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Cubic Derivative Nonlinear Schrödinger Equations*

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Abstract. In this paper we study the Cauchy problem for the cubic derivative nonlinear Schrödinger equation involving at least one derivative in the nonlinear term. We prove the global existence in time of solutions to the Cauchy problem and construct the modified asymptotics for large values of time.

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§1. Introduction

In this paper we study the Cauchy problem for the cubic derivative nonlinear Schrödinger equation

$$\begin{cases} iu_t + \frac{1}{2}u_{xx} = \mathcal{N},\ x \in \mathbf{R},\ t \in \mathbf{R}, \\ u(0,x) = u_0,\ x \in \mathbf{R}, \end{cases} \tag{1.1}$$

where the nonlinear term is

$$\begin{aligned} \mathcal{N} \quad = \quad & \sum_{\boldsymbol{\omega}\neq 0}\Big(a_{1,\boldsymbol{\omega}}\left((i\partial)^{\omega_1} u\right)\left((i\partial)^{\omega_2} u\right)\left((i\partial)^{\omega_3} u\right) \\ & +a_{2,\boldsymbol{\omega}}\overline{\left((i\partial)^{\omega_1} u\right)}\left((i\partial)^{\omega_2} u\right)\left((i\partial)^{\omega_3} u\right) \\ & +a_{3,\boldsymbol{\omega}}\overline{\left((i\partial)^{\omega_1} u\right)\left((i\partial)^{\omega_2} u\right)}\left((i\partial)^{\omega_3} u\right) \\ & +a_{4,\boldsymbol{\omega}}\overline{\left((i\partial)^{\omega_1} u\right)\left((i\partial)^{\omega_2} u\right)\left((i\partial)^{\omega_3} u\right)}\Big), \end{aligned}$$

the coefficients $a_{1,\boldsymbol{\omega}}$, $a_{3,\boldsymbol{\omega}}$, $a_{4,\boldsymbol{\omega}} \in \mathbf{C}$, $a_{2,\boldsymbol{\omega}} \in \mathbf{R}$ and the vector $\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3)$ has the components $\omega_1, \omega_2, \omega_3 = 0, 1$ such that $\boldsymbol{\omega} \neq 0$. This means that at least

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one derivative is included in the nonlinear term. The linear part of equation (1.1) consists of the linear Schrödinger operator, while the nonlinearity involves derivatives of unknown function of the cubic order. Such kinds of equations appear in many areas of Physics (see [13], [14], [15]). The difficulty in the study of the global existence in time of solutions to the Cauchy problem (1.1) is that the cubic nonlinear term of equation (1.1) is critical for large time values, and it is already known that the usual scattering states do not exist for derivative nonlinear Schrödinger equation (1.1) with $a_{1,\omega} = a_{3,\omega} = a_{4,\omega} = 0$, $a_{2,\omega} \neq 0$. There are some works (see, [4], [8], [10], [12], [16], [18]) concerning the large time asymptotics of solutions to the derivative nonlinear Schrödinger equations with cubic nonlinearities which have the self-conjugate property: $\mathcal{N}(e^{i\theta}u) = e^{i\theta}\mathcal{N}(u)$ for all $\theta \in \mathbf{R}$. Recent developments in this direction can be seen in [9]. In our previous paper [7] we considered the cubic derivative nonlinear Schrödinger equation without a self-conjugate property in the case, where the nonlinearity is represented in the form of a full derivative. In [7] we used the techniques developed in our previous work [5], where we introduced an appropriate representation of the solution and instead of the operator $\mathcal{J} = x + it\partial_x$ we used the dilation operator $\mathcal{I}\partial_x^{-1} = x + 2t\partial_t\partial_x^{-1}$, where $\partial_x^{-1} = \int_{-\infty}^{x} dx$. In the present paper we are interested in the asymptotic behavior of solutions to the nonlinear Schrödinger equations with general cubic nonlinearities which do not have a self-conjugate property and do not have the form of a full derivative, however the nonlinearities which are in our scope must contain at least one derivative. In this case the estimates of the operator $\mathcal{J}$ which can be obtained have the growth with time more rapid than $\sqrt{t}$ and so the operator $\mathcal{J}$ can not be used for obtaining the large time estimates of the solution. Instead of the operator $\mathcal{J}$ we use the operator $\mathcal{I} = x\partial_x + 2t\partial_t$ and $\mathcal{P} = -\xi\partial_\xi + 2t\partial_t$, which are considered as the first order differential operators and work well for our problem. We prove the global existence in time of solutions to the Cauchy problem (1.1) and construct the modified asymptotics for large time.

We denote the linear Schrödinger evolution group $\mathcal{U}(t)\phi = \frac{1}{\sqrt{2\pi it}}\int e^{\frac{i}{2t}(x-y)^2}\phi(y)dy = \mathcal{F}^{-1}e^{-\frac{it}{2}\xi^2}\mathcal{F}\phi$, where $\mathcal{F}\phi \equiv \hat{\phi} = \frac{1}{\sqrt{2\pi}}\int e^{-ix\xi}\phi(x)dx$ denotes the Fourier transform of the function ϕ, and the inverse Fourier transformation $\mathcal{F}^{-1}$ is defined by $\mathcal{F}^{-1}\phi = \frac{1}{\sqrt{2\pi}}\int e^{ix\xi}\phi(\xi)d\xi$. The first-order differential operator $\mathcal{P}$ commutes with the exponent $e^{it\xi^2}$ and the operator $\mathcal{I}$ commutes with the exponent $e^{ix^2/t}$, so they are very useful for our applications. Note that the operators $\mathcal{P}$ and $\mathcal{I}$ are related as follows $\mathcal{I} = \mathcal{U}(t)\mathcal{F}^{-1}\mathcal{P}\mathcal{F}\mathcal{U}(-t)$, therefore we have $\|\mathcal{I}\phi\|_{\mathbf{L}^2} = \|\mathcal{P}\mathcal{F}\mathcal{U}(-t)\phi\|_{\mathbf{L}^2}$. We denote the usual Lebesgue space $\mathbf{L}^p = \{\phi \in \mathbf{S}'; \|\phi\|_p < \infty\}$, where the norm $\|\phi\|_p = \left(\int_{\mathbf{R}} |\phi(x)|^p dx\right)^{1/p}$ if $1 \leq p < \infty$ and $\|\phi\|_\infty = \text{ess.sup}\{|\phi(x)|; x \in \mathbf{R}\}$ if $p = \infty$. For simplicity we write $\|\cdot\| = \|\cdot\|_2$. Weighted Sobolev space is

$\mathbf{H}_p^{m,k} = \left\{\phi \in \mathbf{S}' : \|\phi\|_{m,k,p} \equiv \left\|\langle x\rangle^k \langle i\partial\rangle^m \phi\right\|_p < \infty\right\}$, $m,k \in \mathbf{R}$, $1 \le p \le \infty$, $\langle x\rangle = \sqrt{1+x^2}$ are the Japanese brackets. We denote also for simplicity $\{x\} = |x| / \langle x\rangle$, $\mathbf{H}^{m,k} = \mathbf{H}_2^{m,k}$ and the norm $\|\phi\|_{m,k} = \|\phi\|_{m,k,2}$. Different positive constants we denote by the same letter C.

Now we state the result of this paper.

Theorem 1.1. *Let the initial data $u_0 \in \mathbf{H}^{3,4}$ and the norm $\|u_0\|_{3,4}$ be sufficiently small. Then there exists a unique global solution u of the Cauchy problem (1.1) such that $u \in \mathbf{C}\left(\mathbf{R};\mathbf{H}^{3,3}\right)$. Moreover there exist unique functions $Q, W, \vartheta \in \mathbf{L}^\infty$ such that the following asymptotics is valid uniformly with respect to $x \in \mathbf{R}$*

$$\begin{aligned} u(t,x) &= \frac{1}{\sqrt{t}} Q\left(\frac{x}{t}\right) \exp\left(\frac{ix^2}{2t} + ia\left(\frac{x}{t}\right)\left|W\left(\frac{x}{t}\right)\right|^2 \log t + i\vartheta\left(\frac{x}{t}\right)\right) \\ &\quad + O\left(t^{-1/2-\gamma}\right), \end{aligned} \tag{1.2}$$

where $a(\xi) = \sum_{|\boldsymbol{\omega}|\neq 0} a_{2,\boldsymbol{\omega}} \xi^{|\boldsymbol{\omega}|}$, $\gamma \in \left(0, 10^{-4}\right)$.

The result is obtained by estimating the following three norms of the solution

$$\begin{aligned} \|u\|_{\mathbf{X}} &= \|u\|_{1,0} + \sqrt{\langle t\rangle}\, \|u\|_{3,0,\infty} + \langle t\rangle^{1/2-2\gamma} \|\mathcal{I}u\|_{1,0,\infty}, \\ \|u\|_{\mathbf{Y}} &= \sum_{j=0}^{3} \langle t\rangle^{-\gamma-3\gamma j} \left\|\mathcal{I}^j u\right\|_{3-j,0} \end{aligned}$$

and

$$\begin{aligned} \|u\|_{\mathbf{Z}} &= \sum_{k=0}^{2} \langle t\rangle^{-2} \left\|\mathcal{P}^k v\right\|_{1,2-k,\infty} + \sum_{k=0}^{3} \langle t\rangle^{1-\lambda_k} \left\|\left(\mathcal{P}^k v\right)_t\right\|_{0,3-k,\infty} \\ &\quad + \sum_{k=0}^{3} \langle t\rangle^{-\lambda_k} \left\|\mathcal{P}^k v\right\|_{0,3-k,\infty}, \end{aligned}$$

where $v(t) = \mathcal{F}\mathcal{U}(-t)\,u(t)$, $\lambda_0 = 0$, $\lambda_1 = 2\gamma$, $\lambda_2 = 40\gamma$, $\lambda_3 = 20\gamma + \frac{1}{2}$, $\gamma \in \left(0, 10^{-4}\right)$. In order to explain our strategy shortly we consider the equation $iu_t + \frac{1}{2}u_{xx} = u^2 u_x$. In the same spirit as in [5] we apply $\mathcal{F}\mathcal{U}(-t)$ to both sides of the equation to get

$$\begin{aligned} iv_t(t,p) &= \mathcal{F}\mathcal{U}(-t)\left(u^2 u_x\right) = \mathcal{F}\mathcal{U}(-t)\left(\left(\mathcal{U}(t)\,\mathcal{F}^{-1} v\right)^2 \mathcal{U}(t)\,\mathcal{F}^{-1} ipv\right) \\ &= \frac{1}{2\pi} \iint e^{it\Lambda} v(t,p_1) v(t,p_2) ip_3 v(t,p_3)\, dp_1 dp_2, \end{aligned}$$

where $\Lambda = \frac{1}{2}\left(p^2 - p_1^2 - p_2^2 - p_3^2\right)$, $p_3 = p - p_1 - p_2$. Applying the stationary phase method we find one stationary point $(p_1, p_2) = \left(\frac{p}{3}, \frac{p}{3}\right)$, hence the main term of the right hand side is

$$\frac{ip}{6\pi}\iint e^{it\Lambda} v\left(t, \frac{p}{3}\right) v\left(t, \frac{p}{3}\right) v\left(t, \frac{p}{3}\right) dp_1 dp_2.$$

By the change of variables of integration $p_1 = \frac{p}{3} + y - z$, $p_2 = \frac{p}{3} - y - z$ we rewrite the above integral in the form

$$\frac{ip}{6\pi} v\left(t, \frac{p}{3}\right) v\left(t, \frac{p}{3}\right) v\left(t, \frac{p}{3}\right) e^{-\frac{i}{3}tp^2} \iint e^{-3ity^2 - itz^2} dy dz$$
$$= \frac{p}{6t\sqrt{3}} v^3\left(t, \frac{p}{3}\right) e^{-\frac{i}{3}tp^2}.$$

Thus we have

$$\begin{aligned} iv_t(t,p) &= \frac{1}{2\pi} e^{-\frac{1}{3}itp^2} \iint e^{-3ity^2 - itz^2} \left(\Phi\left(p_1, p_2, p_3\right) - \Phi\left(\frac{p}{3}, \frac{p}{3}, \frac{p}{3}\right)\right) dy dz \\ &\quad + \frac{1}{2\pi} e^{-\frac{1}{3}itp^2} \iint e^{-3ity^2 - itz^2} \Phi\left(\frac{p}{3}, \frac{p}{3}, \frac{p}{3}\right) dy dz, \end{aligned} \tag{1.3}$$

where $\Phi(p_1, p_2, p_3) = ip_3 v(t, p_1) v(t, p_2) v(t, p_3)$. If we prove that the first term of the right hand side of (1.3) is the remainder term in our function space, we can get the estimates of the solution in the norm $\mathbf{Z}$. We need Lemmas 2.1 - 2.3 to prove this. In our previous works [5] and [7] we could use the function space involving the operator $\mathcal{J} = x + it\partial_x$ making our proof easier than that of the present paper. More precisely, we used in [7] the following estimate

$$\begin{aligned} &\left\| \frac{1}{2\pi} e^{-\frac{1}{3}itp^2} \iint e^{-3ity^2 - itz^2} \left(\Phi\left(p_1, p_2, p_3\right) - \Phi\left(\frac{p}{3}, \frac{p}{3}, \frac{p}{3}\right)\right) dy dz \right\|_\infty \\ \leq\ & Ct^{-\frac{5}{4}+\varepsilon} \left\|\mathcal{J}u\right\|_{1,0}^2 \left\|u\right\|_{1,0}. \end{aligned}$$

However this estimate does not work for the problem under consideration since the norm $\|\mathcal{J}u\|_{1,0}^2$ grows in time fast enough. In order to prove the estimates of solutions in the norm $\mathbf{X}$ we use the formula

$$
\begin{aligned}
u(t,x) &= \mathcal{U}(t)\mathcal{F}^{-1}v(t,p) = \mathcal{F}^{-1}e^{-\frac{it}{2}p^2}v(t,p) \\
&= \frac{1}{\sqrt{2\pi}}\int e^{-\frac{it}{2}p^2+ipx}v(t,p)dp = \frac{1}{\sqrt{2\pi}}e^{\frac{ix^2}{2t}}\int e^{-\frac{it}{2}\left(p-\frac{x}{t}\right)^2}v(t,p)dp \\
&= \frac{1}{\sqrt{2\pi}}e^{\frac{ix^2}{2t}}\int e^{-\frac{it}{2}\left(p-\frac{x}{t}\right)^2}v(0,p)dp \\
&\quad + \frac{1}{\sqrt{2\pi}}e^{\frac{ix^2}{2t}}\int e^{-\frac{it}{2}\left(p-\frac{x}{t}\right)^2}\int_0^t \frac{1}{2\pi}e^{-\frac{i}{3}\tau p^2}\iint e^{-3i\tau ty^2-i\tau tz^2} \\
&\quad \times \left(\Phi\left(p_1,p_2,p_3\right)-\Phi\left(\frac{p}{3},\frac{p}{3},\frac{p}{3}\right)\right)dy\,dz\,d\tau\,dp \\
&\quad - \frac{i}{\sqrt{24\pi}}e^{\frac{ix^2}{2t}}\int e^{-\frac{it}{2}\left(p-\frac{x}{t}\right)^2}\int_0^t e^{-\frac{i}{3}\tau p^2}\Phi\left(\frac{p}{3},\frac{p}{3},\frac{p}{3}\right)\frac{d\tau}{\tau}dp.
\end{aligned}
$$

The second and third terms of the right hand side of the above representation are considered in Lemma 2.6 and Lemma 2.5, respectively. To understand better the role of Lemmas given in Section 2 we give now the corresponding formulas for the other three types of nonlinearities, for example we take $i\overline{u}uu_x, \overline{u}\overline{u}u_x, \overline{u}\overline{u}\overline{u}_x$. Applying the operator $\mathcal{F}\mathcal{U}(-t)$ we obtain

$$
\mathcal{F}\mathcal{U}(-t)\left(\overline{u}uu_x\right) = \frac{1}{2\pi}\iint e^{it\Lambda}\overline{v(t,p_1)}v(t,p_2)ip_3v(t,p_3)dp_1dp_2,
$$

where $\Lambda = \frac{1}{2}\left(p^2+p_1^2-p_2^2-p_3^2\right)$, $p_3 = p+p_1-p_2$. The stationary point is now $(p_1,p_2) = (p,p)$, hence by the change of the variables $p_1 = p-2z$, $p_2 = p-y-z$ we get

$$
\begin{aligned}
& \mathcal{F}\mathcal{U}(-t)\left(\overline{u}uu_x\right) \\
= \quad & \frac{1}{2\pi}\iint e^{-ity^2+itz^2}\overline{v(t,p_1)}v(t,p_2)ip_3v(t,p_3)dydz \\
= \quad & -\frac{p}{2t}\left|v\left(t,p\right)\right|^2 v\left(t,p\right) + \frac{1}{2\pi}\iint e^{-ity^2+itz^2}\Big(\overline{v(t,p_1)}v(t,p_2)ip_3v(t,p_3) \\
& -ip\left|v\left(t,p\right)\right|^2 v\left(t,p\right)\Big)dydz,
\end{aligned}
$$

whence we see that in order to treat the nonlinearity $i\overline{u}uu_x$ we have to introduce the phase function $\exp\left(\int_1^t -i\frac{p}{2\tau}\left|v\left(\tau,p\right)\right|^2 d\tau\right)$ which helps us to cancel the first summand in the right-hand side of the above formula. The nonlinearities $\overline{u}\overline{u}u_x, \overline{u}\overline{u}\overline{u}_x$ are considered in the same manner as u^2u_x and do not involve any divergent term in their asymptotic representations. We conclude this section by summarizing the content of each Section. In Section 2 we prove some preliminary estimates. We need Lemmas 2.1 -2.3 to estimate the solution in the norm $\mathbf{Z}$. We use Lemmas 2.4-2.6 to estimate the solution in the norm $\mathbf{X}$.

