal{C}, \mathcal{C}, \dots, \mathcal{C}) + \cdots \\ & + f_{r+n-1}(\mathcal{C}, \mathbf{X}_r, \mathbf{X}_{r+1}, \dots, \mathbf{X}_{r+p-2}, \mathcal{C}, \mathbf{Y}_r, \dots, \mathbf{Y}_{r+q-2}) = \mathbf{O}.\end{aligned}$$

If we further substitute $\mathbf{Y}_{r+q-1} = \mathcal{C}$ and $\mathbf{X}_{r+p-1} = \mathcal{C}$ in (2.29), this yields

$$\begin{aligned}(2.30)\quad & f_r(\mathbf{X}_r, \mathbf{X}_{r+1}, \dots, \mathbf{X}_{r+p-2}, \mathcal{C}, \mathbf{Y}_r, \mathbf{Y}_{r+1}, \dots, \mathbf{Y}_{r+q-2}, \mathcal{C}) \\ & + f_{r+1}(\mathbf{X}_{r+1}, \mathbf{X}_{r+2}, \dots, \mathbf{X}_{r+p-2}, \mathcal{C}, \mathcal{C}, \mathbf{Y}_{r+1}, \mathbf{Y}_{r+2}, \dots, \mathbf{Y}_{r+q-2}, \mathcal{C}, \mathcal{C}) + \cdots \\ & + f_{r+p-2}(\mathbf{X}_{r+p-2}, \mathcal{C}, \dots, \mathcal{C}) + f_{r+p-1}(\mathcal{C}, \mathcal{C}, \dots, \mathcal{C}) + f_{r+p}(\mathcal{C}, \mathcal{C}, \dots, \mathcal{C}) + \cdots \\ & + f_{r+n-p}(\mathcal{C}, \mathcal{C}, \dots, \mathcal{C}) + f_{r+n-p+1}(\mathcal{C}, \mathcal{C}, \dots, \mathcal{C}, \mathbf{X}_r, \mathcal{C}, \mathcal{C}, \dots, \mathcal{C}) \\ & + f_{r+n-p+2}(\mathcal{C}, \mathcal{C}, \dots, \mathcal{C}, \mathbf{X}_r, \mathbf{X}_{r+1}, \mathcal{C}, \mathcal{C}, \dots, \mathcal{C}) + \cdots \\ & + f_{r+n-1}(\mathcal{C}, \mathbf{X}_r, \mathbf{X}_{r+1}, \dots, \mathbf{X}_{r+p-2}, \mathcal{C}, \mathbf{Y}_r, \dots, \mathbf{Y}_{r+q-2}) = \mathbf{O}.\end{aligned}$$

Subtracting (2.30) from (2.29), we get the formula

$$\begin{aligned}(2.31)\quad & f_r(\mathbf{X}_r, \mathbf{X}_{r+1}, \dots, \mathbf{X}_{r+p-1}, \mathbf{Y}_r, \mathbf{Y}_{r+1}, \dots, \mathbf{Y}_{r+q-1}) \\ & = f_r(\mathbf{X}_r, \mathbf{X}_{r+1}, \dots, \mathbf{X}_{r+p-2}, \mathcal{C}, \mathbf{Y}_r, \mathbf{Y}_{r+1}, \dots, \mathbf{Y}_{r+q-2}, \mathcal{C}) \\ & - f_{r+1}(\mathbf{X}_{r+1}, \mathbf{X}_{r+2}, \dots, \mathbf{X}_{r+p-1}, \mathcal{C}, \mathbf{Y}_{r+1}, \mathbf{Y}_{r+2}, \dots, \mathbf{Y}_{r+q-1}, \mathcal{C}) \\ & + f_{r+1}(\mathbf{X}_{r+1}, \mathbf{X}_{r+2}, \dots, \mathbf{X}_{r+p-2}, \mathcal{C}, \mathcal{C}, \mathbf{Y}_{r+1}, \mathbf{Y}_{r+2}, \dots, \mathbf{Y}_{r+q-2}, \mathcal{C}, \mathcal{C}) \\ & - f_{r+2}(\mathbf{X}_{r+2}, \mathbf{X}_{r+3}, \dots, \mathbf{X}_{r+p-1}, \mathcal{C}, \mathcal{C}, \mathbf{Y}_{r+2}, \mathbf{Y}_{r+3}, \dots, \mathbf{Y}_{r+q-1}, \mathcal{C}, \mathcal{C}) + \cdots \\ & + f_{r+p-2}(\mathbf{X}_{r+p-2}, \mathcal{C}, \dots, \mathcal{C}) - f_{r+p-1}(\mathbf{X}_{r+p-1}, \mathcal{C}, \dots, \mathcal{C}) \\ & + f_{r+p-1}(\mathcal{C}, \mathcal{C}, \dots, \mathcal{C}),\end{aligned}$$

which holds for every $r = 1, 2, \dots, n$.

Let us put now

$$
\begin{aligned}
(2.32)\qquad & g_r(\mathbf{X}_1,\mathbf{X}_2,\dots,\mathbf{X}_{p-1},\mathbf{Y}_1,\mathbf{Y}_2,\dots,\mathbf{Y}_{q-1})\\
= \;& f_r(\mathbf{X}_1,\mathbf{X}_2,\dots,\mathbf{X}_{p-1},\mathcal{C},\mathbf{Y}_1,\mathbf{Y}_2,\dots,\mathbf{Y}_{q-1},\mathcal{C})\\
+ \;& f_{r+1}(\mathbf{X}_2,\mathbf{X}_3,\dots,\mathbf{X}_{p-1},\mathcal{C},\mathcal{C},\mathbf{Y}_2,\mathbf{Y}_3,\dots,\mathbf{Y}_{q-1},\mathcal{C},\mathcal{C})\\
+ \;& \cdots + f_{r+p-2}(\mathbf{X}_{p-1},\mathcal{C},\dots,\mathcal{C})
\end{aligned}
$$

and

$$(2.33)\qquad A_r = f_{r+p-1}(\mathcal{C},\mathcal{C},\dots,\mathcal{C}).$$

Since $f_r \equiv f_{r+n}$, we conclude that $g_r \equiv g_{r+n}$. Putting in (2.25) $\mathbf{X}_r = \mathbf{Y}_r = \mathcal{C}$ $(1 \le r \le n)$, we find

$$(2.34)\qquad \sum_{r=1}^{n} A_r = \mathbf{O}.$$

According to (2.32) and (2.33), the formula (2.31) can be written in the form

$$
\begin{aligned}
(2.35)\qquad & f_r(\mathbf{X}_1,\mathbf{X}_2,\dots,\mathbf{X}_p,\mathbf{Y}_1,\mathbf{Y}_2,\dots,\mathbf{Y}_q)\\
= \;& g_r(\mathbf{X}_1,\mathbf{X}_2,\dots,\mathbf{X}_{p-1},\mathbf{Y}_1,\mathbf{Y}_2,\dots,\mathbf{Y}_{q-1})\\
- \;& g_{r+1}(\mathbf{X}_2,\mathbf{X}_3,\dots,\mathbf{X}_p,\mathbf{Y}_2,\mathbf{Y}_3,\dots,\mathbf{Y}_q) + A_r \quad (1 \le r \le n).
\end{aligned}
$$

