DEMAZURE OPERATORS FOR COMPLEX REFLECTION GROUPS $G(e, e, n)$

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Abstract This paper is a continuation of the work in [RS], where we studied Demazure operators for the imprimitive complex reflection group $\widetilde{W} = G(e, 1, n)$ and constructed a homogeneous basis of the coinvariant algebra $S_{\widetilde{W}}$. In this paper, we study a similar problem for the reflection subgroup $W = G(e, e, n)$ of $\widetilde{W}$. We prove, by assuming certain conjectures, that the operators $\Delta_w (w \in W)$ are linearly independent over the symmetric algebra $S(V)$. We define a graded space $H_W$ in terms of Demazure operators, and we show that the coinvariant algebra $S_W$ is naturally isomorphic to $H_W$. Then we can define a homogeneous basis of $S_W$ parametrized by $w \in W$.

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§1. Introduction

Let $\widetilde{W} = G(e, 1, n)$ be the imprimitive complex reflection group isomorphic to $S_n \ltimes (\mathbb{Z}/e\mathbb{Z})^n$, regarded as a subgroup of $GL(V)$ with $V \cong \mathbb{C}^n$. (Here $S_n$ denotes the symmetric group of degree $n$). Let $S_{\widetilde{W}}$ be the coinvariant algebra of $\widetilde{W}$, i.e. the quotient of the symmetric algebra $S(V)$ by the ideal generated by the non-constant homogeneous $\widetilde{W}$-invariant polynomials. In [BM1], K. Bremke and G. Malle constructed a length function $n : \widetilde{W} \to \mathbb{N}$ satisfying the property $\sum_{w \in \widetilde{W}} n(w) = P_{\widetilde{W}}(t)$, where $P_{\widetilde{W}}(t)$ is the Poincaré polynomial associated with the graded algebra $S_{\widetilde{W}}$. In [RS], we defined a Demazure operator $\Delta_w$ for each $w \in \widetilde{W}$, which is an endomorphism on $S(V)$ reducing the

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grading by \( n(w) \), and constructed a basis of \( S_{\tilde{W}} \) parametrized by \( w \in \tilde{W} \) by making use of \( \{ \Delta_w \mid w \in \tilde{W} \} \).

In this paper, we consider the group \( W = G(e, e, n) \), which is a subgroup of \( \tilde{W} \) of index \( e \), isomorphic to \( S_n \ltimes (\mathbb{Z}/e\mathbb{Z})^{n-1} \). The length function \( \ell : W \to \mathbb{N} \), satisfying the property \( \sum_{w \in W} t^{\ell(w)} = P_W(t) \), was constructed by [BM2], where \( P_W(t) \) is the Poincaré polynomial associated with the coinvariant algebra \( S_W \) of \( W \). We recall the definition of Demazure operators. For each \( \alpha \in V \), let \( s_\alpha \) be the complex reflection with eigenvector \( \alpha \). A Demazure operator \( \Delta_\alpha : S(V) \to S(V) \) is defined by

\[
\Delta_\alpha(f) = \frac{f - s_\alpha(f)}{\alpha}, \quad \text{for } f \in S(V).
\]