To overcome another difficulty - the so-called derivative loss in the derivative nonlinear Schrödinger equation we follow the idea of Doi [2]. In Section 3 we describe the smoothing property of the linear Schrödinger evolution group and then by virtue of the usual energy type estimates involving special operator $\mathcal{S}$ defined in Section 3 we estimate the solution in the norm $\mathbf{Y}$. Section 4 is devoted to the proof of Theorem 1.1.

§2. Lemmas

In the next lemma we estimate the following integral $\iint e^{it\Lambda}\Phi\left(\boldsymbol{\xi}\right)dydz$, where $\boldsymbol{\xi}=\left(\xi_1,\xi_2,\xi_3\right)$, with $\xi_1=x+y-z$, $\xi_2=x-y-z$, $\xi_3=x+\sigma z$, $\Lambda=\alpha y^2+\beta z^2$, here the constants $\alpha,\beta,\sigma\in\mathbf{R}\backslash\{0\},\sigma\neq-1$, so that ξ_1,ξ_2,ξ_3 are linearly independent; the function $\Phi\left(\boldsymbol{\xi}\right)=A\left(\boldsymbol{\xi}\right)\phi(\xi_1)\psi(\xi_2)\varphi(\xi_3)$, $A\left(\boldsymbol{\xi}\right)=\frac{\langle x\rangle^n\xi_1^{\omega_1}\xi_2^{\omega_2}\xi_3^{\omega_3}}{\langle\xi_1\rangle^{\sigma_1}\langle\xi_2\rangle^{\sigma_2}\langle\xi_3\rangle^{\sigma_3}}$, the powers $\omega_1,\omega_2,\omega_3$ take the values 0 or 1 with the condition $\omega_1+\omega_2+\omega_3\neq0$, the powers σ_1, σ_2, σ_3 are such that $\max\left(0,n-1\right)\leq\sigma_j\leq2$ with the condition $\sigma_1+\sigma_2+\sigma_3=n+3$, $n=0,1,2,3$. Also we denote $\boldsymbol{\xi}_0=\left(x,x,x\right)$. We denote $\|\phi\|_{\mathbf{B}}=\|\phi\|_{0,1,\infty}+\|\phi\|_{1,0,\infty}^{\gamma}\left(\|\mathcal{P}\phi\|_{\infty}+t\left\|\phi_t'\right\|_{0,1,\infty}\right)^{1-\gamma}$.

Lemma 2.1. *We have the following estimate*

$$\left\|\iint e^{it\Lambda}\left(\Phi\left(\boldsymbol{\xi}\right)-\Phi\left(\boldsymbol{\xi}_0\right)\right)dydz\right\|_{\infty}\leq CMt^{\gamma-3/2}$$

for all $t\geq1$, *where* $M=\|\phi\|_{\mathbf{B}}\|\psi\|_{\mathbf{B}}\|\varphi\|_{\mathbf{B}}$, $\gamma\in\left(0,10^{-4}\right)$.

Proof. First we integrate by parts with respect to y using the identity

$$e^{it\Lambda}=\mathcal{Y}\frac{\partial}{\partial y}\left(ye^{it\Lambda}\right), \tag{2.1}$$

where $\mathcal{Y}=\left(1+2i\alpha ty^2\right)^{-1}$, we get

$$\iint e^{it\Lambda}\left(\Phi\left(\boldsymbol{\xi}\right)-\Phi\left(\boldsymbol{\xi}_0\right)\right)dydz=I_1+I_2+I_3, \tag{2.2}$$

where

$$I_1=Ct\iint e^{it\Lambda}\left(\Phi\left(\boldsymbol{\xi}\right)-\Phi\left(\boldsymbol{\xi}_0\right)\right)y^2\mathcal{Y}^2dydz,$$

$$I_2=-\iint e^{it\Lambda}\Phi_{\xi_1}'\left(\boldsymbol{\xi}\right)y\mathcal{Y}dydz\quad\text{and}\quad I_3=\iint e^{it\Lambda}\Phi_{\xi_2}'\left(\boldsymbol{\xi}\right)y\mathcal{Y}dydz.$$

In the first integral I_1 we now integrate by parts with respect to z, using the identity $e^{it\Lambda} = \mathcal{Z}\frac{\partial}{\partial z}\left(ze^{it\Lambda}\right)$, where $\mathcal{Z} = \left(1 + 2i\beta tz^2\right)^{-1}$, we get

$$\begin{aligned} I_1 &= Ct^2 \iint e^{it\Lambda} \left(\Phi\left(\boldsymbol{\xi}\right) - \Phi\left(\boldsymbol{\xi}_0\right)\right) y^2 \mathcal{Y}^2 z^2 \mathcal{Z}^2 dydz \\ &\quad + Ct \iint e^{it\Lambda} \left(\Phi'_{\xi_1}\left(\boldsymbol{\xi}\right) + \Phi'_{\xi_2}\left(\boldsymbol{\xi}\right) - \sigma \Phi'_{\xi_3}\left(\boldsymbol{\xi}\right)\right) y^2 \mathcal{Y}^2 z \mathcal{Z} dydz. \end{aligned} \tag{2.3}$$

Via the operator $\mathcal{P} = -\xi\partial_\xi + 2t\partial_t$ we write $\partial_\xi \phi = -\frac{1}{\xi}\left(\mathcal{P} - 2t\partial_t\right)\phi$, therefore $\left|\partial_\xi \phi\left(\xi\right)\right| \leq C\left(\left|\xi\right|^{-1}\left(\left|\mathcal{P}\phi\right| + 2t\left|\partial_t \phi\right|\right)\right)^{1-\gamma} \left|\partial_\xi \phi\right|^\gamma \leq C\left|\xi\right|^{\gamma-1} \left\|\phi\right\|_{\mathbf{B}}$, whence denoting $\xi_4 = y$, $\xi_5 = z$ and $\{y\} = |y| / \langle y \rangle$ we get for $t \geq 1$

$$\begin{aligned} &\left\|I_1\right\|_\infty \\ &\leq C \sup_{x\in\mathbf{R}} \iint \left|\Phi\left(\boldsymbol{\xi}\right) - \Phi\left(\boldsymbol{\xi}_0\right)\right| \left\langle ty^2\right\rangle^{-1} \left\langle tz^2\right\rangle^{-1} dydz \\ &\quad + C \sup_{x\in\mathbf{R}} \iint \sum_{j=1}^{3} \left|\Phi'_{\xi_j}\left(\boldsymbol{\xi}\right)\right| |z| \left\langle ty^2\right\rangle^{-1} \left\langle tz^2\right\rangle^{-1} dydz \\ &\leq CMt^{\gamma-3/2} \sum_{j,m=1}^{3} \sup_{x\in\mathbf{R}} \iint \frac{\langle x\rangle^n \left(1 + \{\xi_m\}\left\{\xi_j\right\}^{\gamma-1}\right) \{y\}^{\gamma-1} \{z\}^{\gamma-1}}{\langle\xi_1\rangle^{\sigma_1-\gamma} \langle\xi_2\rangle^{\sigma_2-\gamma} \langle\xi_3\rangle^{\sigma_3-\gamma} \langle\xi_4\rangle \langle\xi_5\rangle} dydz \\ &\leq CMt^{\gamma-3/2} \sum_{1\leq l<m\leq 5} \sum_{1\leq k<j\leq 5} \iint \frac{\{\xi_k\}^{\gamma-1}\left\{\xi_j\right\}^{\gamma-1} dydz}{\langle\xi_l\rangle^{2-\gamma} \langle\xi_m\rangle^{2-\gamma}} \leq CMt^{\gamma-3/2}, \end{aligned}$$

where we have used the estimates

$$\{\xi_m\}^{1-\gamma}\left\{\xi_j\right\}^{\gamma-1} \{y\}^{\gamma-1} \{z\}^{\gamma-1} \leq C \sum_{1\leq k<l\leq 5} \{\xi_k\}^{\gamma-1} \{\xi_l\}^{\gamma-1}$$

for $j = 1, 2, 3$, $m = 1, 2, 3, 4, 5$, and

$$\langle x\rangle \langle\xi_k\rangle^{-1} \langle\xi_l\rangle^{-1} \langle\xi_m\rangle^{-1} \leq C\left(\langle\xi_k\rangle^{-1} \langle\xi_l\rangle^{-1} + \langle\xi_k\rangle^{-1} \langle\xi_m\rangle^{-1} + \langle\xi_l\rangle^{-1} \langle\xi_m\rangle^{-1}\right)$$

for $1 \leq k < l < m \leq 5$ so that

$$\frac{\langle x\rangle^n}{\langle\xi_1\rangle^{\sigma_1-\gamma} \langle\xi_2\rangle^{\sigma_2-\gamma} \langle\xi_3\rangle^{\sigma_3-\gamma} \langle\xi_4\rangle \langle\xi_5\rangle} \leq C \sum_{1\leq l<m\leq 5} \langle\xi_l\rangle^{\gamma-2} \langle\xi_m\rangle^{\gamma-2}$$

for $1 \leq l \leq 5$, $k \geq 3$.

Now let us estimate the second integral I_2. For the case $\alpha + \beta \neq 0$ we make a change of variables of integration $y = \beta\zeta - \eta$ and $z = -\alpha\zeta - \eta$ to get

$$I_2 = C \iint e^{it\Omega} \Phi'_{\xi_1}\left(\boldsymbol{\xi}\right) (\beta\zeta - \eta) Y d\zeta d\eta,$$

where $\Omega = \alpha\beta(\alpha+\beta)\zeta^2 + (\alpha+\beta)\eta^2$, $Y = \left(1+2i\alpha t(\beta\zeta-\eta)^2\right)^{-1}$, $\xi_1 = x + (\alpha+\beta)\zeta$, $\xi_2 = x + (\alpha-\beta)\zeta + 2\eta$, $\xi_3 = x - \sigma\alpha\zeta - \sigma\eta$. We now integrate by parts with respect to η via the identity

$$e^{it\Omega} = H\frac{\partial}{\partial\eta}\left(\eta e^{it\Omega}\right), \tag{2.4}$$

where $H = \left(1+2it(\alpha+\beta)\eta^2\right)^{-1}$ to obtain

$$\begin{aligned} I_2 &= Ct\iint e^{it\Omega}\eta^2 H^2 \Phi'_{\xi_1}(\boldsymbol{\xi})(\beta\zeta-\eta)\,Y d\eta d\zeta \\ &\quad + Ct\iint e^{it\Omega}(\beta\zeta-\eta)^2 Y^2 \Phi'_{\xi_1}(\boldsymbol{\xi})\,\eta H d\eta d\zeta \\ &\quad + C\iint e^{it\Omega}\left(2\Phi''_{\xi_1\xi_2}(\boldsymbol{\xi}) - \sigma\Phi''_{\xi_1\xi_3}(\boldsymbol{\xi})\right)(\beta\zeta-\eta)\,Y\eta H d\eta d\zeta, \end{aligned}$$

whence denoting $\xi_4 = y$, $\xi_5 = \eta$ we have by the above identity

$$\begin{aligned} &\|I_2\|_\infty \\ &\le CMt^{\gamma-3/2} \\ &\quad\times \sum_{j=2,3,5}\sum_{m=1}^{3}\sup_{x\in\mathbf{R}}\iint \frac{\langle x\rangle^n\left(1+\{\xi_m\}\{\xi_1\}^{\gamma-1}\right)\{\xi_j\}^{\gamma-1}\{y\}^{\gamma-1}}{\langle\xi_1\rangle^{\sigma_1-\gamma}\langle\xi_2\rangle^{\sigma_2-\gamma}\langle\xi_3\rangle^{\sigma_3-\gamma}\langle\xi_4\rangle\langle\xi_5\rangle}dyd\eta \\ &\le CMt^{\gamma-3/2}\sum_{1\le l<m\le 5}\sum_{1\le k<j\le 5}\iint\frac{\{\xi_k\}^{\gamma-1}\{\xi_j\}^{\gamma-1}dyd\eta}{\langle\xi_l\rangle^{2-\gamma}\langle\xi_m\rangle^{2-\gamma}} \le CMt^{\gamma-3/2}. \end{aligned}$$

In the case $\alpha+\beta=0$ we make the change of the independent variables $y = \eta+\zeta$ and $z = \eta-\zeta$ to get

$$I_2 = C\iint e^{4i\alpha t\zeta\eta}(\zeta+\eta)\mathfrak{Y}\Phi'_{\xi_1}(\boldsymbol{\xi})\,d\eta d\zeta = I_4 + I_5,$$

where $\mathfrak{Y} = \left(1+2i\alpha t(\zeta+\eta)^2\right)^{-1}$, $\xi_1 = x+2\zeta$, $\xi_2 = x-2\eta$, $\xi_3 = x-\sigma\zeta+\sigma\eta$, the integral I_4 is taken over the domain $|\zeta|t \le 1$, so we can easily estimate it as (with $\xi_4 = y$)

$$\begin{aligned} \|I_4\|_\infty &\le CMt^{\gamma-3/2}\sum_{j=1}^{3}\sup_{x\in\mathbf{R}}\iint\frac{\langle x\rangle^n\{\xi_j\}\{\xi_1\}^{\gamma-1}\{\xi_4\}^{\gamma-1}d\zeta dy}{\langle\xi_1\rangle^{\sigma_1-\gamma}\langle\xi_2\rangle^{\sigma_2-\gamma}\langle\xi_3\rangle^{\sigma_3-\gamma}\langle\xi_4\rangle|\zeta|^{1-\gamma}} \\ &\le CMt^{\gamma-3/2}\sum_{1\le l<m\le 4}\sum_{1\le k<j\le 4}\iint\frac{\{\xi_k\}^{\gamma-1}\{\xi_j\}^{\gamma-1}d\zeta dy}{\langle\xi_l\rangle^{2-\gamma}\langle\xi_m\rangle^{2-\gamma}} \le CMt^{\gamma-3/2}, \end{aligned}$$

and the integral I_5 is taken over the domain $|\zeta|t \geq 1$, therefore we can integrate by parts with respect to η to obtain

$$\begin{aligned} I_5 &= \frac{C}{t}\int_{|\zeta|t\geq 1}\int e^{4i\alpha t\zeta\eta}\left(2\Phi''_{\xi_1\xi_2}(\boldsymbol{\xi})-\sigma\Phi''_{\xi_1\xi_3}(\boldsymbol{\xi})\right)(\zeta+\eta)\,\zeta^{-1}\mathfrak{Y}d\eta d\zeta \\ &+C\int_{|\zeta|t\geq 1}\int e^{4i\alpha t\zeta\eta}(\zeta+\eta)^2\,\Phi'_{\xi_1}(\boldsymbol{\xi})\,\zeta^{-1}\mathfrak{Y}^2d\eta d\zeta \\ &+\frac{C}{t}\int_{|\zeta|t\geq 1}\int e^{4i\alpha t\zeta\eta}\Phi'_{\xi_1}(\boldsymbol{\xi})\,\zeta^{-1}\mathfrak{Y}d\eta d\zeta, \end{aligned}$$

whence we have the estimate

$$\begin{aligned} \|I_5\|_\infty &\leq CMt^{\gamma-3/2}\sum_{l=2}^{4}\sum_{j=1}^{3}\sup_{x\in\mathbf{R}}\iint\frac{\langle x\rangle^n\left(1+\{\xi_j\}\{\xi_1\}^{\gamma-1}\right)\{\xi_l\}^{\gamma-1}d\zeta dy}{\langle\xi_1\rangle^{\sigma_1-\gamma}\langle\xi_2\rangle^{\sigma_2-\gamma}\langle\xi_3\rangle^{\sigma_3-\gamma}\langle\xi_4\rangle|\zeta|^{1-\gamma}} \\ &\leq CMt^{\gamma-3/2}\sum_{1\leq l<m\leq 4}\sum_{1\leq k<j\leq 4}\iint\frac{\{\xi_k\}^{\gamma-1}\{\xi_j\}^{\gamma-1}d\zeta dy}{\langle\xi_l\rangle^{2-\gamma}\langle\xi_m\rangle^{2-\gamma}}\leq CMt^{\gamma-3/2}, \end{aligned}$$

where $\xi_4 = y$. The last integral I_3 is considered in the same manner as the integral I_2. Thus the first estimate is true. Lemma 2.1 is proved. ∎

We denote now $\Phi(\boldsymbol{\xi}) = A(\boldsymbol{\xi})\,\phi(\xi_1)\psi(\xi_2)\varphi(\xi_3)$, $A(\boldsymbol{\xi}) = \frac{\xi_1^{\omega_1}\xi_2^{\omega_2}\xi_3^{\omega_3}}{\langle\xi_1\rangle^{\sigma_1}\langle\xi_2\rangle^{\sigma_2}\langle\xi_3\rangle^{\sigma_3}}$, where $0 \leq \sigma_j \leq 2$ and $\sigma_1+\sigma_2+\sigma_3 = 4$.