Finally, with the notations

$$
\begin{aligned}
F_1 &= g_1,\\
F_2 &= g_2 - A_1,\\
F_3 &= g_3 - A_1 - A_2,\\
&\vdots\\
F_n &= g_n - A_1 - A_2 - \cdots - A_{n-1}
\end{aligned}
$$

the expression (2.35) can be written in the form (2.27). Thus we have proved that (2.27) is a consequence of (2.25) if $n+1 \ge 2\max(p,q)$. Conversely, a straightforward computation shows that the functions (2.27) satisfy the equation (2.35) for arbitrary functions F_r from $\mathcal{V}$. $\square$

b) Now we will consider the functional equation

$$(2.36)\qquad \sum_{i=1}^{n} f(\mathbf{X}_i,\mathbf{X}_{i+1},\dots,\mathbf{X}_{i+p-1},\mathbf{Y}_i,\mathbf{Y}_{i+1},\dots,\mathbf{Y}_{i+q-1}) = \mathbf{O}$$

$$(\mathbf{X}_{n+i} \equiv \mathbf{X}_i,\ \mathbf{Y}_{n+i} \equiv \mathbf{Y}_i)$$

which is a particular case of the functional equation (2.25).

In order to determine the general solution of the functional equation (2.36), we will distinguish the following cases:

$$
\begin{aligned}
&1^0 \;\; q < 2q-1 \le p = n, \qquad 2^0 \;\; q < p = n < 2q-1,\\
&3^0 \;\; q < p < n < 2q-1 < 2p-1,\\
&4^0 \;\; q < p < 2q-1 \le n \le p+q-1 < 2p-1,\\
&5^0 \;\; q < p < p+q-1 < n < 2p-1,\\
&6^0 \;\; q < p < 2q-1 < 2p-1 \le n.
\end{aligned}
$$

Theorem 2.7. *If $q < 2q-1 \le p = n$, then the general solution of the functional equation* (2.36) *is given by*

$$
\begin{aligned}
(2.37)\qquad & f(\mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_n, \mathbf{Y}_1, \mathbf{Y}_2, \dots, \mathbf{Y}_q)\\
&= F_0(\mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_n, \mathbf{Y}_1, \mathbf{Y}_2, \dots, \mathbf{Y}_{q-1})\\
&- F_0(\mathbf{X}_2, \mathbf{X}_3, \dots, \mathbf{X}_n, \mathbf{X}_1, \mathbf{Y}_2, \mathbf{Y}_3, \dots, \mathbf{Y}_q),
\end{aligned}
$$

where F_0 is an arbitrary complex vector function from $\mathcal{V}$.

Proof. If we put $k = n$ into (2.2), then by summing up the functions f_r $(1 \le r \le n)$ and putting $f_1 = f_2 = \cdots = f_n = f$, we obtain the formula (2.37). □

In the same way the following theorems can be proved:

Theorem 2.8. *If $q < p = n < 2q-1$, then the general solution of the functional equation* (2.36) *is given by the formula*

$$
\begin{aligned}
(2.38)\quad & f(\mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_n, \mathbf{Y}_1, \mathbf{Y}_2, \dots, \mathbf{Y}_q)\\
&= F_0(\mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_n, \mathbf{Y}_1, \mathbf{Y}_2, \dots, \mathbf{Y}_{q-1})\\
&- F_0(\mathbf{X}_2, \mathbf{X}_3, \dots, \mathbf{X}_n, \mathbf{X}_1, \mathbf{Y}_2, \mathbf{Y}_3, \dots, \mathbf{Y}_q)\\
&+ \sum_{i=1}^{q-[n/2]} \big[F_i(\mathbf{X}_{n-q+i+1}, \mathbf{X}_{n-q+i+2}, \dots, \mathbf{X}_{i+1},\\
&\qquad \mathbf{Y}_1, \mathbf{Y}_2, \dots, \mathbf{Y}_i, \mathbf{Y}_{n-q+i+1}, \mathbf{Y}_{n-q+i+2}, \dots, \mathbf{Y}_q)\\
&- F_i(\mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_n, \mathbf{Y}_{q-i+1}, \mathbf{Y}_{q-i+2}, \dots, \mathbf{Y}_q, \mathbf{Y}_1, \mathbf{Y}_2, \dots, \mathbf{Y}_{2q-i+n})\big],
\end{aligned}
$$

where F_i $(i = 0, 1, \dots, q-[n/2])$ are arbitrary complex vector functions from $\mathcal{V}$.

Theorem 2.9. *If $q < p < n < 2q - 1 < 2p - 1$, then the general solution of the equation* (2.36) *is*

$$\begin{aligned}
(2.39)\quad & f(\mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_p, \mathbf{Y}_1, \mathbf{Y}_2, \dots, \mathbf{Y}_q) \\
&= F_0(\mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_{p-1}, \mathbf{Y}_1, \mathbf{Y}_2, \dots, \mathbf{Y}_{q-1}) \\
&- F_0(\mathbf{X}_2, \mathbf{X}_3, \dots, \mathbf{X}_p, \mathbf{Y}_2, \mathbf{Y}_3, \dots, \mathbf{Y}_q) \\
&+ \sum_{i=1}^{p-q} \big[F_i(\mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_i, \mathbf{X}_{n-p+i+1}, \mathbf{X}_{n-p+i+2}, \dots, \mathbf{X}_p, \\
&\qquad \mathbf{Y}_{n-p+i+1}, \mathbf{Y}_{n-p+i+2}, \dots, \mathbf{Y}_q) \\
&- F_i(\mathbf{X}_{p-i+1}, \mathbf{X}_{p-i+2}, \dots, \mathbf{X}_p, \mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_{2p-n-i}, \mathbf{Y}_1, \mathbf{Y}_2, \dots, \mathbf{Y}_{p+q-n-i})\big] \\
&+ \sum_{i=p-q+1}^{p-[n/2]} \big[F_i(\mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_i, \mathbf{X}_{n-p+i+1}, \mathbf{X}_{n-p+i+2}, \dots, \mathbf{X}_p, \\
&\qquad \mathbf{Y}_1, \mathbf{Y}_2, \dots, \mathbf{Y}_{i+q-p}, \mathbf{Y}_{n-p+i+1}, \mathbf{Y}_{n-p+i+2}, \dots, \mathbf{Y}_q) \\
&\quad - F_i(\mathbf{X}_{p-i+1}, \mathbf{X}_{p-i+2}, \dots, \mathbf{X}_p, \mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_{2p-n-i}, \\
&\qquad \mathbf{Y}_{p-i+1}, \mathbf{Y}_{p-i+2}, \dots, \mathbf{Y}_q, \mathbf{Y}_1, \mathbf{Y}_2, \dots, \mathbf{Y}_{p+q-n-i})\big],
\end{aligned}$$

where F_i $(i = 0, 1, \dots, p - [n/2])$ are arbitrary complex vector functions from $\mathcal{V}$.