We define an operator \( \Delta_w \) for each \( w \in W \) as follows. It is known by [BM2] that there exists a system of representatives \( N \) of the left cosets \( W/S_n \) satisfying the property that \( \ell(u'u'') = \ell(u') + \ell(u'') \) for \( u' \in N \), \( u'' \in S_n \). We define \( \Delta_{w'} \) for \( w' \in N \) as a certain product of various \( \Delta_\alpha \) for \( s_\alpha \in W \). On the other hand, the operator \( \Delta_{w''} \) for \( w'' \in S_n \) is already defined by the theory of Demazure operators for finite Coxeter groups. Then we define, for \( w = u'u'' \in W \) (\( u' \in N \), \( u'' \in S_n \)) the operator \( \Delta_w \) by \( \Delta_w = \Delta_{w'}\Delta_{w''} \). In the case of \( \tilde{W} \), the crucial step for the proof of the main result is to show that the operators \( \{ \Delta_w \mid w \in \tilde{W} \} \) are linearly independent over \( S(V) \). In our situation, we can prove (Theorem 3.10) that the operators \( \{ \Delta_{w'} \mid w' \in N \} \) are linearly independent over \( S(V) \). It is also known by the general theory that the operators \( \{ \Delta_{w''} \mid w'' \in S_n \} \) are linearly independent over \( S(V) \). We expect that \( \{ \Delta_w \mid w \in W \} \) are linearly independent over \( S(V) \). In our paper, we prove this by assuming certain conjectures, (3.12.1) and (3.12.2), concerning the property of \( \Delta_{w'} \) (\( w' \in N \)). Our main result asserts that a similar theorem as in the case of \( \tilde{W} \) holds for \( W \), assuming the above conjectures. More precisely, let \( \mathcal{D}_W \) be the subspace of the dual space of \( S(V) \) generated by \( \epsilon \Delta_w \) (\( w \in W \)), where \( \epsilon : S(V) \to \mathbb{C} \) is the evaluation at 0. Then we can show (Theorem 3.25) that \( \{ \epsilon \Delta_w \mid w \in W \} \) gives a basis of \( \mathcal{D}_W \), and that \( S_W \) is naturally isomorphic to the dual space of \( \mathcal{D}_W \).

The conjecture (3.12.1) is related to the evaluation of \( \Delta_{w_1} \) (\( w_1 \) is the longest element in \( W \) with respect to \( \ell \)) at certain polynomial, and is verified to be true (Theorem 3.14) under the assumption that \( e \geq n \). This theorem leads to the following interesting characterization of \( \Delta_{w_1} \). Let \( J \) be the operator on \( S(V) \) defined by \( J = \sum_{w \in W} \epsilon_w(w)w \), where \( \epsilon_w : W \to \{ \pm 1 \} \) is the sign character of \( W \). Let \( Q \) be the product of all eigenvectors of reflections contained in \( W \). Assume that \( e \geq n \). Then \( \Delta_{w_1} \) is expressed (Proposition 3.18) as \( \Delta_{w_1} = dQ^{-1}J \) for some non-zero constant \( d \in \mathbb{C} \).
§2. Preliminaries

2.1. Let $V$ be the unitary space $\mathbb{C}^n$ with standard basis $x_1, x_2, \ldots, x_n$. Let $\widetilde{W} = G(e, 1, n)$ be the imprimitive complex reflection group contained in $GL(V)$. The group $\widetilde{W}$ is generated by $\{t, s_1, \ldots, s_n\}$, where $s_i$ is a reflection permuting $x_i$ and $x_{i-1}$, and $t$ is a complex reflection of order $e$, which sends $x_1$ to $\zeta x_1$ and leaves all the other $x_i$ unchanged. (Here $\zeta$ is a fixed primitive $e$-th root of unity).

Let $W = G(e, e, n)$ be the subgroup of $\widetilde{W}$ of index $e$ generated by reflections $S = \{s_1, s_2, \ldots, s_n\}$ of order 2, where $s_1 = ts_2t^{-1}$ sends $x_1$ to $\zeta^{-1}x_2$ and $x_2$ to $\zeta x_1$. Note that $W$ is the Weyl group of type $D_n$ if $e = 2$, and $W$ is the dihedral group of order $2e$ if $n = 2$.

Let $S(V) = \oplus_{i \geq 0} S^i(V)$ be the symmetric algebra on $V$, where $S^i(V)$ denotes the $i$-th homogeneous part of $S(V)$. The group $W$ acts naturally on $S(V)$ and we denote by $I_W$ the ideal of $S(V)$ generated by the $W$-invariant homogeneous elements of $S(V)$ of strictly positive degree. The coinvariant algebra associated with $W$ is defined as $S_W = S(V)/I_W$, which has a natural grading $S_W = \oplus_{i \geq 0} S^i_W$ inherited from that of $S(V)$. The Poincaré polynomial $P_W(t)$ is defined by the formula

$$P_W(t) = \sum_{i \geq 0} \dim_{\mathbb{C}}(S^i_W)t^i.$$ 

The group $\widetilde{W}$ acts on $S(V)$, and the coinvariant algebra $S_{\widetilde{W}}$ and the Poincaré polynomial $P_{\widetilde{W}}(t)$ associated with $\widetilde{W}$ are defined similarly.