Lemma 2.2. *We have the following estimates*

$$\left\|\iint e^{it\Lambda}\Phi(\boldsymbol{\xi})\,dydz\right\|_\infty \leq CM_jt^{\gamma-1/2},$$

for all $t \geq 1$, $j = 1,2,3$, *where* $M_1 = \|\phi\|\|\psi\|_{\mathbf{B}}\|\varphi\|_{\mathbf{B}}$, $M_2 = \|\phi\|_{\mathbf{B}}\|\psi\|\|\varphi\|_{\mathbf{B}}$ *and* $M_3 = \|\phi\|_{\mathbf{B}}\|\psi\|_{\mathbf{B}}\|\varphi\|$, $\gamma \in \left(0, 10^{-4}\right)$.

Proof. First let us obtain the estimate for the case $j = 3$. As in Lemma 2.1 we integrate by parts with respect to y using identity (2.1)

$$\begin{aligned} \iint e^{it\Lambda}\Phi(\boldsymbol{\xi})\,dydz &= Ct\iint e^{it\Lambda}\Phi(\boldsymbol{\xi})\,y^2\mathcal{Y}^2dydz-\iint e^{it\Lambda}\Phi'_{\xi_1}(\boldsymbol{\xi})\,y\mathcal{Y}dydz \\ &+\iint e^{it\Lambda}\Phi'_{\xi_2}(\boldsymbol{\xi})\,y\mathcal{Y}dydz. \end{aligned}$$

Whence using the operator $\mathcal{P} = -\xi\partial_\xi + 2t\partial_t$ by the Cauchy inequality we obtain denoting $\xi_4 = y$

$$\begin{aligned}
&\left\| \iint e^{it\Lambda} \Phi(\boldsymbol{\xi})\, dydz \right\|_\infty \\
&\le CM_3 t^{\gamma - 1/2} \sup_{x\in\mathbf{R}} \left(\int d\xi_3 \left(\sum_{m=1,2,4} \int dy \frac{\langle \xi_3 \rangle \{\xi_m\}^{\gamma-1}}{\langle \xi_1 \rangle^{\sigma_1 - \gamma} \langle \xi_2 \rangle^{\sigma_2-\gamma} \langle \xi_3 \rangle^{\sigma_3 - \gamma} \langle \xi_4 \rangle} \right)^2 \right)^{\frac{1}{2}} \\
&\le CM_3 t^{\gamma-1/2} \left(\int dz \langle x - z \rangle^{2\gamma - 2} \left(\sum_{l,m=1,2,4} \int \langle \xi_l \rangle^{\gamma-2} \{\xi_m\}^{\gamma-1} dy \right)^2 \right)^{\frac{1}{2}} \\
&\le CM_3 t^{\gamma-1/2}.
\end{aligned}$$

Now let us prove the estimate for the case $j = 1$. As in the proof of Lemma 2.1 for the case $\alpha + \beta \neq 0$ we make a change of variables of integration $y = \beta\zeta - \eta$ and $z = -\alpha\zeta - \eta$ and integrate by parts with respect to η via identity (2.4) to get

$$\begin{aligned}
\iint e^{it\Lambda} \Phi(\boldsymbol{\xi})\, dydz &= C \iint e^{it\Omega} \Phi(\boldsymbol{\xi})\, d\eta d\zeta = Ct \iint e^{it\Omega} \Phi(\boldsymbol{\xi})\, \eta^2 H^2 d\eta d\zeta \\
&\quad + C \iint e^{it\Omega} \left(2\Phi'_{\xi_2}(\boldsymbol{\xi}) - \sigma \Phi'_{\xi_3}(\boldsymbol{\xi}) \right) \eta H d\eta d\zeta,
\end{aligned}$$

where $\Omega = \alpha\beta(\alpha+\beta)\zeta^2 + (\alpha+\beta)\eta^2$, $\xi_1 = x + (\alpha+\beta)\zeta$, $\xi_2 = x + (\alpha - \beta)\zeta + 2\eta$, $\xi_3 = x - \sigma\alpha\zeta - \sigma\eta$. Whence by the Cauchy inequality we have, denoting $\xi_4 = \eta$

$$\begin{aligned}
&\left\| \iint e^{it\Lambda} \Phi(\boldsymbol{\xi})\, dydz \right\|_\infty \\
&\le CM_1 t^{\gamma-1/2} \sup_{x\in\mathbf{R}} \left(\int d\xi_1 \left(\sum_{l,m=2,3,4} \int d\eta \frac{\langle \xi_1 \rangle \{\xi_m\}^{\gamma-1}}{\langle \xi_1 \rangle^{\sigma_1-\gamma} \langle \xi_2 \rangle^{\sigma_2 - \gamma} \langle \xi_3 \rangle^{\sigma_3-\gamma} \langle \xi_4 \rangle} \right)^2 \right)^{\frac{1}{2}} \\
&\le \frac{CM_1}{t^{1/2-\gamma}} \left(\int d\zeta \sum_{k=1,5,6,7} \langle \xi_k \rangle^{2\gamma-2} \left(\sum_{l,m=2,3,4} \int \langle \xi_l \rangle^{\gamma-2} \{\xi_m\}^{\gamma-1} d\eta \right)^2 \right)^{\frac{1}{2}} \\
&\le CM_1 t^{\gamma-1/2},
\end{aligned}$$

where $\xi_5 = x - \alpha\sigma\zeta$, $\xi_6 = (2+\sigma)x - \sigma(\alpha+\beta)\zeta$, $\xi_7 = x + (\alpha - \beta)\zeta$, if $\alpha - \beta \neq 0$, and $\xi_7 = \xi_5$, if $\alpha = \beta$. And in the case $\alpha + \beta = 0$ we make the change of the independent variables $y = \eta + \zeta$ and $z = \eta - \zeta$ to get

$$\iint e^{it\Lambda} \Phi(\boldsymbol{\xi})\, dydz = C \iint e^{4i\alpha t\zeta\eta} \Phi(\boldsymbol{\xi})\, d\eta d\zeta = I_4 + I_5,$$

where $\xi_1 = x + 2\zeta$, $\xi_2 = x - 2\eta$, $\xi_3 = x - \sigma\zeta + \sigma\eta$, the integral I_4 is taken over the domain $|\zeta| t \leq 1$, so it is easily estimated as

$$\begin{aligned}\|I_4\|_\infty &\leq CM_1 \left(\sup_{x\in\mathbf{R}} \int_{|\zeta|t\leq 1} d\zeta \left(\int \frac{\langle \xi_1 \rangle \, d\eta}{\langle \xi_1 \rangle^{\sigma_1-\gamma} \langle \xi_2 \rangle^{\sigma_2-\gamma} \langle \xi_3 \rangle^{\sigma_3-\gamma}} \right)^2 \right)^{\frac{1}{2}} \\ &\leq CM_1 t^{-1/2}\end{aligned}$$

and the integral I_5 is taken over the domain $|\zeta| t \geq 1$, therefore integrating by parts with respect to η we obtain

$$I_5 = \frac{C}{t} \int_{|\zeta|t\geq 1} \int e^{4i\alpha t\zeta\eta} \left(2\Phi'_{\xi_2}(\boldsymbol{\xi}) - \sigma \Phi'_{\xi_3}(\boldsymbol{\xi}) \right) \zeta^{-1} d\eta d\zeta,$$

whence denoting $\xi_4 = \zeta$ we get

$$\begin{aligned}&\|I_5\|_\infty \\ &\leq CM_1 t^{-1} \sup_{x\in\mathbf{R}} \left(\int_{|\zeta|t\geq 1} \frac{d\zeta}{\{\zeta\}^2} \left(\sum_{j=2,3} \int \frac{\langle \xi_1 \rangle \{\xi_j\}^{\gamma-1} \, d\eta}{\langle \xi_1 \rangle^{\sigma_1-\gamma} \langle \xi_2 \rangle^{\sigma_2-\gamma} \langle \xi_3 \rangle^{\sigma_3-\gamma} \langle \xi_4 \rangle} \right)^2 \right)^{\frac{1}{2}} \\ &\leq CM_1 t^{\gamma-1/2} \sup_{x\in\mathbf{R}} \left(\int_{|\zeta|t\geq 1} \sum_{l=4,5} \frac{\{\zeta\}^{\gamma-1} d\zeta}{\langle \xi_l \rangle^2} \left(\sum_{j,k=2,3} \int \frac{\{\xi_j\}^{\gamma-1} d\eta}{\langle \xi_k \rangle^{2-\gamma}} \right)^2 \right)^{\frac{1}{2}} \\ &\leq CM_1 t^{\gamma-1/2},\end{aligned}$$

where $\xi_5 = (2+\sigma)x - 2\sigma\zeta$. Therefore the estimate for the case $j = 1$ is true. The case $j = 2$ can be reduced to the case $j = 1$ by the change of the variable of integration $y \to -y$. Lemma 2.2 is proved. ∎

Let $\Phi(\boldsymbol{\xi}) = A(\boldsymbol{\xi}) \phi(\xi_1)\psi(\xi_2)\varphi(\xi_3)$, $A(\boldsymbol{\xi}) = \frac{\langle x \rangle^n \xi_1^{\omega_1} \xi_2^{\omega_2} \xi_3^{\omega_3}}{\langle \xi_1 \rangle^{\sigma_1} \langle \xi_2 \rangle^{\sigma_2} \langle \xi_3 \rangle^{\sigma_3}}$, the powers σ_1, σ_2, σ_3 are such that $n \leq \sigma_j \leq 2$ with the condition $\sigma_1 + \sigma_2 + \sigma_3 = n + 4$, $n = 0, 1, 2$.

Lemma 2.3. *We have the estimate*

$$\left\| \partial_x \iint e^{it\Lambda} \Phi(\boldsymbol{\xi}) \, dydz \right\|_\infty \leq CMt^{\gamma-1/2}$$

for all $t \geq 1$, *where* $M = \|\phi\|_{\mathbf{B}} \|\psi\|_{\mathbf{B}} \|\varphi\|_{\mathbf{B}}$, $\gamma \in (0, 10^{-4})$.

Proof. To prove the estimate of the lemma we write $\partial_x \iint e^{it\Lambda} \Phi(\boldsymbol{\xi}) \, dydz = J_1 + J_2 + J_3$, where $J_l = \iint e^{it\Lambda} \Phi'_{\xi_l}(\boldsymbol{\xi}) \, dydz$. As above we integrate by parts with respect to y using identity (2.1)

$$J_3 = Ct \iint e^{it\Lambda} \Phi'_{\xi_3}(\boldsymbol{\xi})\, y^2 \mathcal{Y}^2 dydz - \iint e^{it\Lambda} \Phi''_{\xi_1\xi_3}(\boldsymbol{\xi})\, y\mathcal{Y}dydz$$
$$+ \iint e^{it\Lambda} \Phi''_{\xi_2\xi_3}(\boldsymbol{\xi})\, y\mathcal{Y}dydz.$$

Whence using the operator $\mathcal{P} = -\xi\partial_\xi + 2t\partial_t$ and denoting $\xi_4 = y$ we obtain

$$\|J_3\|_\infty \le CMt^{\gamma-1/2} \sup_{x\in\mathbf{R}} \sum_{m=1,2,4} \iint \frac{\langle x\rangle^n \{\xi_m\}^{\gamma-1} \{\xi_3\}^{\gamma-1} dydz}{\langle\xi_1\rangle^{\sigma_1-\gamma} \langle\xi_2\rangle^{\sigma_2-\gamma} \langle\xi_3\rangle^{\sigma_3-\gamma} \langle\xi_4\rangle}$$
$$\le \quad CMt^{\gamma-1/2} \sum_{1\le k<l\le 4} \sum_{m=1,2,4} \iint \frac{\{\xi_3\}^{\gamma-1} \{\xi_m\}^{\gamma-1} dydz}{\langle\xi_k\rangle^{2-\gamma} \langle\xi_l\rangle^{2-\gamma}} \le CMt^{\gamma-1/2}.$$

Now let us estimate J_1. For the case $\alpha+\beta \ne 0$ we make a change of variables of integration $y = \beta\zeta - \eta$ and $z = -\alpha\zeta - \eta$ and integrate by parts with respect to η via identity (2.4) to get

$$J_1 \;=\; C \iint e^{it\Omega} \Phi'_{\xi_1}(\boldsymbol{\xi})\, d\eta d\zeta = Ct \iint e^{it\Omega} \Phi'_{\xi_1}(\boldsymbol{\xi})\, \eta^2 H^2 d\eta d\zeta$$
$$+C \iint e^{it\Omega} \left(2\Phi''_{\xi_1\xi_2}(\boldsymbol{\xi}) - \sigma\Phi''_{\xi_1\xi_3}(\boldsymbol{\xi})\right) \eta H d\eta d\zeta,$$

where $\Omega = \alpha\beta(\alpha+\beta)\zeta^2 + (\alpha+\beta)\eta^2$, $\xi_1 = x + (\alpha+\beta)\zeta$, $\xi_2 = x + (\alpha-\beta)\zeta + 2\eta$, $\xi_3 = x - \sigma\alpha\zeta - \sigma\eta$. Whence we have the second estimate denoting $\xi_4 = \eta$

$$\|J_1\|_\infty \le CMt^{\gamma-1/2} \sup_{x\in\mathbf{R}} \sum_{m=2,3,4} \int d\xi_1 \int d\eta \frac{\langle x\rangle^n \{\xi_m\}^{\gamma-1} \{\xi_1\}^{\gamma-1}}{\langle\xi_1\rangle^{\sigma_1-\gamma} \langle\xi_2\rangle^{\sigma_2-\gamma} \langle\xi_3\rangle^{\sigma_3-\gamma} \langle\xi_4\rangle}$$
$$\le CMt^{\gamma-1/2} \sum_{1\le k<l\le 4} \sum_{m=2,3,4} \iint \frac{\{\xi_1\}^{\gamma-1} \{\xi_m\}^{\gamma-1} dydz}{\langle\xi_k\rangle^{2-\gamma} \langle\xi_l\rangle^{2-\gamma}} \le CMt^{\gamma-1/2}.$$

And in the case $\alpha+\beta = 0$ we make the change of the independent variables $y = \eta+\zeta$ and $z = \eta-\zeta$ to get

$$\iint e^{it\Lambda} \Phi'_{\xi_1}(\boldsymbol{\xi})\, dydz = C \iint e^{4i\alpha t\zeta\eta} \Phi'_{\xi_1}(\boldsymbol{\xi})\, d\eta d\zeta = I_4 + I_5,$$

where $\xi_1 = x + 2\zeta$, $\xi_2 = x - 2\eta$, $\xi_3 = x - \sigma\zeta + \sigma\eta$, the integral I_4 is taken over the domain $|\zeta|t \le 1$, so we have

$$\|I_4\|_\infty \le CM \sup_{x\in\mathbf{R}} \int_{|\zeta|t\le 1} d\zeta \int \frac{\langle x\rangle^n \{\xi_1\}^{\gamma-1} d\eta}{\langle\xi_1\rangle^{\sigma_1-\gamma} \langle\xi_2\rangle^{\sigma_2-\gamma} \langle\xi_3\rangle^{\sigma_3-\gamma}} \le CMt^{-1/2}$$

and the integral I_5 is taken over the domain $|\zeta| t \geq 1$, where integrating by parts with respect to η we obtain

$$I_5 = \frac{C}{t} \int_{|\zeta| t \geq 1} \int e^{4i\alpha t \zeta \eta} \left(2\Phi''_{\xi_1 \xi_2}(\boldsymbol{\xi}) - \sigma \Phi''_{\xi_1 \xi_3}(\boldsymbol{\xi}) \right) \zeta^{-1} d\eta d\zeta,$$

whence we have the estimate

$$\begin{aligned} & \|I_5\|_\infty \\ & \leq CMt^{\gamma-1} \sup_{x \in \mathbf{R}} \sum_{l=1}^{3} \sum_{j=2,3} \iint \frac{\langle x \rangle^n \left(1 + \{\xi_l\} \{\xi_j\}^{\gamma-1}\right) \{\zeta\}^{\gamma-1} \{\xi_1\}^{\gamma-1}}{\langle \xi_1 \rangle^{\sigma_1 - \gamma} \langle \xi_2 \rangle^{\sigma_2 - \gamma} \langle \xi_3 \rangle^{\sigma_3 - \gamma} \langle \zeta \rangle} d\eta d\zeta \\ & \leq CMt^{\gamma - 1/2}. \end{aligned}$$

Integral J_2 is estimated in the same manner. Therefore the third estimate of the lemma is true. Lemma 2.3 is proved. ∎

In the next simple lemma we give a time - decay estimate for the integral $\int e^{it\mu(x-q)^2} \phi(t,x) x^\omega dx$ uniformly with respect to $q \in \mathbf{R}$, for $\omega = 1, 2, 3$.