Theorem 2.10. *The general solution of the functional equation* (2.36) *for $q < p < 2q - 1 \le n \le p + q - 1 < 2p - 1$ is given by the formula*

$$\begin{aligned}
(2.40)\quad & f(\mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_p, \mathbf{Y}_1, \mathbf{Y}_2, \dots, \mathbf{Y}_q) \\
&= F_0(\mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_{p-1}, \mathbf{Y}_1, \mathbf{Y}_2, \dots, \mathbf{Y}_{q-1}) \\
&- F_0(\mathbf{X}_2, \mathbf{X}_3, \dots, \mathbf{X}_p, \mathbf{Y}_2, \mathbf{Y}_3, \dots, \mathbf{Y}_q) \\
&+ \sum_{i=1}^{p+q-n-1} \big[F_i(\mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_i, \mathbf{X}_{n-p+i+1}, \mathbf{X}_{n-p+i+2}, \dots, \mathbf{X}_p, \\
&\qquad \mathbf{Y}_{n-p+i+1}, \mathbf{Y}_{n-p+i+2}, \dots, \mathbf{Y}_q) \\
&- F_i(\mathbf{X}_{p-i+1}, \mathbf{X}_{p-i+2}, \dots, \mathbf{X}_p, \mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_{2p-n-i}, \mathbf{Y}_1, \mathbf{Y}_2, \dots, \mathbf{Y}_{p+q-n-i})\big] \\
&+ \sum_{i=p+q-n}^{p-[n/2]} \big[F_i(\mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_i, \mathbf{X}_{n-p+i+1}, \mathbf{X}_{n-p+i+2}, \dots, \mathbf{X}_p) \\
&\qquad - F_i(\mathbf{X}_{p-i+1}, \mathbf{X}_{p-i+2}, \dots, \mathbf{X}_p, \mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_{2p-n-i})\big],
\end{aligned}$$

where F_i $(i = 0, 1, \dots, p - [n/2])$ are arbitrary complex vector functions from $\mathcal{V}$.

Next we will prove the following theorem.

Theorem 2.11. *The general solution of the equation* (2.36) *for* $q < p < p+q-1 < n < 2p-1$ *is given by*

$$(2.41)\quad \begin{aligned} & f(\mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_p, \mathbf{Y}_1, \mathbf{Y}_2, \dots, \mathbf{Y}_q) \\ &= F_0(\mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_{p-1}, \mathbf{Y}_1, \mathbf{Y}_2, \dots, \mathbf{Y}_{q-1}) \\ &- F_0(\mathbf{X}_2, \mathbf{X}_3, \dots, \mathbf{X}_p \mathbf{Y}_2, \mathbf{Y}_3, \dots, \mathbf{Y}_q) \\ &+ \sum_{i=1}^{p-[n/2]} \big[F_i(\mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_i, \mathbf{X}_{n-p+i+1}, \mathbf{X}_{n-p+i+2}, \dots, \mathbf{X}_p) \\ &\qquad - F_i(\mathbf{X}_{p-i+1}, \mathbf{X}_{p-i+2}, \dots, \mathbf{X}_p, \mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_{2p-n-i}) \big], \end{aligned}$$

where F_i $(i = 0, 1, \dots, p-[n/2])$ *are arbitrary complex vector functions from* $\mathcal{V}$.

Proof. By summing up the functions f_r $(1 \le r \le n)$ determined by (2.6) and putting $f_1 = f_2 = \cdots f_n = f$, we obtain (2.41), where we introduced the notations

$$\begin{aligned} & F_0(\mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_{p-1}, \mathbf{Y}_1, \mathbf{Y}_2, \dots, \mathbf{Y}_{q-1}) \\ = \ & \frac{1}{n} \left[\sum_{r=1}^{q-1} \sum_{i=1}^{r} G_r(\mathbf{X}_i, \mathbf{X}_{i+1}, \dots, \mathbf{X}_{p-r+i-1}, \mathbf{Y}_i, \mathbf{Y}_{i+1}, \dots, \mathbf{Y}_{q-r+i-1}) \right. \\ & \left. + \sum_{r=q}^{n-p} \sum_{i=1}^{r} G_r(\mathbf{X}_i, \mathbf{X}_{i+1}, \dots, \mathbf{X}_{p-r+i-1}) \right], \end{aligned}$$

$$\begin{aligned} & G_r(\mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_{p-r}, \mathbf{Y}_1, \mathbf{Y}_2, \dots, \mathbf{Y}_{q-r}) \\ = \ & \sum_{i=1}^{n} (-1)^r F_{ir}(\mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_{p-r}, \mathbf{Y}_1, \mathbf{Y}_2, \dots, \mathbf{Y}_{q-r}) \quad (1 \le r \le q-1), \end{aligned}$$

$$G_r(\mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_{p-r}) = \sum_{i=1}^{n} (-1)^r F_{ir}(\mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_{p-r}) \qquad (r = q, q+1, \dots, n-p);$$

$$\begin{aligned} & F_k(\mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_k, \mathbf{X}_{n-p+k+1}, \mathbf{X}_{n-p+k+2}, \dots, \mathbf{X}_p) \\ = \ & \frac{(-1)^{k+1}}{n} \left[\sum_{r=1}^{n-p+k} F_{r,p-k}(\mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_k, \mathbf{X}_{n-p+k+1}, \mathbf{X}_{n-p+k+2}, \dots, \mathbf{X}_p) \right. \\ & \left. - \sum_{r=1}^{p-k} F_{r,n-p+k}(\mathbf{X}_{n-p+k+1}, \mathbf{X}_{n-p+k+2}, \dots, \mathbf{X}_p, \mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_k) \right] \\ & \qquad (1 \le k \le p - [(n+1)/2]). \end{aligned}$$

In particular, if $n = 2m$, we obtain

$$\begin{aligned} & F_{p-m}(\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_{p-m}, \mathbf{X}_{m+1}, \mathbf{X}_{m+2}, \ldots, \mathbf{X}_p) \\ = \; & \frac{(-1)^{p-m+1}}{n} \sum_{r=1}^{m} F_{rm}(\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_{p-m}, \mathbf{X}_{m+1}, \mathbf{X}_{m+2}, \ldots, \mathbf{X}_p), \end{aligned}$$

where

$$\begin{aligned} & F_{r+m,m}(\mathbf{X}_1, \ldots, \mathbf{X}_{p-m}, \mathbf{X}_{m+1}, \ldots, \mathbf{X}_p) \\ & \qquad = -F_{rm}(\mathbf{X}_{m+1}, \ldots, \mathbf{X}_p, \mathbf{X}_1, \ldots, \mathbf{X}_{p-m}) \quad (1 \le r \le m). \, \square \end{aligned}$$

We have not considered the case $q < p < 2q - 1 < 2p - 1 \le n$, because we will give instead the following more general result.