2.2. In [BM1], Bremke and Malle constructed a length function $n : \widetilde{W} \to \mathbb{N}$ by making use of a certain root system, and showed that the sum $\sum_{w \in \widetilde{W}} t^{n(w)}$ coincides with $P_{\widetilde{W}}(t)$. In [BM2], they defined a different type of length function $\ell : \widetilde{W} \to \mathbb{N}$ (the function $\ell_2$ in the notation of [BM2]), in terms of an alternative root system and showed that the restriction of $\ell$ on $W$ satisfies the formula $\sum_{w \in W} \ell^{(w)} = P_W(t)$. Note that the subgroup of $W$ generated by $S' = \{s_2, \ldots, s_n\}$ is identified with $S_n$. The restriction of $\ell$ on $S_n$ coincides with the usual length function of $S_n$ with respect to $S'$.

They found a system of left coset representatives $\mathcal{N}$ of $W/S_n$ having nice properties with respect to the length function $\ell$ on $W$ as follows. For $0 < a \leq e$, $1 \leq i \leq n$ we define an element of $W$ by

$$w(a, i) = \begin{cases} \ s_1 \cdots s_2 f^t & \text{if } 0 < a \leq e/2, \\ \ s_1 \cdots s_2 f^t s_2 \cdots s_i & \text{if } e/2 < a \leq e. \end{cases}$$ (2.2.1)
It is known by Lemma 1.10 in [BM2] that the length of the element \(w(a, i)\) is given as

\[
\ell(w(a, i)) = \begin{cases} (i-1)(2a-1) & \text{if } 0 < a \leq e/2, \\ (i-1)(2e-2a) & \text{if } e/2 < a \leq e. \end{cases}
\]

Put

\[
\mathcal{N} = \{w(a_1, 1) \cdots w(a_n, n) \mid 1 \leq a_i \leq e, \sum_{i=1}^{n} a_i \equiv 0 \pmod{e}\}
\]

They proved the following fact.

**Proposition 2.3 ([BM2, Cor.1.16, Prop. 2.6]).** The set \(\mathcal{N}\) is a system of representatives for the left cosets \(W/S_n\) satisfying the following.

(i) For \(w' \in \mathcal{N}, w'' \in S_n\), we have

\[
\ell(w'w'') = \ell(w') + \ell(w'').
\]

(ii) If \(w' \in \mathcal{N}\) is given as \(w' = w(a_1, 1) \cdots w(a_n, n)\), then \(\ell(w') = \sum_{i=2}^{n} \ell(w(a_i, i))\).

(Note that \(\ell(w(a_1, 1)) = 0\) by (2.2.2)).

2.4. Let \(s_\alpha\) be the reflection in \(W\) with eigenvector \(\alpha \in V\). (Here we assume that the eigenvalue attached to \(\alpha\) is not equal to 1). We define an operator \(\Delta_\alpha : S(V) \rightarrow S(V)\) by the formula

\[
\Delta_\alpha(f) = \frac{f - s_\alpha(f)}{\alpha}, \quad (f \in S(V)).
\]

We call \(\Delta_\alpha\) a Demazure operator on \(S(V)\). Demazure operators are defined for complex reflection groups in general. In the case of finite Coxeter groups, there exists a well established theory for Demazure operators by [BBG], [D]. In the case of (non-real) finite complex reflection groups, not much is known. In [RS], we studied Demazure operators for the group \(\tilde{W}\), and showed that the structure of the coinvariant algebra \(S_{\tilde{W}}\) is described in terms of Demazure operators, as in the case of Coxeter groups, by constructing a certain (non-canonical) basis of \(S_{\tilde{W}}\). Here we take up a similar problem for the group \(W\).

We give some properties of Demazure operators. We have the following.

\[
\Delta_\alpha^2 = 0,
\]

\[
\Delta_\alpha(fh) = \Delta_\alpha(f)h + f\Delta_\alpha(h),
\]
for \( f, h \in S(V) \). If \( f \in S(V) \) is \( s_\alpha \)-invariant, then \( \Delta_\alpha(f) = 0 \). Now let \( S(V)^W \) be the subalgebra of \( S(V) \) consisting of the \( W \)-invariant elements. Then it follows from (2.4.2) that

\[(2.4.3) \quad \Delta_\alpha(fh) = f \Delta_\alpha(h) \quad \text{for} \quad f \in S(V)^W.\]

In particular, we have \( \Delta_\alpha(I_W) \subseteq I_W \) and \( \Delta_\alpha \) induces an operation on \( S_W \).