Lemma 2.4. *We have the estimate*

$$\begin{aligned} \left\| \int e^{it\mu(x-q)^2} \phi(t,x) x^\omega dx \right\|_\infty \leq \; & Ct^{-1/2} \|\phi\|_{0,\omega,\infty} \\ & + Ct^{-3/4} \|\mathcal{P}\phi\|_{0,\omega-1} + C \left\|\phi'_t\right\|_{0,\omega,\infty} \end{aligned}$$

for all $t \geq 1$.

Proof. We have $\int e^{it\mu(x-q)^2} \phi(t,x) x^\omega dx = J_1 + J_2$, where $J_1 = \phi(t,q) q^\omega \int e^{it\mu(x-q)^2} dx$ and $J_2 = \int e^{it\mu(x-q)^2} (\phi(t,x) x^\omega - \phi(t,q) q^\omega) dx$. For the first summand we get $|J_1| \leq Ct^{-1/2} \|\phi\|_{0,\omega,\infty}$. And in the second summand we integrate by parts with respect to x via identity

$$e^{it\mu(x-q)^2} = X \frac{d}{dx} \left((x-q) e^{it\mu(x-q)^2} \right), \tag{2.5}$$

where $X = \left(1 + 2i\mu t (x-q)^2\right)^{-1}$ to get

$$J_2 = -\int e^{i\mu t(x-q)^2} \left(X \left(\phi(t,x) x^\omega - \phi(t,q) q^\omega \right) \right)'_x (x-q) dx.$$

We write $x\phi'_x = -\mathcal{P}\phi + 2t\phi'_t$, then applying the Cauchy inequality we get

$$
\begin{aligned}
|J_2| \le C \int \Bigg(& \left|x^{\omega}\phi'_x\right| + \langle x\rangle^{\omega-1}|\phi| \\
& + \frac{t\,|x-q|}{\left\langle t\,(x-q)^2\right\rangle}\left|\int_x^q \left(\zeta^{\omega}\phi(\zeta)\right)' d\zeta\right|\Bigg)\frac{|x-q|\,dx}{\left\langle t\,(x-q)^2\right\rangle} \\
\le {} & Ct^{-1/2}\|\phi\|_{0,\omega,\infty} + Ct^{-3/4}\|\mathcal{P}\phi\|_{0,\omega-1} + C\left\|\phi'_t\right\|_{0,\omega,\infty}.
\end{aligned}
$$

Whence the estimate of the lemma follows. Lemma 2.4 is proved. ∎

We now consider the asymptotic behavior of the integral

$$\mathcal{N}_1(t,r) = \int dx e^{-\frac{i}{2}t(x-r)^2}\int_1^t \frac{d\tau}{\tau}e^{i\tau\delta x^2}E(t,x)\,\Psi(\tau,x),$$

where $E(t,x) = \exp\left(ia(x)\int_1^t |w(s,x)|^2\frac{ds}{s}\right)$, $a(x) = \sum_{j=1}^3 b_j x^j$, $b_j \in \mathbf{R}$, $\delta \in \mathbf{R}\setminus\{0\}$, $\Psi(t,x) = A(x)\,\phi(t,x)\,\psi(t,x)\,\varphi(t,x)$, $A(x) = \sum_{j=1}^3 B_j x^j$, $B_j \in \mathbf{C}$. We introduce $\mathbf{G} = \{\phi \in \mathbf{C}(\mathbf{R};\mathbf{L}^{\infty}) : \|\phi\|_{\mathbf{G}} < \infty\}$, where $\|\phi\|_{\mathbf{G}} = \sup_{t\ge 1} t^{-\lambda}\left(\|\phi\|_{0,2,\infty} + \|\mathcal{P}\phi\|_{0,1,\infty} + t\left\|\phi'_t\right\|_{0,2,\infty} + \|\phi\|_{\mathbf{B}}\right)$, $\lambda \in (0, 0.025)$.

Lemma 2.5. *Let $\phi,\psi,\varphi,w \in \mathbf{G}$ and*

$$\|w(t) - w(\tau)\|_{0,1,\infty} \le C\tau^{-2\lambda}. \tag{2.6}$$

Then there exists a unique $W \in L^{\infty}$ satisfying the following asymptotics

$$\mathcal{N}_1(t) = \frac{\sqrt{2\pi}}{\sqrt{it}}\exp\left(ia|W|^2\log t + i\vartheta\right)\int_1^{\infty} e^{i\tau\delta r^2}\Psi(\tau)\frac{d\tau}{\tau} + O\left(Mt^{-1/2-\lambda}\right)$$

for all $t \ge 1$ uniformly in $r \in \mathbf{R}$, where $\vartheta = a\int_1^{\infty}(|w(\tau)|^2 - |w(t)|^2)\frac{d\tau}{\tau}$, $M = \|\phi\|_{\mathbf{G}}\|\psi\|_{\mathbf{G}}\|\varphi\|_{\mathbf{G}}\left(1 + \|w\|_{\mathbf{G}}^2\right)$.

Proof. We put $\mu = -\frac{1}{2} + \delta\frac{\tau}{t}$, $q = -\frac{r}{2\mu}$. Then

$$
\begin{aligned}
&\mathcal{N}_1(t,r) \\
&= \int_1^t \frac{d\tau}{\tau}e^{-\frac{i}{2}\tau\delta r^2/\mu}\int dx e^{-i\mu t(x-q)^2}E(t,x)\,\Psi(\tau,x) \\
&= \int_1^t \frac{d\tau}{\tau}e^{-\frac{i}{2}\tau\delta r^2/\mu}\int dx e^{-i\mu t(x-q)^2}\left(E(t,x)\,\Psi(\tau,x) - E(t,q)\,\Psi(\tau,q)\right) \\
&\quad + \frac{\sqrt{2\pi}}{\sqrt{it}}\int_1^t \frac{d\tau}{\tau}e^{-\frac{i}{2}\tau\delta r^2/\mu}\frac{1}{\sqrt{-2\mu}}E(t,q)\,\Psi(\tau,q).
\end{aligned}
$$

Hence we can write the following representation $\mathcal{N}_1 = \sum_{k=1}^{7} J_k$, where

$$J_1 = \frac{\sqrt{2\pi}}{\sqrt{it}} E(t,r) \int_{t^\nu}^{\infty} e^{i\tau\delta r^2} \Psi(\tau,r) \frac{d\tau}{\tau},$$

$$J_2 = \frac{\sqrt{2\pi}}{\sqrt{it}} \int_{t^\nu}^{t} e^{-\frac{i}{2}\tau\delta r^2/\mu} E(t,q) \Psi(\tau,q) \frac{1}{\sqrt{-2\mu}} \frac{d\tau}{\tau},$$

$$J_3 = \frac{\sqrt{2\pi}}{\sqrt{it}} \int_{1}^{t^\nu} e^{-\frac{i}{2}\tau\delta r^2/\mu} E(t,q) \Psi(\tau,q) \left(\frac{1}{\sqrt{-2\mu}} - 1\right) \frac{d\tau}{\tau},$$

$$J_4 = \frac{\sqrt{2\pi}}{\sqrt{it}} E(t,r) \int_{1}^{t^\nu} \left(e^{-\frac{i}{2}\tau\delta r^2/\mu} - e^{i\tau\delta r^2}\right) \Psi(\tau,r) \frac{d\tau}{\tau},$$

$$J_5 = \frac{\sqrt{2\pi}}{\sqrt{it}} \int_{1}^{t^\nu} e^{-\frac{i}{2}\tau\delta r^2/\mu} \left(E(t,q)\Psi(\tau,q) - E(t,r)\Psi(\tau,r)\right) \frac{d\tau}{\tau},$$

$$J_6 = \int_1^t \frac{d\tau}{\tau} e^{-\frac{i}{2}\tau\delta r^2/\mu} \int dx e^{i\mu t(x-q)^2} \left(\Psi(\tau,x) E(t,x) - \Psi(\tau,q) E(t,q)\right)$$

and

$$J_7 = -\frac{\sqrt{2\pi}}{\sqrt{it}} E(t,r) \int_{1}^{\infty} e^{i\tau\delta r^2} \Psi(\tau,r) \frac{d\tau}{\tau},$$

where $\nu \in \left(8\lambda, \frac{1}{2} - 4\lambda\right)$.

Let us integrate by parts with respect to time in the integral J_1 via the identity $\left(1 + i\tau\delta r^2\right) e^{i\tau\delta r^2} = \partial_\tau \left(\tau e^{i\tau\delta r^2}\right)$ we obtain $|J_1| \le CMt^{-1/2-\nu/2+3\lambda} \le CMt^{-1/2-\lambda}$, since $|t\Psi_t(t,x)| \le CM|x|t^{3\lambda}$ and $|tE_t(t,x)| \le CM|x|t^{2\lambda}$, and similarly in the integral J_2 we integrate via identity $\left(1 + \frac{i\tau\delta r^2}{4\mu^2}\right) e^{-\frac{i}{2}\tau\delta r^2/\mu} = \partial_\tau \left(\tau e^{-\frac{i}{2}\tau\delta r^2/\mu}\right)$ to get $|J_2| \le CM\left(t^{3\lambda-1} + t^{-1/2-\nu/2+3\lambda}\right) \le CMt^{-1/2-\lambda}$. Then $|J_3| \le Ct^{\nu-\frac{3}{2}}$. Using the inequality $\left|e^{-\frac{i}{2}\tau\delta r^2/\mu} - e^{i\tau\delta r^2}\right| \le Cr^2\frac{\tau^2}{t}$ we obtain $|J_4| \le CMt^{2\nu+3\lambda-3/2} \le CMt^{-1/2-\lambda}$.

Since $|r-q| \le C|r|\frac{\tau}{t}$, $\|\Psi'_x(t)\|_{0,1,\infty} \le CMt^{3\lambda}$ and $\|E'_x(t)\|_{0,1,\infty} \le CMt^{3\lambda}$ we have for the integral $J_5 : |J_5| \le Ct^{\nu+3\lambda-3/2}M \le CMt^{-1/2-\lambda}$. Integrating by parts via identity (2.5) in the integral J_6 we get

$$\begin{aligned} J_6 &= \int_1^t \frac{d\tau}{\tau} e^{-\frac{i}{2}\tau\delta r^2/\mu} \int e^{i\mu t(x-q)^2} \left(\left(\Psi(\tau,x) E(t,x)\right.\right. \\ &\quad \left.\left. -\Psi(\tau,q) E(t,q)\right) X\right)'_x (x-q)\, dx, \end{aligned}$$

where X is the same one as defined in the proof of Lemma 2.4. Since $w \in \mathbf{G}$ we have $\left\|(\Psi E)'_x\right\|_{0,1,\infty} \le CMt^{3\lambda}$, hence

$$|J_6| \le C \int_1^t \frac{d\tau}{\tau} \int \left(\left|(\Psi E)'_x\right|\right) \frac{|x-q|\, dx}{\left\langle t(x-q)^2 \right\rangle} \le CMt^{3\lambda-1} \le CMt^{-1/2-\lambda}.$$

Via (2.6) we see that there exists a unique limit function $W \in \mathbf{L}^{\infty}$ such that $\|W - w(t)\|_{0,1,\infty} \le Ct^{-2\lambda}$. We now denote $\chi(t) = ia\int_1^t(|w(\tau)|^2 - |w(t)|^2))\frac{d\tau}{\tau}$. Then $\chi(t) - \chi(s) = ia\int_s^t(|w(\tau)|^2 - |w(t)|^2)\frac{d\tau}{\tau} + ia(|w(t)|^2 - |w(s)|^2)\log s$, where $1 < s < \tau < t$. Using (2.6) we get

$$\|\chi(t) - \chi(s)\|_{\infty} \le C\|w\|_{\mathbf{B}}\int_s^t \langle\tau\rangle^{-1-2\lambda}d\tau + C\|w\|_{\mathbf{G}}s^{-2\lambda}\log s \le C\|w\|_{\mathbf{G}}s^{-\lambda}.$$

Therefore, we see that there exists a unique function $\vartheta \in \mathbf{L}^{\infty}, i\vartheta = \lim_{t\to\infty}\chi(t)$ satisfying $\|i\vartheta - \chi(t)\|_{\infty} \le Ct^{-\lambda}$. Then via the identity

$$\begin{aligned} ia\int_1^t |w(\tau)|^2\frac{d\tau}{\tau} &= ia|W|^2\log t + i\vartheta + (\chi(t) - i\vartheta) \\ +ia(|w(t)|^2 - |W|^2))\log t &= ia|W|^2\log t + i\vartheta + O\left(\|w\|_{\mathbf{G}}t^{-\lambda}\right) \end{aligned}$$

we obtain

$$E(t,r) = \exp\left(ia|W(r)|^2\log t + i\vartheta(r)\right) + O\left(\|w\|_{\mathbf{G}}t^{-\lambda}\right) \tag{2.7}$$

Therefore the integral J_7 gives us the main term of asymptotics. Thus the result of the lemma is true. ∎

In order to get uniform decay estimates of the solution we consider now the integral

$$\begin{aligned} \mathcal{N}_2(t,r) &= \int_1^t d\tau\int dx e^{-\frac{i}{2}t(x-r)^2 + i\tau\delta x^2}E(t,x)\overline{E(\tau,x)} \\ &\quad\times\iint e^{i\tau\Lambda}\left(\Phi(\tau,\boldsymbol{\xi}) - \Phi(\tau,\boldsymbol{\xi}_0)\right)dydz \end{aligned}$$

for all $r \in \mathbf{R}$, $t \ge 1$, where $\delta \in \mathbf{R}$, and as above we put $\boldsymbol{\xi} = (\xi_1, \xi_2, \xi_3)$, $\boldsymbol{\xi}_0 = (x,x,x)$ with $\xi_1 = x + y - z$, $\xi_2 = x - y - z$, $\xi_3 = x + \sigma z$, $\Lambda = \alpha y^2 + \beta z^2$, here the constants $\alpha, \beta, \sigma \in \mathbf{R}\backslash\{0\}, \sigma \ne -1$, the function $\Phi(\boldsymbol{\xi}) = \xi_1^{\omega_1}\xi_2^{\omega_2}\xi_3^{\omega_3}\phi(\xi_1)\psi(\xi_2)\varphi(\xi_3)$, the powers ω_1, ω_2, ω_3 take the values 0 or 1 with the condition $\omega_1 + \omega_2 + \omega_3 \ne 0$.

Lemma 2.6. *Let $\phi, \psi, \varphi, w \in \mathbf{G}$ and (2.6) be fulfilled. Then there exists a unique $W \in L^{\infty}$ satisfying the following asymptotics*

$$\mathcal{N}_2(t) = \frac{\sqrt{2\pi}}{\sqrt{it}}\exp\left(ia|W|^2\log t + i\vartheta\right)\int_1^{\infty}e^{i\tau\delta r^2}V(\tau)\,d\tau + O\left(t^{-1/2-\lambda}M\right)$$

for all $t \ge 1$, where $V(\tau) = \overline{E(\tau)}\iint e^{i\tau\Lambda}\left(\Phi(\tau,\boldsymbol{\xi}) - \Phi(\tau,\boldsymbol{\xi}_0)\right)dydz$, $M = \|\phi\|_{\mathbf{G}}\|\psi\|_{\mathbf{G}}\|\varphi\|_{\mathbf{G}}\left(1 + \|w\|_{\mathbf{G}}^2\right)$, $\lambda \in (0, 0.025)$.