Theorem 2.12. *If* $n + 1 \ge 2 \max(p, q)$, *then the general solution of the functional equation* (2.36) *is given by*

$$\begin{aligned} & f(\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_p, \mathbf{Y}_1, \mathbf{Y}_2, \ldots, \mathbf{Y}_q) \\ & = F(\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_{p-1}, \mathbf{Y}_1, \mathbf{Y}_2, \ldots, \mathbf{Y}_{q-1}) \\ & - F(\mathbf{X}_2, \mathbf{X}_3, \ldots, \mathbf{X}_p, \mathbf{Y}_2, \mathbf{Y}_3, \ldots, \mathbf{Y}_q), \end{aligned} \tag{2.42}$$

where F *is an arbitrary complex vector function from* $\mathcal{V}$.

Proof. A straightforward calculation shows that every function f of the form (2.42) satisfies the functional equation (2.36). We have to prove the converse, i.e. that from (2.36) it follows that f has the form (2.42).

Let $\mathcal{C}$ be a fixed vector from $\mathcal{V}$. For $\mathbf{X}_i = \mathbf{Y}_i = \mathcal{C}$ $(1 \le i \le n)$ the equation (2.36) yields

$$f(\mathcal{C}, \mathcal{C}, \ldots, \mathcal{C}) = \mathbf{O}. \tag{2.43}$$

Assuming that $p > q$ (for $p = q$ there are just slight modifications in the following formulas) and $n \ge 2p - 1$ and putting $\mathbf{Y}_{q+1} = \mathbf{Y}_{q+2} = \cdots = \mathbf{Y}_n = \mathcal{C}$ and $\mathbf{X}_{p+1} = \mathbf{X}_{p+2} = \cdots = \mathbf{X}_n = \mathcal{C}$ into (2.36), by virtue of (2.43) we obtain

$$\begin{aligned} & f(\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_p, \mathbf{Y}_1, \mathbf{Y}_2, \ldots, \mathbf{Y}_q) \\ + \; & f(\mathbf{X}_2, \mathbf{X}_3, \ldots, \mathbf{X}_p, \mathcal{C}, \mathbf{Y}_2, \mathbf{Y}_3, \ldots, \mathbf{Y}_q, \mathcal{C}) + \cdots \\ + \; & f(\mathbf{X}_p, \mathcal{C}, \ldots, \mathcal{C}) + f(\mathcal{C}, \mathcal{C}, \ldots, \mathcal{C}, \mathbf{X}_1, \mathcal{C}, \mathcal{C}, \ldots, \mathcal{C}) \\ + \; & f(\mathcal{C}, \mathcal{C}, \ldots, \mathcal{C}, \mathbf{X}_1, \mathbf{X}_2, \mathcal{C}, \mathcal{C}, \ldots, \mathcal{C}) + \cdots \\ + \; & f(\mathcal{C}, \mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_{p-1}, \mathcal{C}, \mathbf{Y}_1, \mathbf{Y}_2, \ldots, \mathbf{Y}_{q-1}) = \mathbf{O}. \end{aligned} \tag{2.44}$$

If we further substitute $\mathbf{Y}_q = \mathcal{C}$ and $\mathbf{X}_p = \mathcal{C}$ in the last equation, we get

$$
\begin{aligned}
(2.45) \qquad & f(\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_{p-1}, \mathcal{C}, \mathbf{Y}_1, \mathbf{Y}_2, \ldots, \mathbf{Y}_{q-1}, \mathcal{C}) \\
+ \;& f(\mathbf{X}_2, \mathbf{X}_3, \ldots, \mathbf{X}_{p-1}, \mathcal{C}, \mathcal{C}, \mathbf{Y}_2, \mathbf{Y}_3, \ldots, \mathbf{Y}_{q-1}, \mathcal{C}, \mathcal{C}) + \cdots \\
+ \;& f(\mathbf{X}_{p-1}, \mathcal{C}, \ldots, \mathcal{C}) + f(\mathcal{C}, \mathcal{C}, \ldots, \mathcal{C}, \mathbf{X}_1, \mathcal{C}, \mathcal{C}, \ldots, \mathcal{C}) \\
+ \;& f(\mathcal{C}, \mathcal{C}, \ldots, \mathcal{C}, \mathbf{X}_1, \mathbf{X}_2, \mathcal{C}, \mathcal{C}, \ldots, \mathcal{C}) + \cdots \\
+ \;& f(\mathcal{C}, \mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_{p-1}, \mathcal{C}, \mathbf{Y}_1, \mathbf{Y}_2, \ldots, \mathbf{Y}_{q-1}) = \mathbf{O}.
\end{aligned}
$$

Subtracting (2.45) from (2.44), we find

$$
\begin{aligned}
(2.46) \qquad & f(\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_p, \mathbf{Y}_1, \mathbf{Y}_2, \ldots, \mathbf{Y}_q) \\
= \;& f(\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_{p-1}, \mathcal{C}, \mathbf{Y}_1, \mathbf{Y}_2, \ldots, \mathbf{Y}_{q-1}, \mathcal{C}) \\
- \;& f(\mathbf{X}_2, \mathbf{X}_3, \ldots, \mathbf{X}_p, \mathcal{C}, \mathbf{Y}_2, \mathbf{Y}_3, \ldots, \mathbf{Y}_q, \mathcal{C}) \\
+ \;& f(\mathbf{X}_2, \mathbf{X}_3, \ldots, \mathbf{X}_{p-1}, \mathcal{C}, \mathcal{C}, \mathbf{Y}_2, \mathbf{Y}_3, \ldots, \mathbf{Y}_{q-1}, \mathcal{C}, \mathcal{C}) \\
- \;& f(\mathbf{X}_3, \mathbf{X}_4, \ldots, \mathbf{X}_p, \mathcal{C}, \mathcal{C}, \mathbf{Y}_3, \mathbf{Y}_4, \ldots, \mathbf{Y}_q, \mathcal{C}, \mathcal{C}) \\
+ \;& \cdots + f(\mathbf{X}_{p-1}, \mathcal{C}, \ldots, \mathcal{C}) - f(\mathbf{X}_p, \mathcal{C}, \ldots, \mathcal{C}).
\end{aligned}
$$