2.5. Let \( S_n \) be the subgroup of \( W \) as in 2.2. Then \( (S_n, S') \) is a Coxeter system, with associated length function \( \ell : S_n \to \mathbb{N} \). Hence, by the general theory of Demazure operators for finite Coxeter groups, we have the following facts. Let \( w = s_i s_{i_2} \cdots s_{i_k} \) (\( s_i \in S' \)) be a reduced expression of \( w \in S_n \). Then we define

\[(2.5.1) \quad \Delta_w = \Delta_{i_1} \cdots \Delta_{i_k},\]

where \( \Delta_i = \Delta_{\alpha_i} \) with \( \alpha_i = x_i - x_{i-1} \). It is known that the operator \( \Delta_w \) is independent of the choice of the reduced expression. (See, for example [H, IV, Prop. 1.7]).

Let \( w_0 \) be the longest element in \( S_n \). We define a polynomial \( Q_0 \) by \( Q_0 = \prod_{i>j} (x_i - x_j) \). The following facts are known.

**Proposition 2.6 ([H, IV, Prop. 1.6]).** \( \Delta_{w_0}(Q_0) = 1 \).

**Proposition 2.7 ([H, IV, Cor. 2.3]).** For any \( w, w' \in W \) such that \( \ell(w) \leq \ell(w') \), we have \( \Delta_{w'} \Delta_{w^{-1}w_0} = \delta_{w, w'} \Delta_{w_0} \).

Note that the condition \( \ell(w) \leq \ell(w') \) is dropped in the statement of Corollary 2.3 in [H].

3. Demazure operators for \( G(e, e, n) \)

3.1. From now on we identify \( S(V) \) with the polynomial algebra \( \mathbb{C}[x_1, \ldots, x_n] \) with indeterminates \( x_i \). The group \( W = G(e, e, n) \) acts on \( \mathbb{C}[x_1, \ldots, x_n] \) as in 2.1.

For \( i = 2, 3, \ldots, n \) we define inductively the element \( s'_i \) as follows; Let \( s'_2 = s_1 \) and \( s'_i = s_{i-1} s'_i s_{i-1} \). Then \( s'_i \) is the complex reflection of order 2, which sends \( x_i \) to \( \zeta x_{i-1} \), and \( x_{i-1} \) to \( \zeta^{-1} x_i \). We note that if we put \( y_i = \zeta^{-1/2} x_i \) and \( y_{i-1} = \zeta^{1/2} x_{i-1} \), then we can regard \( s'_i \) as a permutation of \( y_i, y_{i-1} \). We define two operators \( \Delta_{s'_i}, \Delta_{s'_i} \) on \( S(V) \) by the formulas

\[(3.1.1) \quad \Delta_{s'_i}(f) = \frac{f - s_i(f)}{x_i - x_{i-1}}, \quad \Delta_{s'_i}(f) = \frac{f - s'_i(f)}{\zeta^{-1/2} x_i - \zeta^{1/2} x_{i-1}}, \quad (f \in S(V)).\]
Then the following two formulas hold:

\[
\Delta_s(x_i^a x_{i-1}^b) = \varepsilon \sum x_i^j x_{i-1}^{a+b-1-j},
\]

\[
\Delta_{s'}(x_i^a x_{i-1}^b) = \varepsilon \zeta^{(2n-1)/2} \sum \zeta^{-j} x_i^j x_{i-1}^{a+b-1-j},
\]

where in both formulas the sum is taken over \( j \) such that \( \min\{a,b\} \leq j \leq \max\{a,b\}-1 \), and \( \varepsilon = 1 \) (resp. \( \varepsilon = -1 \)) if \( a > b \) (resp. \( a < b \)). The first formula is contained in [R&S], and the second one is obtained from the first by changing the variables \( x_i \mapsto y_i, x_{i-1} \mapsto y_{i-1} \).