Proof. We now write $\mathcal{N}_2 = J_1 + J_2 + J_3$, where

$$J_1 = \int_1^{t^\nu} d\tau e^{-\frac{i}{2}\tau\delta r^2/\mu} \int dx e^{i\mu t(x-q)^2} E(t,r) V(\tau,r),$$

$$J_2 = \int_1^{t^\nu} d\tau e^{-\frac{i}{2}\tau\delta r^2/\mu} \int dx e^{i\mu t(x-q)^2} \left(E(t,x) V(\tau,x) - E(t,r) V(\tau,r)\right)$$

and

$$J_3 = \int dx e^{-\frac{i}{2}t(x-r)^2} \int_{t^\nu}^{t} d\tau e^{i\tau\delta x^2} E(t,x) V(\tau,x),$$

here $V(\tau,x) = \overline{E(\tau,x)} \iint e^{i\tau\Lambda} \left(\Phi(\tau,\boldsymbol{\xi}) - \Phi(\tau,\boldsymbol{\xi}_0)\right) dydz$, $\mu = -\frac{1}{2} + \delta\frac{\tau}{t}$, $q = -\frac{r}{2\mu}$, $\nu = \frac{1}{2} + 10\lambda$. In view of the estimate of Lemma 2.1, Lemma 2.5 and asymptotics (2.7) the integral J_1 gives us the main term of the asymptotics

$$\begin{aligned}
J_1 &= \int_1^{t^\nu} d\tau e^{-\frac{i}{2}\tau\delta r^2/\mu} \int dx e^{i\mu t(x-q)^2} E(t,r) V(\tau,r) \\
&= \frac{\sqrt{2\pi}}{\sqrt{it}} E(t,r) \int_1^{t^\nu} e^{-\frac{i}{2}\tau\delta r^2/\mu} V(\tau,r) \frac{d\tau}{\sqrt{-2\mu}} \\
&= \frac{\sqrt{2\pi}}{\sqrt{it}} E(t,r) \int_1^{t^\nu} e^{i\tau\delta r^2} V(\tau,r) d\tau + O\left(Mt^{-1/2-\lambda}\right) \\
&= \frac{\sqrt{2\pi}}{\sqrt{it}} \exp\left(ia|W|^2 \log t + i\vartheta\right) \int_1^{\infty} e^{i\tau\delta r^2} V(\tau,r) d\tau + O\left(Mt^{-1/2-\lambda}\right)
\end{aligned}$$

since by Lemma 2.1 we have $|V(\tau)| \le CM\tau^{\gamma+3\lambda-3/2}$. In the remainder integral J_2 we integrate by parts via (2.5)

$$\begin{aligned}
J_2 = & \int_1^{t^\nu} d\tau e^{-\frac{i}{2}\tau\delta r^2/\mu} \int e^{i\mu t(x-q)^2} \left(X\left(E(t,x) V(\tau,x)\right.\right. \\
& \left.\left. - E(t,r) V(\tau,r)\right)\right)'_x (x-q)\, dx.
\end{aligned}$$

By Lemma 2.3 we have $\|V'_x(\tau)\|_{0,1,\infty} \le CMt^{(3\lambda+\gamma-1/2)\nu}$, hence we obtain

$$\begin{aligned}
|J_2| &\le C \int_1^{t^\nu} d\tau \int \left(|E| \left|V'_x(\tau,x)\right| + \left|E'_x\right| |V(\tau,x)|\right) \frac{(|x-q| + |q-r|)\, dx}{\left\langle t(x-q)^2 \right\rangle} \\
&\le CMt^{(3\lambda+\gamma+1/2)\nu-1} \le CMt^{-1/2-\lambda}.
\end{aligned}$$

Now let us prove the following estimate $\|J_3\|_\infty \le CMt^{-1/2-\lambda}$. Using representations (2.2) and (2.3) of Lemma 2.1 we get $J_3 = \sum_{j=1}^{4} I_j$, where

$$I_1 = \int_{t^\nu}^{t} d\tau \int dx e^{-\frac{i}{2}t(x-r)^2 + i\tau\delta x^2} \iint e^{i\tau\Lambda} \Psi \mathcal{Y}\mathcal{Z} dy dz,$$

$$I_2 = \int_{t^\nu}^{t} d\tau \int dx e^{-\frac{i}{2}t(x-r)^2 + i\tau\delta x^2} \iint e^{i\tau\Lambda} \Omega \mathcal{Y}\mathcal{Z} dy dz,$$

$$I_3 = -\int_{t^\nu}^{t} d\tau \int dx e^{-\frac{i}{2}t(x-r)^2 + i\tau\delta x^2} \widetilde{E} \iint e^{i\tau\Lambda} \Phi'_{\xi_1}(\tau, \boldsymbol{\xi})\, y\mathcal{Y} dy dz$$

and

$$I_4 = \int_{t^\nu}^{t} d\tau \int dx e^{-\frac{i}{2}t(x-r)^2 + i\tau\delta x^2} \widetilde{E} \iint e^{i\tau\Lambda} \Phi'_{\xi_2}(\tau, \boldsymbol{\xi})\, y\mathcal{Y} dy dz,$$

where $\widetilde{E} = \widetilde{E}(t,\tau,x) = E(t,x)\overline{E(\tau,x)}$, $\Psi = C\widetilde{E}t\left(\Phi'_{\xi_1}(\tau,\boldsymbol{\xi}) + \Phi'_{\xi_2}(\tau,\boldsymbol{\xi})\right) \times zy^2\mathcal{Y} + C\widetilde{E}t^2\left(\Phi(\tau,\xi_1,\xi_2,\xi_3) - \Phi(\tau,x,x,\xi_3)\right)^2 y^2 z^2 \mathcal{Y}\mathcal{Z}$, $\Omega = C\widetilde{E}t\Phi'_{\xi_3}(\tau,\boldsymbol{\xi}) \times zy^2\mathcal{Y} + C\widetilde{E}t^2\left(\Phi(\tau,x,x,\xi_3) - \Phi(\tau,\boldsymbol{\xi}_0)\right) y^2 z^2 \mathcal{Y}\mathcal{Z}$.

Note that

$$|\Psi| \le CM\tau^{6\lambda} \frac{(\{\xi_1\} + \{\xi_2\} + \{\xi_3\})}{\langle\xi_1\rangle\langle\xi_2\rangle\langle\xi_3\rangle} (|y| + |z|) \left(|\xi_1|^{\gamma-1} + |\xi_2|^{\gamma-1}\right).$$

Let us first consider the integral I_1. We make a change of the variable of integration $z = x - \zeta$, then we get

$$I_1 = \int_{t^\nu}^{t} d\tau \iint dy d\zeta Z\mathcal{Y} \int e^{it\mathcal{Q}} \Psi dx,$$

where $\xi_1 = y + \zeta$, $\xi_2 = -y + \zeta$, $\xi_3 = (1+\sigma)x - \sigma\zeta$, $\mathcal{Q} = \mu(x-q)^2 - \mu q^2 - \frac{1}{2}r^2 + \frac{\tau}{t}\left(\alpha y^2 + \beta\zeta^2\right)$, $q = -\frac{1}{2\mu}r + \frac{\tau}{t}\frac{\beta}{\mu}\zeta$, $\mu = -\frac{1}{2} + \tilde{\delta}\frac{\tau}{t}$, $\tilde{\delta} = \delta + \beta$, $Z = \left(1 + 2i\beta\tau(x-\zeta)^2\right)^{-1}$. Now we can integrate by parts with respect to x via the identity (2.5). We get

$$I_1 = C\int_{t^\nu}^{t} d\tau \iint dy d\zeta \mathcal{Y} \int e^{it\mathcal{Q}} (XZ\Psi)'_x (x-q)\, dx.$$

Since

$$\begin{aligned} |\Psi'_x| \le\ & CM\tau^{6\lambda} \frac{(\{\xi_1\} + \{\xi_2\} + \{\xi_3\})}{\langle\xi_1\rangle\langle\xi_2\rangle\langle\xi_3\rangle} \left(|\xi_1|^{\gamma-1} + |\xi_2|^{\gamma-1}\right) \\ & \times \left(1 + \left(|\xi_3|^{\gamma-1} + \sqrt{\tau}\right)(|y| + |z|)\right), \end{aligned}$$

we have

$$
\begin{aligned}
& \left|(x-q)\,\mathcal{Y}\,(EXZ\Psi)'_x\right| \\
\le\ & CM\tau^{6\lambda}\frac{|x-q|\left(\{\xi_1\}+\{\xi_2\}+\{\xi_3\}\right)}{\langle\xi_1\rangle\langle\xi_2\rangle\langle\xi_3\rangle\langle\tau y^2\rangle\langle\tau z^2\rangle\left\langle t\mu\,(x-q)^2\right\rangle}\Big(|\xi_1|^{\gamma-1} \\
& +|\xi_2|^{\gamma-1}\Big)\left(1+\left(|\xi_3|^{\gamma-1}+\sqrt{\tau}+\sqrt{|t\mu|}\right)(|y|+|z|)\right) \\
\le\ & \frac{CM}{\tau^{3/2-7\lambda}\sqrt{|t\mu|}\,\langle\xi_1\rangle\langle\xi_2\rangle\langle\xi_3\rangle}\Big(\left(|\xi_1|^{\gamma-1}+|\xi_2|^{\gamma-1}\right)\left(|y|^{\gamma-1}+|z|^{\gamma-1}\right) \\
& +|y|^{\gamma-1}|z|^{\gamma-1}\Big)\left(|\xi_3|^{\gamma-1}+\left(\sqrt{\tau}+\sqrt{|t\mu|}\right)|t\mu|^{\frac{\gamma-1}{2}}|x-q|^{\gamma-1}\right).
\end{aligned}
$$

Hence we obtain

$$
\begin{aligned}
|I_1|\ \le\ & CM\int_{t^\nu}^{t}\frac{d\tau}{\tau^{3/2-7\lambda}\left|-\frac{t}{2}+(\delta+\beta)\,\tau\right|^{\frac12}} \\
& \times\iiint dydzdx\frac{\left(|\xi_1|^{\gamma-1}+|\xi_2|^{\gamma-1}\right)\left(|y|^{\gamma-1}+|z|^{\gamma-1}\right)+|y|^{\gamma-1}|z|^{\gamma-1}}{\langle\xi_1\rangle\langle\xi_2\rangle\langle\xi_3\rangle} \\
& \times\left(|\xi_3|^{\gamma-1}+\left(\sqrt{\tau}\left|-\frac{t}{2}+\tilde{\delta}\tau\right|^{\frac{\gamma-1}{2}}+\left|-\frac{t}{2}+\tilde{\delta}\tau\right|^{\frac{\gamma}{2}}\right)|x-q|^{\gamma-1}\right) \\
\le\ & CMt^{-1/2-\lambda}.
\end{aligned}
$$

Analogously, in the integral I_2 we make a change $z=\frac{x-\zeta}{\sigma}$, then we get

$$
I_2=\int_{t^\nu}^{t}d\tau\iint dyd\zeta Z\mathcal{Y}\int e^{it\mathcal{Q}}\Omega dx,
$$

where $\xi_1=\frac{1+\sigma}{\sigma}x+y-\frac{1}{\sigma}\zeta$, $\frac{1+\sigma}{\sigma}x-y-\frac{1}{\sigma}\zeta$, $\xi_3=-\zeta$, $\mathcal{Q}=\mu\,(x-q)^2-\mu q^2-\frac12 r^2+\frac{\tau}{t}\left(\alpha y^2+\tilde{\beta}\zeta^2\right)$, $q=-\frac{1}{2\mu}r+\frac{\tau}{t}\frac{\tilde{\beta}}{\mu}\zeta$, $\mu=-\frac12+\tilde{\delta}\frac{\tau}{t}$, $\tilde{\delta}=\delta+\tilde{\beta}$, $\tilde{\beta}=\frac{\beta}{\sigma^2}$, $Z=\left(1+2i\tilde{\beta}\tau\,(x-\zeta)^2\right)^{-1}$. Then we integrate by parts with respect to x via the identity (2.5) to get the estimate

$$
|I_2|\le Cgt^{-1/2-\lambda}.
$$

Now let us estimate the integral $I_{3.}$ For the case $\alpha+\beta\neq0$ we make a change of variables of integration $y=\beta\zeta-\eta$ and $z=-\alpha\zeta-\eta$ to obtain

$$
I_3=C\int_{t^\nu}^{t}d\tau\int dxe^{-\frac{i}{2}t(x-r)^2+i\tau\delta x^2}\widetilde{E}\iint e^{i\tau\mathcal{W}}\Phi'_{\xi_1}(\boldsymbol{\xi})\,(\beta\zeta-\eta)Y d\zeta d\eta,
$$

where $\mathcal{W} = \alpha\beta(\alpha+\beta)\zeta^2 + (\alpha+\beta)\eta^2$, $Y = \left(1+2i\alpha\tau(\beta\zeta-\eta)^2\right)^{-1}$, $\xi_1 = x+(\alpha+\beta)\zeta$, $\xi_2 = x+(\alpha-\beta)\zeta+2\eta$, $\xi_3 = x-\sigma\alpha\zeta-\sigma\eta$. We now integrate by parts with respect to η via the identity (2.4) to obtain $I_3 = I_5 + I_6$, where

$$I_5 = C\int_{t^\nu}^t d\tau \int dx e^{-\frac{i}{2}t(x-r)^2+i\tau\delta x^2}\widetilde{E}\iint e^{i\tau\mathcal{W}}\Big(2\Phi''_{\xi_1\xi_2}(\tau,\boldsymbol{\xi}) \\ -\sigma\Phi''_{\xi_1\xi_3}(\tau,\boldsymbol{\xi})\Big)\eta\,(\beta\zeta-\eta)\,YHd\eta d\zeta$$

$$I_6 = C\int_{t^\nu}^t d\tau \int dx e^{-\frac{i}{2}t(x-r)^2+i\tau\delta x^2}\widetilde{E}\iint e^{i\tau\mathcal{W}}\mathcal{K}\Phi'_{\xi_1}(\tau,\boldsymbol{\xi})\,YHd\eta d\zeta,$$

where $H = \left(1+2i\tau(\alpha+\beta)\eta^2\right)^{-1}$, $\mathcal{K} = -\eta - 4i(\alpha+\beta)\tau\eta^2(\beta\zeta-\eta)H - 4i\alpha\tau\eta(\beta\zeta-\eta)^2Y$. Hence we have

$$\begin{aligned}|I_5| &\le CM\int_{t^\nu}^t d\tau\tau^{8\lambda-2} \\ &\quad\times\iiint dyd\eta dx\frac{\left(|\xi_2|^{\gamma-1}+|\xi_3|^{\gamma-1}\right)(\{\xi_1\}+\{\xi_2\}+\{\xi_3\})}{|y|^{1-\gamma}|\eta|^{1-\gamma}|\xi_1|^{1-\gamma}\langle\xi_1\rangle\langle\xi_2\rangle\langle\xi_3\rangle} \\ &\le CMt^{-1/2-\lambda}.\end{aligned}$$

In the integral I_6 we change the variable $x+(\alpha+\beta)\zeta=\theta$, then $\xi_1=\theta$, $\xi_2 = \frac{2\beta}{\alpha+\beta}x+\frac{\alpha-\beta}{\alpha+\beta}\theta+2\eta$, $\xi_3 = \frac{(1+\sigma)\alpha+\beta}{\alpha+\beta}x-\frac{\sigma\alpha}{\alpha+\beta}\theta-\sigma\eta$, and we get

$$I_6 = C\int_{t^\nu}^t d\tau\iint d\eta d\theta\int dx e^{it\mathcal{Q}}H\widetilde{E}\mathcal{K}\Phi'_{\xi_1}(\boldsymbol{\xi})\,Y,$$

where $Y = \left(1+2i\tau\alpha\left(\frac{\beta}{\alpha+\beta}\theta-\frac{\beta}{\alpha+\beta}x-\eta\right)^2\right)^{-1}$, $\mathcal{Q} = \mu(x-q)^2-\mu q^2-\frac{1}{2}r^2+\frac{\tau}{t}\left(\tilde{\alpha}\eta^2+\tilde{\beta}\theta^2\right)$, $q = -\frac{1}{2\mu}r+\frac{\tau}{t}\frac{\tilde{\beta}}{\mu}\theta$, $\mu = -\frac{1}{2}+\tilde{\delta}\frac{\tau}{t}$, $\tilde{\alpha}=\alpha+\beta$, $\tilde{\beta}=\frac{\alpha\beta}{\alpha+\beta}$, $\tilde{\delta}=\delta+\tilde{\beta}$. As above we integrate by parts with respect to x via the identity (2.5) to get

$$I_6 = C\int_{t^\nu}^t d\tau\iint d\eta d\theta H\int dx e^{it\mathcal{Q}}(x-q)\left(\mathcal{K}\widetilde{E}HXY\Phi'_{\xi_1}\right)'_x.$$

Since $|H_x'| \leq C\frac{\sqrt{\tau}}{\langle\tau\eta^2\rangle}$, $|X_x'| \leq C\frac{\sqrt{|t\mu|}}{\langle t\mu(x-q)^2\rangle}$, $|Y_x'| \leq C\frac{\sqrt{\tau}}{\langle\tau y^2\rangle}$ we get

$$\begin{aligned}
&\left|(x-q)\left(\mathcal{K}\widetilde{E}HXY\Phi'_{\xi_1}\right)'_x\right| \\
\leq\ & \frac{CM\tau^{6\lambda}|x-q|\left(\{\xi_1\}+\{\xi_2\}+\{\xi_3\}\right)}{\langle\xi_1\rangle\langle\xi_2\rangle\langle\xi_3\rangle\langle\tau\eta^2\rangle\left\langle t\mu(x-q)^2\right\rangle\langle\tau y^2\rangle|\xi_1|^{1-\lambda}} \\
&\times\left(1+(|y|+|\eta|)\left(\sqrt{\tau}+\sqrt{|t\mu|}+|\xi_2|^{\gamma-1}+|\xi_3|^{\gamma-1}\right)\right) \\
\leq\ & \frac{CM\tau^{7\lambda-3/2}}{\sqrt{|t\mu|}\langle\xi_1\rangle\langle\xi_2\rangle\langle\xi_3\rangle}\left(|\xi_1|^{\gamma-1}\left(|y|^{\gamma-1}+|\eta|^{\gamma-1}\right)+|y|^{\gamma-1}|\eta|^{\gamma-1}\right) \\
&\times\left(|\xi_2|^{\gamma-1}+|\xi_3|^{\gamma-1}+\left(\sqrt{\tau}+\sqrt{|t\mu|}\right)|t\mu|^{\frac{\gamma-1}{2}}|x-q|^{\gamma-1}\right).
\end{aligned}$$