Putting

$$
\begin{aligned}
& F(\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_{p-1}, \mathbf{Y}_1, \mathbf{Y}_2, \ldots, \mathbf{Y}_{q-1}) \\
= \;& f(\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_{p-1}, \mathcal{C}, \mathbf{Y}_1, \mathbf{Y}_2, \ldots, \mathbf{Y}_{q-1}, \mathcal{C}) \\
+ \;& f(\mathbf{X}_2, \mathbf{X}_3, \ldots, \mathbf{X}_{p-1}, \mathcal{C}, \mathcal{C}, \mathbf{Y}_2, \mathbf{Y}_3, \ldots, \mathbf{Y}_{q-1}, \mathcal{C}, \mathcal{C}) \\
+ \;& \cdots + f(\mathbf{X}_{p-1}, \mathcal{C}, \ldots, \mathcal{C}),
\end{aligned}
$$

the equality (2.46) takes on the form (2.42). □

For particular cases see the results obtained in [8, 14].

§3. General Semicyclic Complex Vector Functional Equation

Let $\mathcal{V}$ be a complex vector space, and let $\mathbf{Z}_i$ $(1 \leq i \leq n)$ be complex vectors as in §1.

Also, let S_k^n $(0 < k < n)$ be the set of all strictly increasing mappings of the set $\{1, 2, \ldots, k\}$ into $\{1, 2, \ldots, n\}$. Let f_r $(r \in S_k^n)$ be mappings $\mathcal{V}^k \mapsto \mathcal{V}$.

We will solve the following semicyclic complex vector functional equation

$$
(3.1) \qquad \sum_{r \in S_k^n} f_r(\mathbf{Z}_{r(1)}, \mathbf{Z}_{r(2)}, \ldots, \mathbf{Z}_{r(k)}) = \mathbf{O}.
$$

Theorem 3.1. *The general solution of the functional equation* (3.1) *is given by*

$$f_r(\mathbf{Z}_{r(1)}, \mathbf{Z}_{r(2)}, \dots, \mathbf{Z}_{r(k)}) = \sum_{p \in S_{k-1}^k} F_{rp}(\mathbf{Z}_{rp(1)}, \mathbf{Z}_{rp(2)}, \dots, \mathbf{Z}_{rp(k-1)}) \qquad (r \in S_k^n), \tag{3.2}$$

where $rp(i) \equiv r(p(i))$*, and* F_{rp} *are arbitrary complex vector functions from the vector space* $\mathcal{V}$ *such that*

$$\sum_{rp=t} F_{rp}(\mathbf{Z}_{t(1)}, \mathbf{Z}_{t(2)}, \dots, \mathbf{Z}_{t(k-1)}) = \mathbf{O} \qquad (t \in S_{k-1}^n), \tag{3.3}$$

where the sum is extended over all $r \in S_k^n$ *and* $p \in S_{k-1}^k$ *such that* $rp = t$.

Proof. Let f_r be defined by (3.2) and let (3.3) hold. Then we have

$$\begin{aligned} &\sum_{r \in S_k^n} f_r(\mathbf{Z}_{r(1)}, \mathbf{Z}_{r(2)}, \dots, \mathbf{Z}_{r(k)}) \\ = &\sum_{r \in S_k^n} \sum_{p \in S_{k-1}^k} F_{rp}(\mathbf{Z}_{rp(1)}, \mathbf{Z}_{rp(2)}, \dots, \mathbf{Z}_{rp(k-1)}) \\ = &\sum_{t \in S_{k-1}^n} \sum_{rp=t} F_{rp}(\mathbf{Z}_{t(1)}, \mathbf{Z}_{t(2)}, \dots, \mathbf{Z}_{t(k-1)}) \;=\; \mathbf{O}. \end{aligned}$$

Hence, such functions satisfy the functional equation (3.1).

Conversely, if f_r $(r \in S_k^n)$ is any solution of (3.1), we have to prove that the functions f_r admit the representation (3.2) with the conditions (3.3). For fixed r let us put $\mathbf{Z}_i = \mathcal{C}$ into (3.1) for $i \neq r(j)$ $(1 \leq j \leq k)$. Then (3.1) yields the following (not unique) representation

$$f_r(\mathbf{Z}_{r(1)}, \mathbf{Z}_{r(2)}, \dots, \mathbf{Z}_{r(k)}) = \sum_{p \in S_{k-1}^k} G_{rp}(\mathbf{Z}_{rp(1)}, \mathbf{Z}_{rp(2)}, \dots, \mathbf{Z}_{rp(k-1)}). \tag{3.4}$$

For an arbitrary $t \in S_{k-1}^n$ let

$$H_t(\mathbf{Z}_{t(1)}, \mathbf{Z}_{t(2)}, \dots, \mathbf{Z}_{t(k-1)}) = \sum_{rp=t} G_{rp}(\mathbf{Z}_{t(1)}, \mathbf{Z}_{t(2)}, \dots, \mathbf{Z}_{t(k-1)}). \tag{3.5}$$

The equation (3.1) can be written in the form

$$\sum_{t \in S_{k-1}^n} H_t(\mathbf{Z}_{t(1)}, \mathbf{Z}_{t(2)}, \dots, \mathbf{Z}_{t(k-1)}) = \mathbf{O}. \tag{3.6}$$

Let P' be the set of all $t \in S^n_{k-1}$ such that $H_t \neq \mathbf{O}$ and $P'' = S^n_{k-1} \setminus P'$. The equation (3.6) is reduced to

$$\sum_{t\in P'} H_t(\mathbf{Z}_{t(1)}, \mathbf{Z}_{t(2)}, \dots , \mathbf{Z}_{t(k-1)}) = \mathbf{O}. \tag{3.7}$$

We can suppose that among all representations of the form (3.4) of the functions f_r we have taken that (or one of those) for which the number s of the elements of the set P' is minimal. If $s = 0$, we can take $F_{rp} = G_{rp}$ and the theorem is proved. The case $s = 1$ is impossible in view of (3.7). Thus we can suppose that $s > 1$. Let t be some fixed element from P'. Putting $\mathbf{Z}_i = \mathcal{C}$ into (3.7) for $i \neq t(j)$ $(1 \leq j \leq k-1)$, we obtain

$$H_t(\mathbf{Z}_{t(1)}, \mathbf{Z}_{t(2)}, \dots , \mathbf{Z}_{t(k-1)}) = -\sum J_{rp}(\mathbf{Z}_{t(i_1)}, \mathbf{Z}_{t(i_2)}, \dots , \mathbf{Z}_{t(i_m)}), \tag{3.8}$$

where $J_{rp}(\mathbf{Z}_{t(i_1)}, \mathbf{Z}_{t(i_2)}, \dots , \mathbf{Z}_{t(i_m)})$ $(m \leq k-2)$ is obtained from $G_{rp}(\mathbf{Z}_{rp(1)}, \mathbf{Z}_{rp(2)}, \dots , \mathbf{Z}_{rp(k-1)})$ by putting $\mathbf{Z}_i = \mathcal{C}$ for all i but $t(1), t(2), \dots , t(k-1)$. The sum on the right-hand side of (3.8) is extended over certain (not all) pairs of indices r and p.