For \( i = 2, \ldots, n \), we define operators \( \Delta_i^{[a]}, \Delta_{s'}^{[a]} \) in the following way

\[
\Delta_i^{[a]} = \cdots \Delta_{s'}^{[a]} \Delta_i^{[a]}, \quad \Delta_{s'}^{[a]} = \cdots \Delta_{s'}^{[a]} \Delta_i^{[a]}.
\]

### 3.2.

In order to study the above operators in a more detailed way, we need to evaluate them at various polynomials. For this we prepare some notation.

Let \( a, b \) be two positive integers such that \( 1 \leq a \leq b \). We put

\[
c(a, b) = (-1)^{[a+1]/2} \prod_{j=1}^{a-1} (\zeta^{(b-j)/2} - \zeta^{-(b-j)/2}),
\]

where \( [a] \) denotes the smallest integer which does not exceed \( a \). We have \( c(a, b) = -1 \) if \( a = 1 \). The following two lemmas will be used in our later discussion.

**Lemma 3.3.** Let \( a, b \) be integers such that \( 1 \leq a \leq b \).

(i) Assume that \( a < b \). Then we have

\[
\Delta_i^{[a]}(x_i^b x_{i-1}^b) = \begin{cases} c(a, b)(x_i^{b-a} + x_{i-1}^{b-a}) + f, & \text{if } a \text{ is odd}, \\ c(a, b)(y_i^{b-a} + y_{i-1}^{b-a}) + f, & \text{if } a \text{ is even}, \end{cases}
\]

\[
\Delta_{s'}^{[a]}(x_i^b x_{i-1}^b) = \begin{cases} (-1)^{a-1} \zeta^{-b/2} c(a, b)(y_i^{b-a} + y_{i-1}^{b-a}) + f, & \text{if } a \text{ is odd}, \\ (-1)^{a-1} \zeta^{-b/2} c(a, b)(x_i^{b-a} + x_{i-1}^{b-a}) + f, & \text{if } a \text{ is even}, \end{cases}
\]

where in each case, \( f \) denotes a polynomial divisible by \( x_i x_{i-1} = y_i y_{i-1} \).

(ii) Assume that \( a = b \). Then we have

\[
\Delta_i^{[a]}(x_i^a x_{i-1}^a) = c(a, a),
\]

\[
\Delta_{s'}^{[a]}(x_i^a x_{i-1}^a) = (-1)^{a-1} \zeta^{-a/2} c(a, a).
\]
Proof. We prove only the formula (i). The proof of (ii) is similar, and simpler. We show the first formula in (i). The case where $a = 1$ is straightforward from (3.1.2). The following two formulas are obtained by using the definition of $\Delta_s, \Delta_t$ and the fact that $y_i = \zeta^{-1/2} x_i$ and $y_{i-1} = \zeta^{1/2} x_{i-1}$.

$$\Delta_t(x^{b-a+1}_i + x^{b-a+1}_{i-1}) = (\zeta^{(b-a+1)/2} - \zeta^{-(b-a+1)/2})(y_t^{b-a} + y_t^{b-a} + f_1,$$

$$\Delta_s(y^{b-a+1}_i + y^{b-a+1}_{i-1}) = (\zeta^{-(b-a+1)/2} - \zeta^{(b-a+1)/2})(x_s^{b-a} + x_s^{b-a} + f_1),$$

where $f_1$ is a polynomial divisible by $x_i x_{i-1} = y_i y_{i-1}$. We also notice that since $x_i x_{i-1} = y_i y_{i-1}$ is stable by the reflections $s_i$ and $s_t$, if a polynomial $f$ is divisible by $x_i x_{i-1} = y_i y_{i-1}$, then so are $\Delta_s(f)$ and $\Delta_t(f)$. The first formula in (i) follows from the above formulas by induction on $a$. Next we show the second formula in (i). If we note that $x^b_{i-1} = \zeta^{-b/2} y^b_{i-1}$, it is easy to see that $\Delta_t^a(y^b_{i-1})$ coincides with the polynomial which is obtained from $\Delta_t^a(x^b_{i-1})$ by replacing $x_i, x_{i-1}$ by $y_i, y_{i-1}$, by replacing $\zeta$ by $\zeta^{-1}$, and then by multiplying by $\zeta^{-b/2}$. Hence the second formula follows immediately from the first one.