Whence we get

$$\begin{aligned}
|I_6| \ \leq\ & CM\int_{t^\nu}^{t}\frac{d\tau}{\tau^{3/2-7\lambda}\left|-\frac{t}{2}+\tilde{\delta}\tau\right|^{\frac{1}{2}}} \\
&\times\iiint dyd\eta dx\frac{\left(\left(|y|^{\gamma-1}+|\eta|^{\gamma-1}\right)|\xi_1|^{\gamma-1}+|y|^{\gamma-1}|\eta|^{\gamma-1}\right)}{\langle\xi_1\rangle\langle\xi_2\rangle\langle\xi_3\rangle}\left(|\xi_2|^{\gamma-1}\right. \\
&\left.+|\xi_3|^{\gamma-1}+\left(\sqrt{\tau}\left|-\frac{t}{2}+\tilde{\delta}\tau\right|^{\frac{\gamma-1}{2}}+\left|-\frac{t}{2}+\tilde{\delta}\tau\right|^{\frac{\gamma}{2}}\right)|x-q|^{\gamma-1}\right) \\
\leq\ & CMt^{-1/2-\lambda}.
\end{aligned}$$

In the case $\alpha+\beta=0$ we make the change of the independent variables $y=\eta+\zeta$ and $z=\eta-\zeta$ to get $I_3=I_7+I_8$,

$$\begin{aligned}
I_7 = C&\int_{t^\nu}^{t}d\tau\int dx e^{-\frac{i}{2}t(x-r)^2+i\tau\delta x^2}\widetilde{E}\iint e^{4i\alpha\tau\zeta\eta}\left(1-\chi\left(\zeta\tau^\rho\right)\right) \\
&\times\Phi'_{\xi_1}(\tau,\boldsymbol{\xi})(\zeta+\eta)\mathfrak{Y}d\eta d\zeta,
\end{aligned}$$

$$\begin{aligned}
I_8 = C&\int_{t^\nu}^{t}d\tau\int dx e^{-\frac{i}{2}t(x-r)^2+i\tau\delta x^2}\widetilde{E} \\
&\times\iint e^{4i\alpha\tau\zeta\eta}\chi\left(\zeta\tau^\rho\right)\Phi'_{\xi_1}(\tau,\boldsymbol{\xi})(\zeta+\eta)\mathfrak{Y}d\eta d\zeta,
\end{aligned}$$

where $\chi(\zeta)\in\mathbf{C}^1(\mathbf{R})$, $\chi(\zeta)=1$ for $|\zeta|\geq 1$ and $\chi(\zeta)=0$ for $|\zeta|\leq\frac{1}{2}$, $\mathfrak{Y}=\left(1+2i\alpha t(\zeta+\eta)^2\right)^{-1}$, $\xi_1=x+2\zeta$, $\xi_2=x-2\eta$, $\xi_3=x-\sigma\zeta+\sigma\eta$,

$\rho = 1 - 11\lambda$. The integral I_7 is taken over the domain $|\zeta|\tau^{\rho} \leq 1$, so it is estimated as

$$|I_7| \leq CM \int_{t^{\nu}}^{t} d\tau \tau^{6\lambda} \iiint \frac{(1-\chi(\zeta\tau^{\rho}))\,|y|\,dxd\zeta dy}{|\xi_1|^{1-\gamma}\langle\xi_1\rangle^2\langle\xi_2\rangle^2\langle\xi_3\rangle^2\langle\tau y^2\rangle} \leq CMt^{-1/2-\lambda},$$

and the integral I_8 is taken over the domain $|\zeta|\tau^{\rho} \geq 1$, therefore we can integrate by parts with respect to η to obtain $I_8 = I_9 + I_{10}$, where

$$I_9 = C\int_{t^{\nu}}^{t} \frac{d\tau}{\tau} \int dx e^{-\frac{i}{2}t(x-r)^2 + i\tau\delta x^2} \widetilde{E} \iint d\eta d\zeta e^{4i\alpha\tau\zeta\eta} \zeta^{-1} \mathfrak{Y}$$
$$\times \left(2\Phi''_{\xi_1\xi_2}(\tau,\boldsymbol{\xi}) - \sigma\Phi''_{\xi_1\xi_3}(\tau,\boldsymbol{\xi})\right)\chi(\zeta\tau^{\rho})(\zeta+\eta),$$

$$I_{10} = C\int_{t^{\nu}}^{t} \frac{d\tau}{\tau} \int dx e^{-\frac{i}{2}t(x-r)^2 + i\tau\delta x^2} \iint d\eta d\zeta e^{4i\alpha\tau\zeta\eta} \zeta^{-1} \mathfrak{Y}\mathcal{M}\Phi'_{\xi_1}(\tau,\boldsymbol{\xi}),$$

here $\mathcal{M} = \left(1 - 4i\alpha\tau(\zeta+\eta)^2\mathfrak{Y}\right)\widetilde{E}\chi(\zeta\tau^{\rho})$. We have the estimate

$$\begin{aligned} &|I_9| \\ &\leq CM\int_{t^{\nu}}^{t} \frac{d\tau}{\tau^{1-7\lambda}} \iiint dxd\zeta dy\,|y|\,\frac{(\{\xi_1\}+\{\xi_2\}+\{\xi_3\})\left(|\xi_2|^{\gamma-1}+|\xi_3|^{\gamma-1}\right)}{\langle\xi_1\rangle\langle\xi_2\rangle\langle\xi_3\rangle|\zeta|^{1-\gamma}|\xi_1|^{1-\gamma}\langle\tau y^2\rangle} \\ &\leq CMt^{-1/2-\lambda}. \end{aligned}$$

In the integral I_{10} we make the change of the variable $x + 2\zeta = \theta$ then $\xi_1 = \theta$, $\xi_2 = x - 2\eta$, $\xi_3 = \left(1+\frac{\sigma}{2}\right)x - \frac{\sigma}{2}\theta - \sigma\eta$, and

$$I_{10} = C\int_{t^{\nu}}^{t} \frac{d\tau}{\tau} \iint d\eta d\theta \int dx e^{it\mathcal{Q}} \frac{\tilde{\mathfrak{Y}}\mathcal{M}}{\theta - x}\Phi'_{\xi_1}(\tau,\boldsymbol{\xi}),$$

where $\tilde{\mathfrak{Y}} = \left(1 + i\frac{\alpha}{2}\tau(\theta - x + 2\eta)^2\right)^{-1}$, $\mathcal{Q} = \mu(x-\theta)^2 - \mu q^2 - \frac{1}{2}r^2 - 2\alpha\frac{\tau}{t}\theta\eta$, $q = -\frac{1}{2\mu}r + \frac{\alpha\tau}{\mu t}\eta$, $\mu = -\frac{1}{2} + \delta\frac{\tau}{t}$. Via identity (2.5) we obtain

$$I_{10} = C\int_{t^{\nu}}^{t} \frac{d\tau}{\tau} \iiint d\eta d\theta dx e^{it\mathcal{Q}}(x-q)\left(\frac{\tilde{\mathfrak{Y}}\mathcal{M}X}{\theta - x}\Phi'_{\xi_1}(\tau,\boldsymbol{\xi})\right)'_x.$$

Since

$$
\begin{aligned}
& \left|(x-q)\left(\frac{\tilde{\mathfrak{Y}}\mathcal{M}X}{\theta-x}\Phi'_{\xi_1}(\tau,\boldsymbol{\xi})\right)'_x\right| \\
\leq\ & \frac{Cg\tau^{7\lambda}|x-q|}{\langle\xi_1\rangle\langle\xi_2\rangle\langle\xi_3\rangle\left\langle t\mu(x-q)^2\right\rangle\langle\tau y^2\rangle|\zeta|^{1-\gamma}|\xi_1|^{1-\gamma}} \\
& \times\left(1+\left(\tau^{\lambda+\rho}+\sqrt{|t\mu|}+|\xi_2|^{\gamma-1}+|\xi_3|^{\gamma-1}\right)(\{\xi_1\}+\{\xi_2\}+\{\xi_3\})\right) \\
\leq\ & \frac{Cg\tau^{7\lambda}\left(\tau^{2\lambda+\rho-1/2}+\tau^{\lambda-1/2}\sqrt{|t\mu|}\right)}{\langle\xi_1\rangle\langle\xi_2\rangle\langle\xi_3\rangle|t\mu|^{1-\gamma/2}|x-q|^{1-\gamma}}\left(|\xi_1\xi_2|^{1-\gamma}+|\xi_1\xi_3|^{1-\gamma}\right. \\
& \left.+|\xi_1\zeta|^{1-\gamma}+|\xi_2\zeta|^{1-\gamma}+|\xi_3\zeta|^{1-\gamma}+|\xi_1 y|^{1-\gamma}+|\zeta y|^{1-\gamma}\right).
\end{aligned}
$$

Thus we obtain

$$
\begin{aligned}
|I_{10}| \leq\ & CM\int_{t^\nu}^{t}\frac{d\tau\left(\tau^{2\lambda+\rho-1/2}+\tau^{\lambda-1/2}\sqrt{-\frac{t}{2}+\tau\delta}\right)}{\tau^{1-7\lambda}\left|-\frac{t}{2}+\tau\delta\right|^{1-\lambda/2}} \\
& \times\iiint dyd\zeta dx\frac{|x-q|^{\gamma-1}}{\langle\xi_1\rangle\langle\xi_2\rangle\langle\xi_3\rangle}\left(|\xi_1\xi_2|^{1-\gamma}+|\xi_1\xi_3|^{1-\gamma}\right. \\
& \left.+|\xi_1\zeta|^{1-\gamma}+|\xi_2\zeta|^{1-\gamma}+|\xi_3\zeta|^{1-\gamma}+|\xi_1 y|^{1-\gamma}+|\zeta y|^{1-\gamma}\right) \\
\leq\ & CM\int_{t^\nu}^{t}d\tau\left(\tau^{9\lambda+\rho-3/2}\left|-\frac{t}{2}+\tau\delta\right|^{\lambda/2-1}+\tau^{8\lambda-3/2}\left|-\frac{t}{2}+\tau\delta\right|^{\lambda-1/2}\right) \\
\leq\ & CMt^{-1/2-\lambda}.
\end{aligned}
$$

Whence the result follows. Lemma 2.6 is proved. ∎

§3. Linear smoothing effect

In this section we represent the smoothing property of Doi [2] for the solutions of the Cauchy problem for the linear Schrödinger equations

$$
\begin{cases} iu_t+\frac{1}{2}u_{xx}=f, & x\in\mathbf{R},\quad t\in\mathbf{R}, \\ u(0,x)=u_0(x), & x\in\mathbf{R}, \end{cases} \tag{3.1}
$$

where the function $f(t,x)$ is a force.

The Hilbert transformation with respect to the variable x is defined as follows $\mathcal{H}\phi(x)=\frac{1}{\pi}\mathrm{PV}\int_{\mathbf{R}}\frac{\phi(z)}{x-z}dz=-i\mathcal{F}^{-1}\frac{\xi}{|\xi|}\mathcal{F}\phi$, where PV means the principal value of the singular integral. We widely use the fact that the Hilbert transformation $\mathcal{H}$ is a bounded operator from $\mathbf{L}^2$ to $\mathbf{L}^2$. The fractional derivative $|\partial|^\alpha$,

$\alpha \in (0,1)$ is equal to $|\partial|^\alpha \phi = \mathcal{F}^{-1}|\xi|^\alpha \mathcal{F}\phi = C\int_{\mathbf{R}}(\phi(x+z)-\phi(x))\frac{dz}{|z|^{1+\alpha}}$ and similarly we have $|\partial|^\alpha \mathcal{H}\phi = -i\mathcal{F}^{-1}\text{sign}\xi|\xi|^\alpha \mathcal{F}\phi = C\int_{\mathbf{R}}(\phi(x+z)-\phi(x))\frac{dz}{z|z|^{\alpha}}$, with some constants C. The next lemma shows that the commutators $[|\partial|^\alpha, \phi]$, and $[|\partial|^\alpha \mathcal{H}, \phi]$ are continuous operators from $\mathbf{L}^2$ to $\mathbf{L}^2$.

Lemma 3.1. *The following inequalities*

$$\|[|\partial_x|^\alpha, \phi]\,\psi\| \le C\,\|\phi\|_{1,0,\infty}\,\|\psi\| \quad \textit{and} \quad \|[|\partial_x|^\alpha \mathcal{H}, \phi]\,\psi\| \le C\,\|\phi\|_{1,0,\infty}\,\|\psi\|$$

are valid, provided that the right hand sides are bounded.

We define the operator $\mathcal{S}(\varphi) = \cosh(\varphi) + i\sinh(\varphi)\mathcal{H}$, where the real-valued function $\varphi(t,x) \in \mathbf{L}^\infty\left(0,T;\mathbf{H}^{2,0}_\infty\right) \cap \mathbf{C}^1([0,T];\mathbf{L}^\infty)$ and is positive. From its definition we easily see that the operator $\mathcal{S}$ acts continuously from $\mathbf{L}^2$ to $\mathbf{L}^2$ with the following estimate $\|\mathcal{S}(\varphi)\psi\| \le 2\|\psi\|\exp\|\varphi\|_\infty$. Since $\|\tanh(\varphi)\psi\| \le \|\psi\|\tanh\|\varphi\|_\infty < \|\psi\|$ the inverse operator $\mathcal{S}^{-1}(\varphi) = (1+i\tanh(\varphi)\mathcal{H})^{-1}\frac{1}{\cosh(\varphi)}$ also exists and is continuous $\|\mathcal{S}^{-1}(\varphi)\psi\| \le (1-\tanh\|\varphi\|_\infty)^{-1}\|\psi\| \le \|\psi\|\exp\|\varphi\|_\infty$. The operator $\mathcal{S}$ helps us to obtain a smoothing property of the Schrödinger - type equation (3.1) by virtue of the usual energy estimates. In the next lemma we present an energy estimate, involving the operator $\mathcal{S}$, in which we have an additional positive term giving us the norm of the half derivative of the unknown function u. We also assume that $\varphi(x)$ is written as $\varphi(x) = \partial_x^{-1}(w^2)$, so that $w(x) = \sqrt{(\partial_x\varphi)}$.

Lemma 3.2. *The following inequality*

$$\frac{d}{dt}\|\mathcal{S}u\|^2 + \frac{1}{2}\left\|w\mathcal{S}\sqrt{|\partial_x|}u\right\|^2 \le 2\,|\Im(\mathcal{S}u,\mathcal{S}f)| + C\|u\|^2 e^{2\|\varphi\|_\infty}\left(\|w\|^4_\infty + \|w\|^2_{1,0,\infty} + \|\varphi_t\|_\infty\right)$$

is valid for the solution u of the Cauchy problem (3.1).

In the next lemma we give the estimate for the nonlinearity.

Lemma 3.3. *We have the following estimates*

$$\begin{aligned}
&|(\mathcal{S}u, \mathcal{S}\phi\psi\partial_x v)| \\
&\quad\le \left\||\phi|\mathcal{S}\sqrt{|\partial_x|}u\right\|^2 + \left\||\psi|\mathcal{S}\sqrt{|\partial_x|}v\right\|^2 \\
&\qquad + C\|u\|\|v\|\exp(6\|\varphi\|_\infty)\left(\|\phi\|^2_{1,0,\infty} + \|\psi\|^2_{1,0,\infty}\right)\left(1+\|\varphi\|^2_{1,0,\infty}\right),
\end{aligned}$$

and

$$\begin{aligned}
&|(\mathcal{S}u, \mathcal{S}\phi\psi\partial_x \bar{v})| \\
&\quad\le \left\||\phi|\mathcal{S}\sqrt{|\partial_x|}u\right\|^2 + \exp(2\|\varphi\|_\infty)\left\||\psi|\mathcal{S}\sqrt{|\partial_x|}v\right\|^2 \\
&\qquad + C\|u\|\|v\|\exp(6\|\varphi\|_\infty)\left(\|\phi\|^2_{1,0,\infty} + \|\psi\|^2_{1,0,\infty}\right)\left(1+\|\varphi\|^2_{1,0,\infty}\right),
\end{aligned}$$

provided that the right hand sides are bounded.

For the proofs of Lemmas 3.1 - 3.3, see [8].