Consider a certain summand $J_{r_0p_0}$ on the right-hand side of (3.8). Let $u_0 \in S^n_k$ and $v_0 \in S^k_{k-1}$ be such that $u_0v_0 = t$. We can construct a sequence of ordered pairs

$$(u_0, v_0),\ (u_0, w_0),\ (u_1, v_1),\ (u_1, w_1),\ \dots ,\ (u_q, w_q)$$

which satisfy the following conditions

1^o $u_i \in S^n_k$, $v_i \in S^k_{k-1}$, $w_i \in S^k_{k-1}$;

2^o $(u_q, w_q) = (r_0, p_0)$;

3^o $u_{i-1}w_{i-1} = u_iv_i$ $(1 \leq i \leq q)$;

4^o the sequence $u_iw_i(1), \dots , u_iw_i(k-1)$ $(0 \leq i \leq q)$ contains the sequence $t(i_1), \dots , t(i_m)$ as a subsequence.

Let us put

$$G^*_{u_i,v_i} = G_{u_i,v_i} + J_{r_0,p_0}, \qquad G^*_{u_i,w_i} = G_{u_i,w_i} - J_{r_0,p_0} \qquad (0 \leq i \leq q). \tag{3.9}$$

We note that the representation (3.4) is still valid with $G^*_{u_i,v_i}$ and $G^*_{u_i,w_i}$ instead of G_{u_i,v_i} and G_{u_i,w_i}, respectively. We have $H^*_t = H_t + J_{r_0p_0}$. On the other hand, if $t \in P''$, i.e. $H_t \equiv \mathbf{O}$, then also $H^*_t \equiv \mathbf{O}$.

If the same procedure is applied to all summands of the right-hand side of (3.8), we conclude that the new function H_t is identically the zero vector.

This contradicts the minimum property of the number s. Hence, $s = 0$ which proves the theorem. □

This theorem generalizes the results given in [3].

Example. If $n = 5$ and $k = 4$, the equation (3.1) is

$$\begin{aligned} f(\mathbf{Z}_1, \mathbf{Z}_2, \mathbf{Z}_3, \mathbf{Z}_4) + \quad & g(\mathbf{Z}_1, \mathbf{Z}_2, \mathbf{Z}_3, \mathbf{Z}_5) + h(\mathbf{Z}_1, \mathbf{Z}_2, \mathbf{Z}_4, \mathbf{Z}_5) \\ + \quad & i(\mathbf{Z}_1, \mathbf{Z}_3, \mathbf{Z}_4, \mathbf{Z}_5) + j(\mathbf{Z}_2, \mathbf{Z}_3, \mathbf{Z}_4, \mathbf{Z}_5) \quad = \mathbf{O}. \end{aligned}$$

Its general solution is given by

$$\begin{aligned} f(\mathbf{Z}_1, \mathbf{Z}_2, \mathbf{Z}_3, \mathbf{Z}_4) &= f_1(\mathbf{Z}_1, \mathbf{Z}_2, \mathbf{Z}_3) + f_2(\mathbf{Z}_1, \mathbf{Z}_2, \mathbf{Z}_4) \\ &+ f_3(\mathbf{Z}_1, \mathbf{Z}_3, \mathbf{Z}_4) + f_4(\mathbf{Z}_2, \mathbf{Z}_3, \mathbf{Z}_4), \\ g(\mathbf{Z}_1, \mathbf{Z}_2, \mathbf{Z}_3, \mathbf{Z}_5) &= g_1(\mathbf{Z}_1, \mathbf{Z}_2, \mathbf{Z}_3) + g_2(\mathbf{Z}_1, \mathbf{Z}_2, \mathbf{Z}_5) \\ &+ g_3(\mathbf{Z}_1, \mathbf{Z}_3, \mathbf{Z}_5) + g_4(\mathbf{Z}_2, \mathbf{Z}_3, \mathbf{Z}_5), \\ h(\mathbf{Z}_1, \mathbf{Z}_2, \mathbf{Z}_4, \mathbf{Z}_5) &= h_1(\mathbf{Z}_1, \mathbf{Z}_2, \mathbf{Z}_4) + h_2(\mathbf{Z}_1, \mathbf{Z}_2, \mathbf{Z}_5) \\ &+ h_3(\mathbf{Z}_1, \mathbf{Z}_4, \mathbf{Z}_5) + h_4(\mathbf{Z}_2, \mathbf{Z}_4, \mathbf{Z}_5), \\ i(\mathbf{Z}_1, \mathbf{Z}_3, \mathbf{Z}_4, \mathbf{Z}_5) &= i_1(\mathbf{Z}_1, \mathbf{Z}_3, \mathbf{Z}_4) + i_2(\mathbf{Z}_1, \mathbf{Z}_3, \mathbf{Z}_5) \\ &+ i_3(\mathbf{Z}_1, \mathbf{Z}_4, \mathbf{Z}_5) + i_4(\mathbf{Z}_3, \mathbf{Z}_4, \mathbf{Z}_5), \\ j(\mathbf{Z}_2, \mathbf{Z}_3, \mathbf{Z}_4, \mathbf{Z}_5) &= j_1(\mathbf{Z}_2, \mathbf{Z}_3, \mathbf{Z}_4) + j_2(\mathbf{Z}_2, \mathbf{Z}_3, \mathbf{Z}_5) \\ &+ j_3(\mathbf{Z}_2, \mathbf{Z}_4, \mathbf{Z}_5) + j_4(\mathbf{Z}_3, \mathbf{Z}_4, \mathbf{Z}_5), \end{aligned}$$