Next we compute the values $\Delta_t^a(x^b_i)$ and $\Delta_t^a(x^b_i)$. By (3.1.2) we see that

$$\Delta_s(x^b_i) = -\Delta_s(x^b_{i-1}), \quad \Delta_t(y^b_i) = -\Delta_t(y^b_{i-1}).$$

Therefore we have

$$\Delta_t(x^b_i) = \zeta^{b/2} \Delta_t(y^b_i)$$

$$= -\zeta^{b/2} \Delta_t(y^b_{i-1})$$

$$= -\zeta^b \Delta_t(x^b_{i-1}).$$

This implies that the value $\Delta_t^a(x^b_i)$ (resp. $\Delta_t^a(x^b_i)$) coincides with $-\Delta_t^a(x^b_{i-1})$ (resp. $-\zeta^{b \Delta_t^a(x^b_{i-1})}$). Therefore as a corollary to Lemma 3.3 we obtain the following result.

Lemma 3.4. Let $a, b$ as in Lemma 3.3.

(i) Assume that $a < b$. Then we have

$$\Delta_t^a(x^b_i) = \begin{cases}
-c(a, b)(x^{b-a}_i + x^{b-a}_{i-1}) + f & \text{if } a \text{ is odd}, \\
-c(a, b)(y^{b-a}_i + y^{b-a}_{i-1}) + f & \text{if } a \text{ is even,}
\end{cases}$$

$$\Delta_t^a(x^b_i) = \begin{cases}
(1-a) \zeta^{b/2} c(a, b)(x^{b-a}_i + x^{b-a}_{i-1}) + f & \text{if } a \text{ is odd}, \\
(1-a) \zeta^{b/2} c(a, b)(x^{b-a}_i + x^{b-a}_{i-1}) + f & \text{if } a \text{ is even.}
\end{cases}$$
(ii) Assume that \( a = b \). Then we have
\[
\Delta^{[a]}_i(x_1^a) = -c(a,a),
\]
\[
\Delta^{[a]}_x(x_1^a) = (-1)^{a-1} \frac{1}{a} c(a,a).
\]

3.5. We fix an integer \( a \geq 0 \). We define, for \( 2 \leq i \leq n \), an operator \( \Delta_i[a] \) on \( S(V) \) by the formula
\[
\Delta_i[a] = \begin{cases} 
\Delta_2^{[a]} \cdots \Delta_n^{[a]} & \text{if } a \geq 1, \\
1 & \text{if } a = 0.
\end{cases}
\]
The operator \( \Delta_i[a] \) reduces the grading by \((i-1)a\). For each \( a \geq 0 \), we define a polynomial \( g_{i,a}(x) \) of degree \((i-1)a\) by \( g_{i,a}(x) = (x_1 \cdots x_{i-1})^a \). Then the following lemma holds.

**Lemma 3.6.** Assume that \( a \geq 1 \). Let \( \Delta_i[a] \), \( g_{i,a}(x) \) be defined as above. Then
\[
\Delta_i[a](g_{i,a}) = \{(a-1)^{a-1} \frac{1}{a} c(a,a)\}^{i-1}.
\]
In particular, \( \Delta_i[a](g_{i,a}) \neq 0 \) for \( 1 \leq a \leq e-1 \).

**Proof.** First we note that the operator \( \Delta^{[a]}_i \) affects only the variables \( x_i \) and \( x_{i-1} \) and leaves all the others unchanged. Therefore we have
\[
(3.6.1) \quad \Delta_i[a](g_{i,a}) = (x_1 \cdots x_{i-1})^a \Delta^{[a]}_x(x_1^a).
\]
But we have \( \Delta^{[a]}_x(x_1^a) = (-1)^{a-1} \frac{1}{a} c(a,a) \) by Lemma 3.3 (ii). Hence the right hand side of (3.6.1) can be written as \( \gamma g_{i-1,a} \) with \( \gamma = (-1)^{a-1} \frac{1}{a} c(a,a) \). Repeating this procedure for the operators \( \Delta^{[a]}_{i-1}, \ldots, \Delta^{[a]}_2 \) we obtain the result. \( \square \)