§4. Proof of Theorem 1.1

By virtue of the method of papers [1], [3], [11], [16] (see also the proof of a-priori estimates below in Lemma 4.2) we easily obtain the existence of local solutions in the functional space

$$\mathbf{F}=\left\{\phi\in\mathbf{L}^{\infty}\left((-T,T),\mathbf{H}^{3,3}\right):\|\phi\|_{\mathbf{X}}+\|\phi\|_{\mathbf{Y}}+\|\phi\|_{\mathbf{Z}}<\infty\right\},$$

where the norms $\mathbf{X},\mathbf{Y}$ and $\mathbf{Z}$ are defined in the Introduction, for the convenience of the reader here we repeat their definition

$$\begin{aligned}\|u\|_{\mathbf{X}} &= \|u\|_{1,0}+\sqrt{\langle t\rangle}\,\|u\|_{3,0,\infty}+\langle t\rangle^{1/2-2\gamma}\,\|\mathcal{I}u\|_{1,0,\infty},\\ \|u\|_{\mathbf{Y}} &= \sum_{j=0}^{3}\langle t\rangle^{-\gamma-3\gamma j}\left\|\mathcal{I}^{j}u\right\|_{3-j,0}\end{aligned}$$

and

$$\begin{aligned}\|u\|_{\mathbf{Z}} = &\sum_{k=0}^{2}\langle t\rangle^{-2}\left\|\mathcal{P}^{k}v\right\|_{1,2-k,\infty}+\sum_{k=0}^{3}\langle t\rangle^{-1-\lambda_k}\left\|\left(\mathcal{P}^{k}v\right)_t\right\|_{0,3-k,\infty}\\ &+\sum_{k=0}^{3}\langle t\rangle^{-\lambda_k}\left\|\mathcal{P}^{k}v\right\|_{0,3-k,\infty},\end{aligned}$$

where $v(t)=\mathcal{FU}(-t)\,u(t),\lambda_0=0,\ \lambda_1=2\gamma,\ \lambda_2=40\gamma,\lambda_3=20\gamma+\frac{1}{2},\ \gamma\in\left(0,10^{-4}\right).$

Theorem 4.1. *Let the initial data $u_0\in\mathbf{H}^{3,4}$. Then for some time $T>0$ there exists a unique solution $u\in\mathbf{F}$ of the Cauchy problem (1.1). If we assume in addition that the norm of the initial data $\|u_0\|_{3,4}=\varepsilon$ is sufficiently small, then there exists a unique solution $u\in\mathbf{F}$ of (1.1) on a finite time interval $[0,T]$ with $T>1/\varepsilon$, such that the following estimates $\sup_{t\in[0,T]}\|u\|_{\mathbf{X}}<\sqrt{\varepsilon}$ and $\sup_{t\in[0,T]}(\|u\|_{\mathbf{Y}}+\|u\|_{\mathbf{Z}})<\varepsilon^{3/4}$ are valid.*

In the next lemma we obtain the optimal time decay estimate $\|u(t)\|_{3,0,\infty}\leq C\langle t\rangle^{-1/2}$ of global solutions to the Cauchy problem (1.1) and the a-priori estimate of solutions in the norms $\mathbf{Y}$ and $\mathbf{Z}$.

Lemma 4.2. *Let the initial data $u_0\in\mathbf{H}^{3,4}$ and the norm $\|u_0\|_{3,4}=\varepsilon$ be sufficiently small. Then there exists a unique global solution of the Cauchy problem (1.1) such that $u\in\mathbf{C}\left(\mathbf{R};\mathbf{H}^{3,3}\right)$ and $\mathcal{I}^{k}u\in\mathbf{L}^{\infty}_{loc}\left(\mathbf{R};\mathbf{H}^{3-k,0}\right)$, $0\leq k\leq 3$. Moreover the following estimates are valid*

$$\sup_{t>0}\|u\|_{\mathbf{X}}<\sqrt{\varepsilon}\ \text{and}\ \sup_{t>0}\left(\|u\|_{\mathbf{Y}}+\|u\|_{\mathbf{Z}}\right)<\varepsilon^{3/4}. \tag{4.1}$$

Proof. Applying the result of Theorem 3.1 and using a standard continuation argument we can find a maximal time $T > 1$ such that the inequalities

$$\|u\|_{\mathbf{X}} < 2\sqrt{\varepsilon} \text{ and } (\|u\|_{\mathbf{Y}} + \|u\|_{\mathbf{Z}}) < 2\varepsilon^{3/4} \tag{4.2}$$

are true for all $t \in [0, T]$. If we prove (4.1) on the whole time interval $[0, T]$, then by the contradiction argument we obtain the desired result of the lemma. In view of the local existence Theorem 3.1 it is sufficient to consider only estimates of the solution for the time interval $t \geq 1$. Note that from (4.2) we get the estimate:

$$\sum_{k=0}^{2} \langle t \rangle^{-2\gamma-\lambda_k} \left\| \langle x \rangle^{2-k} \mathcal{P}^k v \right\|_{\mathbf{B}} \leq C\varepsilon^{3/4}. \tag{4.3}$$

Let us start with the norm $\mathbf{Y}$. Differentiating three times equation (1.1) we get $\mathcal{L}u_{xxx} = \mathcal{N}_{u_x} u_{xxxx} + \mathcal{N}_{\bar{u}_x} \bar{u}_{xxxx} + R_0$, where $\mathcal{L} = i\partial_t + \frac{1}{2}\partial_x^2$ and in view of (4.2) the remainder term R_0 is estimated as $\|R_0\| \leq C \|u\|_{2,0,\infty}^2 \|u\|_{3,0} \leq C\varepsilon^{7/4} \langle t \rangle^{\gamma-1}$. Applying the operator $\mathcal{I}$ to both sides of equation (1.1) and using the commutator relation $\mathcal{L}\mathcal{I}^k = (\mathcal{I}+2)^k \mathcal{L}$ we find $\mathcal{L}\partial_x^{3-k}\mathcal{I}^k u = \mathcal{N}_{u_x}\partial_x^{4-k}\mathcal{I}^k u + \mathcal{N}_{\overline{u}_x}\partial_x^{4-k}\mathcal{I}^k \overline{u} + R_k$, $k = 1, 2, 3$, where by virtue of (4.2) we get

$$\begin{aligned}
\|R_1\| &\leq C \|u\|_{3,0,\infty}^2 \left(\|u\|_{3,0} + \|\mathcal{I}u\|_{2,0} \right) \leq C\varepsilon^{7/2} \langle t \rangle^{4\gamma-1}, \\
\|R_2\| &\leq C \|u\|_{2,0,\infty}^2 \left(\|u\|_{2,0} + \|\mathcal{I}u\|_{2,0} + \left\|\mathcal{I}^2 u\right\|_{1,0} \right) \\
&\quad + \|u\|_{2,0,\infty} \|\mathcal{I}u\|_{1,0,\infty} \|\mathcal{I}u\|_{2,0} \\
&\leq C\varepsilon^{7/4} \langle t \rangle^{7\gamma-1}
\end{aligned}$$

and

$$\begin{aligned}
\|R_3\| &\leq C \|u\|_{2,0,\infty}^2 \left(\|u\|_{2,0} + \|\mathcal{I}u\|_{2,0} + \left\|\mathcal{I}^2 u\right\|_{1,0} + \left\|\mathcal{I}^3 u\right\| \right) \\
&\quad + \|u\|_{2,0,\infty} \|\mathcal{I}u\|_{1,0,\infty} \left(\|\mathcal{I}u\|_{1,0} + \left\|\mathcal{I}^2 u\right\|_{1,0} \right) + \|\mathcal{I}u\|_{1,0,\infty}^2 \|\mathcal{I}u\|_{1,0} \\
&\leq C\varepsilon^{7/4} \langle t \rangle^{10\gamma-1}.
\end{aligned}$$

Thus we find

$$\mathcal{L}h_k = \mathcal{N}_{u_x}\partial h_k + \mathcal{N}_{\overline{u}_x}\partial \overline{h}_k + R_k, \tag{4.4}$$

where $h_k = \left(1 + \partial_x^{3-k}\right)\mathcal{I}^k u$ and $\|R_k\| \leq C\varepsilon^{7/4} \langle t \rangle^{\gamma+3k\gamma-1}$, $k = 0, 1, 2, 3$. We use the operator $\mathcal{S}(\varphi) = \cosh(\varphi) + i\sinh(\varphi)\mathcal{H}$, introduced in Section 3, where we take now $\varphi(t, x) = \frac{1}{\varepsilon}\partial_x^{-1}\left(|u(t,x)|^2 + |u_x(t,x)|^2\right)$ and as in Section 3 we define $w(t,x) = \frac{1}{\sqrt{\varepsilon}}\sqrt{|u(t,x)|^2 + |u_x(t,x)|^2}$. Then applying Lemma 3.2 we obtain the energy type inequality for the functions h_k (index k we omit)

$$
\begin{aligned}
(4.5)\quad & \frac{d}{dt}\|\mathcal{S}h\|^2 + \left\|w\mathcal{S}\sqrt{|\partial_x|}h\right\|^2 \\
&\quad \le 2\left|\Im\left(\mathcal{S}h, \mathcal{S}\left(\mathcal{N}_{u_x}h_x + \mathcal{N}_{\bar{u}_x}\bar{h}_x\right)\right)\right| + 2\left|\Im\left(\mathcal{S}h, \mathcal{S}R\right)\right| \\
&\qquad + Ce^{\|\varphi\|_\infty}\left(\|u\|_\infty^4 + \|u\|_{1,0,\infty}^2 + \|\varphi_t\|_\infty\right)\|h\|^2,
\end{aligned}
$$

where the functions R are bounded in $\mathbf{L}^2$. Hence we get via (4.2)

$$
(4.6)\qquad \left|\Im\left(\mathcal{S}h_k, \mathcal{S}R_k\right)\right| \le e^{2\|\varphi\|_\infty}\|h_k\|\|R_k\| \le C\varepsilon^{5/2}\langle t\rangle^{6k\gamma+2\gamma-1}.
$$

To estimate the first summand $\Im\left(\mathcal{S}h, \mathcal{S}\left(\mathcal{N}_{u_x}h_x + \mathcal{N}_{\bar{u}_x}\bar{h}_x\right)\right)$ in the left hand side of (4.4) we apply Lemma 3.3 to obtain

$$
\begin{aligned}
&(4.7)\\
&\left|\left(\mathcal{S}h_k, \mathcal{S}\left(\mathcal{N}_{u_x}h_{kx} + \mathcal{N}_{\bar{u}_x}\bar{h}_{kx}\right)\right)\right| \le C\sqrt{\varepsilon}\left\|w\mathcal{S}\sqrt{|\partial_x|}h_k\right\|^2 + C\varepsilon^{5/2}\langle t\rangle^{6k\gamma+\gamma-1}.
\end{aligned}
$$

Substitution of (4.6) and (4.7) into (4.5) yields

$$
\begin{aligned}
(4.8)\quad & \frac{d}{dt}\|\mathcal{S}h_k\|^2 + (1 - C\sqrt{\varepsilon})\left\|w\mathcal{S}\sqrt{|\partial_x|}h_k\right\|^2 \\
&\le C^2e^{2\|u\|_{1,0}^2}\left(\|u\|_{1,0,\infty}^4 + \|u\|_{2,0,\infty}^2 + \|\varphi_t\|_\infty\right)\|\mathcal{S}h_k\|^2 + C\varepsilon^{5/2}\langle t\rangle^{6k\gamma+2\gamma-1}.
\end{aligned}
$$

We also have

$$
\begin{aligned}
\|\varphi_t\|_\infty &= \left\|\partial_t\int_{-\infty}^x\left(\left|u(t,x')\right|^2 + \left|u_x(t,x')\right|^2\right)dx'\right\|_\infty \\
&= \left\|\int_{-\infty}^x\left(u_t\bar{u} + \bar{u}_t u + u_{xt}\bar{u}_x + \bar{u}_{xt}u_x\right)dx\right\|_\infty \\
&= \left\|\int_{-\infty}^x\big((u_{xx} - \mathcal{N})\bar{u} - (\bar{u}_{xx} - \overline{\mathcal{N}})u + \right. \\
&\qquad \left.(u_{xxx} - \mathcal{N}_x)\bar{u}_x - (\bar{u}_{xxx} - \overline{\mathcal{N}}_x)u_x\big)dx\right\|_\infty \\
&= \left\|u_x\bar{u} - \bar{u}_x u + u_{xx}\bar{u}_x - \bar{u}_{xx}u_x\right. \\
&\qquad \left. - \int_{-\infty}^x\left(\mathcal{N}\bar{u} - \overline{\mathcal{N}}u + \mathcal{N}_x\bar{u}_x - \overline{\mathcal{N}}_x u_x\right)dx\right\|_\infty \\
&\le C\|u\|_{2,0,\infty}^2\left(1 + \|u\|_{1,0}^2\right) \le C\varepsilon\langle t\rangle^{-1}.
\end{aligned}
$$

Therefore from (4.8) and the Gronwall inequality we get $\|h_k\| < C\varepsilon\langle t\rangle^{3k\gamma+\gamma}$. Thus we have the estimate

$$
(4.9)\qquad \|u(t)\|_{\mathbf{Y}} < \frac{1}{2}\varepsilon^{3/4}\langle t\rangle^{3k\gamma+\gamma}
$$

for all $t \in [0,T]$. To prove the estimate in the norm $\mathbf{Z}$ we represent the solution u of the Cauchy problem (3.1) in the form $u(t) = \mathcal{U}(t)\mathcal{F}^{-1}v(t)$. Applying the Fourier transformation to equation (1.1) and then changing the dependent variables $\hat{u} = e^{-\frac{it}{2}\xi^2}v$ we get

$$iv_t(t,x) = \sum_{j=1}^{4} e^{it\delta_j x^2} \iint e^{it\Lambda_j}\Phi_{0,j}\left(t,\boldsymbol{\xi}_j\right) dydz$$

where $\delta_1 = \frac{1}{3}$, $\delta_2 = 0$, $\delta_3 = 1$, $\delta_4 = \frac{2}{3}$, $\Lambda_j = \alpha_j y^2 + \beta_j z^2$, $(\alpha_1,\beta_1) = (-1,-3)$, $(\alpha_2,\beta_2) = (-1,1)$, $(\alpha_3,\beta_3) = (1,-1)$, $(\alpha_4,\beta_4) = (-1,-3)$, $\Phi_{0,j}(t,\boldsymbol{\xi}) = A_j(\boldsymbol{\xi})\prod_{k=1}^{3} v_{\sigma_{k,j}}(t,\sigma_{k,j}\xi_k)$, $A_j(\boldsymbol{\xi}) = \frac{1}{2\pi}\sum_{\omega\neq 0} a_{j,\boldsymbol{\omega}}\prod_{k=1}^{3}\xi_k^{\omega_k}$, $v_\sigma(t,\xi) = v(t,\xi)$ if $\sigma = 1$ and $v_\sigma(t,\xi) = \overline{v(t,\xi)}$ if $\sigma = -1$, $\boldsymbol{\xi} = (\xi_1,\xi_2,\xi_3)$, $\boldsymbol{\xi}_j = \left(\xi_{j,1},\xi_{j,2},\xi_{j,3}\right)$, $\xi_{1,1} = \frac{x}{3}+y-z$, $\xi_{1,2} = \frac{x}{3}-y-z$, $\xi_{1,3} = \frac{x}{3}+2z$, $\xi_{2,1} = x+y-z$, $\xi_{2,2} = x-y-z$, $\xi_{2,3} = x-2z$, $\xi_{3,1} = -x+y-z$, $\xi_{3,2} = -x-y-z$, $\xi_{3,3} = -x-2z$, $\xi_{4,1} = -\frac{x}{3}+y-z$, $\xi_{4,2} = -\frac{x}{3}-y-z$, $\xi_{4,3} = -\frac{x}{3}+2z$, $\boldsymbol{\sigma}_j = (\sigma_{1,j},\sigma_{2,j},\sigma_{3,j})$, $\boldsymbol{\sigma}_1 = (1,1,1)$, $\boldsymbol{\sigma}_2 = (1,1,-1)$, $\boldsymbol{\sigma}_3 = (-1,-1,1)$, $\boldsymbol{\sigma}_4 = (-1,-1,-1)$. Similarly we get

$$i(\mathcal{P}v)_t = 2iv_t + \sum_{j=1}^{4} e^{it\delta_j x^2}\iint e^{it\Lambda_j}\Phi_{1,j}\left(t,\boldsymbol{\xi}_j\right) dydz,$$

$$i\left(\mathcal{P}^2 v\right)_t = 4i(\mathcal{P}v)_t - 4iv_t + \sum_{j=1}^{4} e^{it\delta_j x^2}\iint e^{it\Lambda_j}\Phi_{2,j}\left(t,\boldsymbol{\xi}_j\right) dydz$$

and

$$i\left(\mathcal{P}^3 v\right)_t = 6i\left(\mathcal{P}^2 v\right)_t - 12i(\mathcal{P}v)_t + 8iv_t + \sum_{j=1}^{4} e^{it\delta_j x^2}\iint e^{it\Lambda_j}\Phi_{3,j}\left(t,\boldsymbol{\xi}_j\right) dydz,$$

where $\Phi_{k,j}(t,\boldsymbol{\xi}) = A_j(\boldsymbol{\xi})\sum_{|\boldsymbol{\omega}|=k}\frac{k!}{\omega_1!\omega_2!\omega_3!}\prod_{l=1}^{3}\mathcal{P}^{\omega_l}v_{\sigma_{l,j}}(t,\sigma_{l,j}\xi_l)$, $k = 1,2,3$, $\boldsymbol{\omega} = (\omega_1,\omega_2,\omega_3)$, $|\boldsymbol{\omega}| = \omega_1+\omega_2+\omega_3$.