where

$$\begin{aligned} f_1(\mathbf{Z}_1, \mathbf{Z}_2, \mathbf{Z}_3) + g_1(\mathbf{Z}_1, \mathbf{Z}_2, \mathbf{Z}_3) &= \mathbf{O}, \\ f_2(\mathbf{Z}_1, \mathbf{Z}_2, \mathbf{Z}_4) + h_1(\mathbf{Z}_1, \mathbf{Z}_2, \mathbf{Z}_4) &= \mathbf{O}, \\ f_3(\mathbf{Z}_1, \mathbf{Z}_3, \mathbf{Z}_4) + i_1(\mathbf{Z}_1, \mathbf{Z}_3, \mathbf{Z}_4) &= \mathbf{O}, \\ f_4(\mathbf{Z}_2, \mathbf{Z}_3, \mathbf{Z}_4) + j_1(\mathbf{Z}_2, \mathbf{Z}_3, \mathbf{Z}_4) &= \mathbf{O}, \\ g_2(\mathbf{Z}_1, \mathbf{Z}_2, \mathbf{Z}_5) + h_2(\mathbf{Z}_1, \mathbf{Z}_2, \mathbf{Z}_5) &= \mathbf{O}, \\ g_3(\mathbf{Z}_1, \mathbf{Z}_3, \mathbf{Z}_5) + i_2(\mathbf{Z}_1, \mathbf{Z}_3, \mathbf{Z}_5) &= \mathbf{O}, \\ g_4(\mathbf{Z}_2, \mathbf{Z}_3, \mathbf{Z}_5) + j_2(\mathbf{Z}_2, \mathbf{Z}_3, \mathbf{Z}_5) &= \mathbf{O}, \\ h_3(\mathbf{Z}_1, \mathbf{Z}_4, \mathbf{Z}_5) + i_3(\mathbf{Z}_1, \mathbf{Z}_4, \mathbf{Z}_5) &= \mathbf{O}, \\ h_4(\mathbf{Z}_2, \mathbf{Z}_4, \mathbf{Z}_5) + j_3(\mathbf{Z}_2, \mathbf{Z}_4, \mathbf{Z}_5) &= \mathbf{O}, \\ i_4(\mathbf{Z}_3, \mathbf{Z}_4, \mathbf{Z}_5) + j_4(\mathbf{Z}_3, \mathbf{Z}_4, \mathbf{Z}_5) &= \mathbf{O}. \end{aligned}$$

Hence we may take $f_1, f_2, f_3, f_4, g_2, g_3, g_4, h_3, h_4, i_4$ to be arbitrary complex vector functions from the complex vector space $\mathcal{V}$ and

$$\begin{aligned} g_1 = -f_1, \quad h_1 = -f_2, \quad i_1 = -f_3, \quad j_1 = -f_4, \\ h_2 = -g_2, \quad i_2 = -g_3, \quad j_2 = -g_4, \\ i_3 = -h_3, \quad j_3 = -h_4, \\ j_4 = -i_4. \end{aligned}$$

§4. General Special Cyclic Complex Vector Functional Equation

The notations for the vectors in this section are the same as in §1.

Let $\mathcal{V}$ be the vector space and let there exist mappings

$$f_i : \mathcal{V}^{i+1} \mapsto \mathcal{V} \quad (1 \le i \le n) \quad \text{and} \quad g_i : \mathcal{V}^{i+2} \mapsto \mathcal{V} \quad (1 \le i \le n-1).$$

Now we will prove the following result.

Theorem 4.1. *The general solution of the functional equation*

$$\sum_{i=1}^{n} f_i(\mathbf{Z}_i, \mathbf{Z}_{i+1}, \dots, \mathbf{Z}_{2i}) + \sum_{i=1}^{n-1} g_i(\mathbf{Z}_1, \mathbf{Z}_3, \dots, \mathbf{Z}_{2i+1}, \mathbf{Z}_{2i+2}) = \mathbf{O} \tag{4.1}$$

is given by

$$\begin{gathered} f_1(\mathbf{Z}_1, \mathbf{Z}_2) = H_1(\mathbf{Z}_1) - F_1(\mathbf{Z}_2), \\ f_i(\mathbf{Z}_i, \mathbf{Z}_{i+1}, \dots, \mathbf{Z}_{2i}) = (-1)^i F_{i-1}(\mathbf{Z}_i, \mathbf{Z}_{i+1}, \dots, \mathbf{Z}_{2i-2}) \\ +(-1)^{i+1} G_{i-1}(\{\mathbf{Z}_i, \mathbf{Z}_{i+1}, \dots, \mathbf{Z}_{2i}\} \cap \{\mathbf{Z}_1, \mathbf{Z}_3, \dots, \mathbf{Z}_{2i-1}\}, \mathbf{Z}_{2i}) \\ +(-1)^i F_i(\mathbf{Z}_{i+1}, \mathbf{Z}_{i+2}, \dots, \mathbf{Z}_{2i}) \quad (2 \le i \le n), \end{gathered}$$

$$F_n(\mathbf{Z}_{n+1}, \mathbf{Z}_{n+2}, \dots, \mathbf{Z}_{2n}) = \mathbf{O}, \tag{4.2}$$

$$\begin{gathered} g_i(\mathbf{Z}_1, \mathbf{Z}_3, \dots, \mathbf{Z}_{2i+1}, \mathbf{Z}_{2i+2}) = (-1)^i H_i(\mathbf{Z}_1, \mathbf{Z}_3, \dots, \mathbf{Z}_{2i-1}) \\ +(-1)^{i+1} G_i(\{\mathbf{Z}_{i+1}, \mathbf{Z}_{i+2}, \dots, \mathbf{Z}_{2i+2}\} \cap \{\mathbf{Z}_1, \mathbf{Z}_3, \dots, \mathbf{Z}_{2i+1}\}, \mathbf{Z}_{2i+2}) \\ +(-1)^i H_{i+1}(\mathbf{Z}_1, \mathbf{Z}_3, \dots, \mathbf{Z}_{2i+1}) \quad (1 \le i \le n-1), \\ H_n(\mathbf{Z}_1, \mathbf{Z}_3, \cdots, \mathbf{Z}_{2n-1}) = \mathbf{O}, \end{gathered}$$

where $F_i,\ G_i,\ H_i$ $(1 \le i \le n-1)$ *are arbitrary functions with values in* $\mathcal{V}$.

Proof. The proof is by mathematical induction on n.

If $n = 2$, then equation (4.1) becomes

$$f_1(\mathbf{Z}_1, \mathbf{Z}_2) + f_2(\mathbf{Z}_2, \mathbf{Z}_3, \mathbf{Z}_4) + g_1(\mathbf{Z}_1, \mathbf{Z}_3, \mathbf{Z}_4) = \mathbf{O}, \tag{4.3}$$

whose solution according to [6] is

$$\begin{aligned} f_1(\mathbf{Z}_1, \mathbf{Z}_2) &= H_1(\mathbf{Z}_1) - F_1(\mathbf{Z}_2), \\ f_2(\mathbf{Z}_2, \mathbf{Z}_3, \mathbf{Z}_4) &= F_1(\mathbf{Z}_2) - G_1(\mathbf{Z}_3, \mathbf{Z}_4), \\ g_1(\mathbf{Z}_1, \mathbf{Z}_3, \mathbf{Z}_4) &= -H_1(\mathbf{Z}_1) + G_1(\mathbf{Z}_3, \mathbf{Z}_4). \end{aligned} \tag{4.4}$$

Thus, the theorem holds for $n = 2$.