3.7. Let \( \mathcal{M} = [0, e-1]^{n-1} \) (\( n-1 \) copies of the interval \([0, e-1]\)). For each \( \lambda = (\lambda_2, \ldots, \lambda_n) \in \mathcal{M} \), we define an operator \( \Delta_\lambda \) on \( S(V) \) by
\[
\Delta_\lambda = \Delta_n[\lambda_n] \cdots \Delta_2[\lambda_2].
\]
Also for \( \lambda \in \mathcal{M} \) we define a polynomial \( P_\lambda(x) \) by \( P_\lambda = \prod_{i=2}^n g_{i,\lambda_i} \). Let \( \lambda = (\lambda_2, \cdots, \lambda_n), \mu = (\mu_2, \cdots, \mu_n) \in \mathcal{M} \). We define a total order \( \lambda > \mu \) on \( \mathcal{M} \) by \( \lambda_2 = \mu_2, \ldots, \lambda_{i-1} = \mu_{i-1} \) and \( \lambda_i > \mu_i \) for some \( i \geq 1 \). Then we have the following proposition.
Proposition 3.8. Let $\lambda, \mu \in \mathcal{M}$. Then there exists a non-zero element $c_\lambda \in \mathbb{C}$ such that

$$\Delta_\lambda(P_\mu) = \begin{cases} c_\lambda & \text{if } \lambda = \mu, \\ 0 & \text{if } \lambda > \mu. \end{cases}$$

Proof. First we note that $\Delta_i[\lambda_i]$ leaves $g_{i, \mu_i} = (x_1 \cdots x_{i-1})^{\mu_i}$ invariant for $j < i$. In fact, $\Delta_i[\lambda_i]$ consists of various products of the operators $\Delta_{s_1}, \ldots, \Delta_{s_j}, \Delta_{s'_j}, \ldots, \Delta_{s'_{i-1}}$ and these operators leave $g_{i, \mu_i}$ invariant, since $s_j$ and $s'_j$ stabilize $x_{j-1}x_j = y_{j-1}y_j$ (in the notation of 3.1).

First assume that $\lambda = \mu$. Then by Lemma 3.6 $\Delta_i[\lambda_i](g_{i, \lambda_i})$ is a non-zero constant for each $i$. Combining with the above remark, we see that

$$\Delta_\lambda(P_\lambda) = \prod_{i=2}^n \Delta_i[\lambda_i](g_{i, \lambda_i}),$$

and the right hand side is a non-zero constant, which we write as $c_\lambda$.

Next assume that $\lambda > \mu$. Then there exists $i$ such that $\lambda_2 = \mu_2, \ldots, \lambda_{i-1} = \mu_{i-1}$ and $\lambda_i > \mu_i$. Then we have

$$\Delta_\lambda(P_\mu) = c \Delta_n[\lambda_n] \cdots \Delta_i[\lambda_i](\prod_{j=i}^n g_{j, \mu_j}),$$

with some $c \in \mathbb{C} - \{0\}$ by a similar argument as in the previous case. But then

$$\Delta_i[\lambda_i](\prod_{j=i}^n g_{j, \mu_j}) = (\prod_{j=i+1}^n g_{j, \mu_j}) \Delta_i[\lambda_i](g_{i, \mu_i}),$$

and $\Delta_i[\lambda_i](g_{i, \mu_i}) = 0$, since $\Delta_i[\lambda_i]$ reduces the degree by $(i-1)\lambda_i$, which is bigger than the degree of $g_{i, \mu_i}$. Hence $\Delta_\lambda(P_\mu) = 0$. \hfill $\square$

3.9. Let $\mathcal{D}_W$ be the subalgebra of $\text{End}_{\mathbb{C}} S(V)$ generated by $\Delta_i (s \in S)$ and $\alpha^* (\alpha \in V)$, where $\alpha^* : S(V) \to S(V)$ denotes the multiplication by the vector $\alpha$. Then $\mathcal{D}_W$ becomes a left $S(V)$-module. We also note that for any $w \in W$ the endomorphism $w$ on $S(V)$ is contained in $\mathcal{D}_W$, since $s_\alpha = 1 - \alpha^* \Delta_\alpha \in \mathcal{D}_W$ for any $s_\alpha \in S$. Since $\Delta_{s'_i} = w \Delta_{s'_i} w^{-1}$ for some $w \in S_n$, we see that $\Delta_{s'_i}$ ($2 \leq i \leq n$) are also contained in $\mathcal{D}_W$. Therefore $\Delta_\lambda \in \mathcal{D}_W$ for any $\lambda \in \mathcal{M}$.