By (4.3) and Lemma 2.2 with $\sigma_1 = 0$, $\sigma_2 = \sigma_3 = 2$ we get

$$\begin{aligned} &\left\|\iint e^{it\Lambda_1}A_1(\boldsymbol{\xi})\,\mathcal{P}^3 v(t,\xi_1)\,v(t,\xi_2)\,v(t,\xi_3)\,dydz\right\|_\infty \\ \leq\;& C\langle t\rangle^{\gamma-1/2}\left\|\mathcal{P}^3 v\right\|\left\|\langle x\rangle^2 v\right\|_{\mathbf{B}}^2 \leq C\langle t\rangle^{19\gamma-1/2}\|u\|_{\mathbf{Y}}\|u\|_{\mathbf{Z}}^2 \leq C\varepsilon^2\langle t\rangle^{\lambda_3-1} \end{aligned}$$

and by Lemma 2.1 with $\sigma_1 = \sigma_2 = 1$, $\sigma_3 = 2$ we get

$$\begin{aligned}
&\left\| \iint e^{it\Lambda_1} A_1(\boldsymbol{\xi}) \mathcal{P}^2 v(t,\xi_1) \mathcal{P}v(t,\xi_2) v(t,\xi_3)\, dydz \right\|_\infty \\
\leq\ & C\langle t\rangle^{-1} \left\|\mathcal{P}^2 v\right\|_{0,1,\infty} \|\mathcal{P}v\|_{0,2,\infty} \|v\|_{0,3,\infty} \\
&+C\langle t\rangle^{\gamma-3/2} \left\|\mathcal{P}^2 v\right\|_{\mathbf{B}} \|\langle x\rangle \mathcal{P}v\|_{\mathbf{B}} \left\|\langle x\rangle^2 v\right\|_{\mathbf{B}} \\
\leq\ & C\langle t\rangle^{70\gamma-1} \|u\|_{\mathbf{Y}} \|u\|_{\mathbf{Z}}^2 \leq C\varepsilon^2 \langle t\rangle^{\lambda_3-1}
\end{aligned}$$

The rest terms are considered in the same manner. Therefore we obtain

$$\left\|\left(\mathcal{P}^3 v\right)_t\right\|_\infty \leq C\varepsilon^2 \langle t\rangle^{\lambda_3-1} \text{ and } \left\|\mathcal{P}^3 v\right\|_\infty \leq C\varepsilon \langle t\rangle^{\lambda_3}. \tag{4.10}$$

Applying (4.3), Lemma 2.3 with $n=2$ and Lemma 2.1 with $n=3$ we get

$$\begin{aligned}
\left\|v''_{xt}\right\|_{0,2,\infty} \leq\ & C\sum_{j=1}^4 \left\| \partial_x \iint e^{it\Lambda_j} \Phi_{0,j}\left(t,\boldsymbol{\xi}_j\right) dydz \right\|_{0,2,\infty} \\
&+Ct\sum_{j=1}^4 \left\| x \iint e^{it\Lambda_j} \Phi_{0,j}\left(t,\boldsymbol{\xi}_j\right) dydz \right\|_{0,2,\infty} \\
\leq\ & C\left\|\langle x\rangle^2 v\right\|_{\mathbf{B}}^3 \leq C\langle t\rangle \|v\|_{\mathbf{Z}}^3 \leq C\varepsilon^2 \langle t\rangle,
\end{aligned}$$

then applying Lemma 2.3 with $n=1$ and Lemma 2.1 with $n=2$ we obtain

$$\begin{aligned}
\left\|(\mathcal{P}v)''_{xt}\right\|_{0,1,\infty} \leq\ & C\left\|v''_{xt}\right\|_{0,1,\infty} + C\sum_{j=1}^4 \left\| \partial_x \iint e^{it\Lambda_j} \Phi_{1,j}\left(t,\boldsymbol{\xi}_j\right) dydz \right\|_{0,1,\infty} \\
&+Ct\sum_{j=1}^4 \left\| x \iint e^{it\Lambda_j} \Phi_{1,j}\left(t,\boldsymbol{\xi}_j\right) dydz \right\|_{0,1,\infty} \\
\leq\ & C\|\langle x\rangle \mathcal{P}v\|_{\mathbf{B}} \left\|\langle x\rangle^2 v\right\|_{\mathbf{B}}^2 \leq C\langle t\rangle \|v\|_{\mathbf{Z}}^3 \leq C\varepsilon^2 \langle t\rangle,
\end{aligned}$$

finally via Lemma 2.3 with $n=0$ and Lemma 2.1 with $n=1$ we obtain

$$\begin{aligned}
&\left\|\left(\mathcal{P}^2 v\right)''_{xt}\right\|_\infty \\
&\leq C\left\|v''_{xt}\right\|_\infty + C\left\|(\mathcal{P}v)''_{xt}\right\|_\infty + C\sum_{j=1}^4 \left\| \partial_x \iint e^{it\Lambda_j} \Phi_{2,j}\left(t,\boldsymbol{\xi}_j\right) dydz \right\|_\infty \\
&\quad + Ct\sum_{j=1}^4 \left\| x \iint e^{it\Lambda_j} \Phi_{2,j}\left(t,\boldsymbol{\xi}_j\right) dydz \right\|_\infty \\
&\leq C\left\|\mathcal{P}^2 v\right\|_{\mathbf{B}} \left\|\langle x\rangle^2 v\right\|_{\mathbf{B}}^2 + C\|\langle x\rangle \mathcal{P}v\|_{\mathbf{B}}^2 \left\|\langle x\rangle^2 v\right\|_{\mathbf{B}} \\
&\leq C\langle t\rangle \|v\|_{\mathbf{Z}}^3 \leq C\varepsilon^2 \langle t\rangle,
\end{aligned}$$

Thus we have

$$\sum_{k=0}^{2} \langle t\rangle^{-2} \left\|\mathcal{P}^k v\right\|_{1,2-k,\infty} \le \frac{1}{2}\varepsilon^{3/4}. \tag{4.11}$$

Now by (4.3) and applying Lemma 2.1 with $n=3$ we get

$$\begin{aligned}
\left\|v_t'\right\|_{0,3,\infty} &\le C\sum_{j=1}^{4}\left\|\iint e^{it\Lambda_j}\Phi_{0,j}\left(t,\boldsymbol{\xi}_j\right)dydz\right\|_{0,3,\infty}\\
&\le C\langle t\rangle^{-1}\|v\|_{0,3,\infty}^3 + C\langle t\rangle^{\gamma-3/2}\left\|\langle x\rangle^2 v\right\|_{\mathbf{B}}^3\\
&\le C\langle t\rangle^{-1}\|v\|_{\mathbf{Z}}^3 \le C\varepsilon^2\langle t\rangle^{-1},
\end{aligned}$$

then by Lemma 2.1 with $n=2$ we obtain

$$\begin{aligned}
\left\|(\mathcal{P}v)_t'\right\|_{0,2,\infty} &\le C\left\|v_t'\right\|_{0,2,\infty} + C\sum_{j=1}^{4}\left\|\iint e^{it\Lambda_j}\Phi_{1,j}\left(t,\boldsymbol{\xi}_j\right)dydz\right\|_{0,2,\infty}\\
&\le C\varepsilon^2\langle t\rangle^{-1} + C\langle t\rangle^{-1}\|\mathcal{P}v\|_{0,1,\infty}\|v\|_{0,2,\infty}^2\\
&\quad + C\langle t\rangle^{\gamma-3/2}\|\langle x\rangle\mathcal{P}v\|_{\mathbf{B}}\left\|\langle x\rangle^2 v\right\|_{\mathbf{B}}^2 \le C\varepsilon^2\langle t\rangle^{\lambda_1-1},
\end{aligned}$$

whence $\|\mathcal{P}v\|_{0,2,\infty} \le C\varepsilon\langle t\rangle^{\lambda_1}$. Finally via Lemma 2.1 with $n=1$ we have

$$\begin{aligned}
&\left\|\left(\mathcal{P}^2 v\right)_t'\right\|_{0,1,\infty}\\
&\quad\le C\left\|v_t'\right\|_{0,1,\infty} + C\left\|(\mathcal{P}v)_t'\right\|_{0,1,\infty}\\
&\qquad + C\sum_{j=1}^{4}\left\|\iint e^{it\Lambda_j}\Phi_{2,j}\left(t,\boldsymbol{\xi}_j\right)dydz\right\|_{0,1,\infty}\\
&\quad\le C\varepsilon^2\langle t\rangle^{2\gamma-1} + C\langle t\rangle^{-1}\left\|\mathcal{P}^2 v\right\|_{\infty}\|v\|_{0,2,\infty}^2\\
&\qquad + C\langle t\rangle^{-1}\|\mathcal{P}v\|_{0,1,\infty}^2\|v\|_{0,2,\infty} + C\langle t\rangle^{\gamma-3/2}\left\|\mathcal{P}^2 v\right\|_{\mathbf{B}}\left\|\langle x\rangle^2 v\right\|_{\mathbf{B}}^2\\
&\qquad + C\langle t\rangle^{\gamma-3/2}\|\langle x\rangle\mathcal{P}v\|_{\mathbf{B}}^2\left\|\langle x\rangle^2 v\right\|_{\mathbf{B}} \le C\varepsilon^2\langle t\rangle^{\lambda_2-1},
\end{aligned}$$

whence $\left\|\mathcal{P}^2 v\right\|_{0,1,\infty} \le C\varepsilon\langle t\rangle^{\lambda_2}$. To get the uniform estimate of v we exclude the worst term with $j=2$ containing $\delta_2=0$ by the change of the dependent variable $v=hE$, where $E=\exp\left(ia\int_1^t|v(\tau,x)|^2\frac{d\tau}{\tau}\right)=\exp\left(ia\int_1^t|h(\tau,x)|^2\frac{d\tau}{\tau}\right)$,

$a(x) = \sum_{j=1}^{3} b_j x^j$, $b_j = \sum_{|\boldsymbol{\omega}|=j} a_{2,\boldsymbol{\omega}}$. Then we have

$$\begin{aligned} ih_t &= \overline{E}\sum_{j=1}^{4} e^{it\delta_j x^2} \iint e^{it\Lambda_j}\left(\Phi_{0,j}\left(t,\boldsymbol{\xi}_j\right) - \Phi_{0,j}\left(t,\boldsymbol{\xi}_{j,0}\right)\right) dydz \\ &+\overline{E}\sum_{j=1}^{4} \frac{C}{t} e^{it\delta_j x^2}\Phi_{0,j}\left(t,\boldsymbol{\xi}_{j,0}\right) = J_1 + J_2, \end{aligned}$$

for the first summand by Lemma 2.1 we obtain $\|J_1\|_\infty \le C\langle t\rangle^{\gamma-3/2} \times \left\|\langle x\rangle^2 v\right\|_{\mathbf{B}}^3 \le C\langle t\rangle^{13\gamma-3/2}\|u\|_{\mathbf{Z}}^3 \le C\varepsilon^2\langle t\rangle^{13\gamma-3/2}$. And in the second summand we integrate by parts with respect to time t via identity $\left(1+i\delta_j\tau x^2\right)e^{i\tau\delta_j x^2} = \frac{\partial}{\partial\tau}\left(\tau e^{i\tau\delta_j x^2}\right)$

$$\begin{aligned} \left|\int_s^t J_2 d\tau\right| &\le C\sum_{j=1}^{4}\left|\int_s^t e^{i\tau\delta_j x^2}\overline{E}(\tau,x)\,\Phi_{0,j}\left(\tau,\boldsymbol{\xi}_{j,0}\right)\frac{d\tau}{\tau}\right| \\ &\le C\sum_{j=1}^{4}\left|\int_s^t e^{i\tau\delta_j x^2}\frac{\partial}{\partial\tau}\left(\frac{\overline{E}(\tau,x)\,\Phi_{0,j}\left(\tau,\boldsymbol{\xi}_{j,0}\right)}{\tau\left(1+i\delta_j\tau x^2\right)}\right)\tau d\tau\right| \\ &\le C\varepsilon^2\left|\sum_{j=1}^{4}\int_s^t \frac{|x|\,d\tau}{\tau\left(1+\tau x^2\right)}\right| \le C\varepsilon^2 s^{-40\gamma}, \end{aligned}$$

since $\left|\Phi_{0,j}\left(\tau,\boldsymbol{\xi}_{j,0}\right)\right| \le C|x|\,\|v\|_{0,1,\infty}^3 \le C\varepsilon^2|x|$, $\left|\partial_\tau\Phi_{0,j}\left(\tau,\boldsymbol{\xi}_{j,0}\right)\right| \le \frac{C}{\tau}\varepsilon^2|x|$ and $|\partial_\tau E(\tau,x)| \le \frac{C}{\tau}\varepsilon^{3/2}|x|$. Thus $\|v\|_\infty = \|h\|_\infty \le C\varepsilon$, $\|h(t)-h(s)\| \le C\varepsilon t^{-100\gamma}$ and we get

$$\sum_{k=0}^{3}\langle t\rangle^{1-\lambda_k}\left\|\left(\mathcal{P}^k v\right)_t\right\|_{0,3-k,\infty} + \sum_{k=0}^{3}\langle t\rangle^{-\lambda_k}\left\|\mathcal{P}^k v\right\|_{0,3-k,\infty} < \frac{1}{2}\varepsilon^{3/4}. \tag{4.12}$$

By (4.11) and (4.12) we obtain

$$\|u\|_{\mathbf{Z}} < \varepsilon^{3/4}. \tag{4.13}$$

Now let us obtain the estimate of the solution in the norm $\mathbf{X}$. By Lemma 2.4 with $\omega = 3$ we get

$$\begin{aligned} \left\|\partial_x^3 u\right\|_\infty &\le C\left\|\int e^{\frac{i}{2}t(\zeta-x)^2} v(t,\zeta)\,\zeta^3 d\zeta\right\|_\infty \\ &\le Ct^{-1/2}\|v\|_{0,3,\infty} + Ct^{-3/4}\|\mathcal{P}v\|_{0,2} + C\|v_t\|_{0,3,\infty} \\ &\le C\varepsilon^{3/4}\langle t\rangle^{-1/2}, \end{aligned}$$

since $\|\mathcal{P}v\|_{0,2} = \|\mathcal{I}u\|_{2,0} \le C\langle t\rangle^{4\gamma}\|u\|_{\mathbf{Y}} \le C\varepsilon^{3/4}\langle t\rangle^{4\gamma}$. And we obtain the estimate of $\|u\|_\infty$ by Lemmas 2.5-2.6 since we have

$$
\begin{aligned}
& e^{-ir^2t/2}u(t,r) \\
= & \frac{1}{\sqrt{2\pi}}\int e^{-\frac{i}{2}t(r-x)^2}v(t,x)\,dx = C\int e^{-\frac{i}{2}t(r-x)^2}E(t,x)\,h(t,x)\,dx \\
= & C\int e^{-\frac{i}{2}t(r-x)^2}E(t,x)\,h(0,x)\,dx \\
& +C\sum_{j=1}^{4}\int dx e^{-\frac{i}{2}t(r-x)^2}E(t,x)\int_0^1\overline{E}(\tau,x)\,e^{i\tau\delta_jx^2}d\tau \\
& \times\iint e^{i\tau\Lambda_j}\Phi_{0,j}\left(\tau,\boldsymbol{\xi}_j\right)dydz \\
& +\sum_{j=1}^{4}C_j\int e^{-\frac{i}{2}t(r-x)^2}E(t,x)\int_1^t\overline{E}(\tau,x)\,e^{i\tau\delta_jx^2}\Phi_{0,j}\left(\tau,\boldsymbol{\xi}_{j,0}\right)\frac{d\tau}{\tau}dx \\
& +\sum_{j=1}^{4}C_j\int dx e^{-\frac{i}{2}t(r-x)^2}E(t,x)\int_1^t d\tau\overline{E}(\tau,x)\,e^{i\tau\delta_jx^2} \\
& \times\iint e^{i\tau\Lambda_j}\left(\Phi_{0,j}\left(\tau,\boldsymbol{\xi}_j\right)-\Phi_{0,j}\left(\tau,\boldsymbol{\xi}_{j,0}\right)\right)dydz,
\end{aligned}
\tag{4.14}
$$

here in the first and second summands we easily can integrate with respect to x since the estimate of the derivative provided by Lemma 2.3 does not give us any growth, when $\tau\in(0,1)$. Now we have $\|u\|_{1,0} = \|v\|_{0,1} = \|h\|_{0,1} \le \|h\|_{0,2,\infty} \le C\varepsilon^{3/4}$. Finally writing the representation similar to (4.14) for the function $\mathcal{I}u$ with $E=1$, (i.e. $a=0$) via Lemmas 2.5 -2.6 we have the estimate $\|\mathcal{I}u\|_\infty = \left\|\int e^{\frac{i}{2}t(x-\xi)^2}\mathcal{P}v(t,\xi)\,d\xi\right\|_\infty \le C\varepsilon\langle t\rangle^{2\gamma-1/2}$ and by Lemma 2.4 we get $\|\partial\mathcal{I}u\|_\infty = \left\|\int e^{\frac{i}{2}t(x-\xi)^2}\mathcal{P}v(t,\xi)\,\xi d\xi\right\|_\infty \le C\varepsilon\langle t\rangle^{2\gamma-1/2}$. Hence

$$\|u\|_{\mathbf{X}} \le \sqrt{\varepsilon}. \tag{4.15}$$

Inequalities (4.13), (4.15) imply estimate (4.1) for all $t>0$. thus Lemma 4.2 is true. Applying Lemma 4.2 and also Lemmas 2.5 -2.6 we get the asymptotics (1.2). Theorem 1.1 is proved. ∎

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