Suppose that the theorem holds for any fixed n and let us consider the equation

$$\sum_{i=1}^{n+1} f_i(\mathbf{Z}_i, \mathbf{Z}_{i+1}, \dots, \mathbf{Z}_{2i}) + \sum_{i=1}^{n} g_i(\mathbf{Z}_1, \mathbf{Z}_3, \dots, \mathbf{Z}_{2i+1}, \mathbf{Z}_{2i+2}) = \mathbf{O}. \tag{4.5}$$

By putting $\mathbf{Z}_i = \mathcal{C}_i$ $(1 \le i \le n)$, where $\mathcal{C}_i = \text{const}$ into (4.5), we see that f_{n+1} can be represented in the form

$$f_{n+1}(\mathbf{Z}_{n+1}, \mathbf{Z}_{n+2}, \dots, \mathbf{Z}_{2n+2}) = (-1)^{n+1} F_n(\mathbf{Z}_{n+1}, \mathbf{Z}_{n+2}, \dots, \mathbf{Z}_{2n})$$

$$+(-1)^n G_n(\{\mathbf{Z}_{n+1}, \mathbf{Z}_{n+2}, \dots, \mathbf{Z}_{2n+2}\} \cap \{\mathbf{Z}_1, \mathbf{Z}_3, \dots, \mathbf{Z}_{2n+1}\}, \mathbf{Z}_{2n+2}), \tag{4.6}$$

where F_n and G_n are arbitrary functions.

By a substitution of (4.6) into (4.5), we obtain the equation

$$\sum_{i=1}^{n} A_i(\mathbf{Z}_i, \mathbf{Z}_{i+1}, \dots, \mathbf{Z}_{2i}) + \sum_{i=1}^{n} B_i(\mathbf{Z}_1, \mathbf{Z}_3, \dots, \mathbf{Z}_{2i+1}, \mathbf{Z}_{2n+2}) = \mathbf{O}, \tag{4.7}$$

where we introduced the notations

$$\begin{gathered} A_i = f_i, \quad B_i = g_i \quad (1 \le i \le n-1), \\ A_n = f_n + (-1)^{n+1} F_n, \quad B_n = g_n + (-1)^n G_n. \end{gathered} \tag{4.8}$$

By putting $\mathbf{Z}_i = \mathcal{C}_i$ $(i = 2, 4, \cdots, 2n)$ into (4.7) we obtain

$$B_n(\mathbf{Z}_1, \mathbf{Z}_3, \dots, \mathbf{Z}_{2n+1}, \mathbf{Z}_{2n+2}) = (-1)^n H_n(\mathbf{Z}_1, \mathbf{Z}_3, \dots, \mathbf{Z}_{2n-1}), \tag{4.9}$$

where H_n is an arbitrary function.

On the basis of the expression (4.9), the equation (4.7) takes on the following form

$$\sum_{i=1}^{n} A_i(\mathbf{Z}_i, \mathbf{Z}_{i+1}, \dots, \mathbf{Z}_{2i}) + \sum_{i=1}^{n-1} D_i(\mathbf{Z}_1, \mathbf{Z}_3, \dots, \mathbf{Z}_{2i+1}, \mathbf{Z}_{2i+2}) = \mathbf{O}, \tag{4.10}$$

where we introduced the notations

$$D_i = B_i \quad (1 \le i \le n-2), \qquad D_{n-1} = B_{n-1} + (-1)^n H_n. \tag{4.11}$$

The functional equation (4.10) is an equation of the form (4.1). According to the inductive hypothesis, the general solution of the equation (4.10) is given by equalities of the form (4.2) with f_i replaced by A_i and g_i replaced by D_i.

By virtue of (4.11), (4.9), (4.8) and (4.6) we deduce that the general solution of the equation (4.5) is given by

$$f_i(\mathbf{Z}_i, \mathbf{Z}_{i+1}, \dots, \mathbf{Z}_{2i}) = A_i(\mathbf{Z}_i, \mathbf{Z}_{i+1}, \cdots, \mathbf{Z}_{2i}) \qquad (1 \le i \le n-1),$$

$$f_n(\mathbf{Z}_n, \mathbf{Z}_{n+1}, \dots, \mathbf{Z}_{2n}) = A_n(\mathbf{Z}_n, \mathbf{Z}_{n+1}, \dots, \mathbf{Z}_{2n}) + (-1)^n F_n(\mathbf{Z}_{n+1}, \mathbf{Z}_{n+2}, \dots, \mathbf{Z}_{2n}),$$

$$f_{n+1}(\mathbf{Z}_{n+1}, \mathbf{Z}_{n+2}, \dots, \mathbf{Z}_{2n+2}) = (-1)^{n+1} F_n(\mathbf{Z}_{n+1}, \mathbf{Z}_{n+2}, \dots, \mathbf{Z}_{2n}) + (-1)^n G_n(\{\mathbf{Z}_{n+1}, \mathbf{Z}_{n+2}, \dots, \mathbf{Z}_{2n+2}\} \cap \{\mathbf{Z}_1, \mathbf{Z}_3, \dots, \mathbf{Z}_{2n+1}\}, \mathbf{Z}_{2n+2}),$$

$$g_i(\mathbf{Z}_1, \mathbf{Z}_3, \dots, \mathbf{Z}_{2i+1}, \mathbf{Z}_{2i+2}) = D_i(\mathbf{Z}_1, \mathbf{Z}_3, \dots, \mathbf{Z}_{2i+1}, \mathbf{Z}_{2i+2}) \quad (1 \le i \le n-2),$$

$$g_{n-1}(\mathbf{Z}_1, \mathbf{Z}_3, \dots, \mathbf{Z}_{2n-1}, \mathbf{Z}_{2n}) = D_{n-1}(\mathbf{Z}_1, \mathbf{Z}_3, \dots, \mathbf{Z}_{2n-1}, \mathbf{Z}_{2n}) + (-1)^{n+1} H_n(\mathbf{Z}_1, \mathbf{Z}_3, \dots, \mathbf{Z}_{2n-1}),$$

$$g_n(\mathbf{Z}_1, \mathbf{Z}_3, \dots, \mathbf{Z}_{2n+1}, \mathbf{Z}_{2n+2}) = (-1)^n H_n(\mathbf{Z}_1, \mathbf{Z}_3, \dots, \mathbf{Z}_{2n-1}) + (-1)^{n+1} G_n(\{\mathbf{Z}_{n+1}, \mathbf{Z}_{n+2}, \dots, \mathbf{Z}_{2n+2}\} \cap \{\mathbf{Z}_1, \mathbf{Z}_3, \dots, \mathbf{Z}_{2n+1}\}, \mathbf{Z}_{2n+2}). \qquad \square$$

This theorem generalizes the result given in [4].

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