As a corollary to Proposition 3.8 we have the following theorem. The proof is immediate from Proposition 3.8.

Theorem 3.10. The set $\{\Delta_\lambda | \lambda \in \mathcal{M}\}$ of operators in $\mathcal{D}_W$ is linearly independent over $S(V)$. 

3.11. In the case of \( \tilde{W} = G(e, 1, n) \), the operator \( \Delta_w \) was constructed in [RS] for each \( w \in \tilde{W} \) by making use of a particular reduced expression of \( w \). Here \( \Delta_w \) is an operator which reduces the grading by \( n(w) \). In our case, the operators \( \Delta_\lambda \) with \( \lambda \in \mathcal{M} \) are not directly related to the elements of \( W \). However, one gets a bijection between the set \( \{ \Delta_\lambda | \lambda \in \mathcal{M} \} \) and the set \( \mathcal{N} \) in \( W \) as follows. For each \( 0 < a \leq e \), we set

\[
\varphi(a) = \begin{cases} 
2a - 1 & \text{if } 0 < a \leq e/2, \\
2e - 2a & \text{if } e/2 < a \leq e.
\end{cases}
\]

Then the map \( \varphi \) gives rise to a bijection from the set \([1, e]\) to the set \([0, e - 1]\), and one can define a bijection \( \tilde{\varphi} : \mathcal{N} \to \mathcal{M} \) by \( \tilde{\varphi}(w) = (\varphi(a_2), \ldots, \varphi(a_n)) \). Hence the set \( \{ \Delta_\lambda | \lambda \in \mathcal{M} \} \) is in bijection with the set \( \mathcal{N} \). It is easily checked, by using (2.2.2), that if \( \lambda \in \mathcal{M} \) corresponds to \( w \in \mathcal{N} \), then \( \Delta_\lambda \) reduces the degree by \( \ell(w) \).

3.12. In the case of \( \tilde{W} \), it was shown in [RS, Prop. 2.14] that \( D_{\tilde{W}} \) is a free \( S(V) \)-module with basis \( \{ \Delta_w | w \in \tilde{W} \} \). In order to obtain a similar result for \( W \), we try to construct operators \( \Delta_w \) for any \( w \in W \). In view of Proposition 2.3, any element \( w \in W \) can be expressed uniquely as \( w = w'w'' \), with \( w' \in \mathcal{N}, w'' \in S_n \) with \( \ell(w) = \ell(w') + \ell(w'') \). We now define \( \Delta_w \) \((w \in W)\) by \( \Delta_w = \Delta_\lambda \Delta_{w''} \), where \( \lambda \in \mathcal{M} \) is given by \( \lambda = \tilde{\varphi}(w') \). (Note that the operator \( \Delta_{w''} \) corresponding to \( w'' \in S_n \) is defined without ambiguity, see 2.5.)

We know, by Theorem 3.10, that the set \( \{ \Delta_\lambda | \lambda \in \mathcal{M} \} \) is linearly independent over \( S(V) \). It is also known that the set \( \{ \Delta_{w''} | w'' \in S_n \} \) is linearly independent over \( S(V) \). We expect that the set \( \{ \Delta_w | w \in W \} \) gives rise to a basis of \( D_W \). In what follows, we show that this conjecture is reduced to some properties of \( \Delta_\lambda \). Here we prepare some notation. For each \( \lambda \in \mathcal{M} \) we define the length \( \ell(\lambda) \) by \( \ell(\lambda) = \ell(\lambda') \) whenever \( \lambda \) corresponds to \( \lambda' \in \mathcal{N} \). Hence \( \ell(w) = \ell(\lambda) + \ell(w'') \) if \( w \in W \) corresponds to the pair \( (\lambda, w'') \in \mathcal{M} \times S_n \). For each integer \( c \geq 1 \), we put \( \mathcal{M}_c = \{ \lambda \in \mathcal{M} | \ell(\lambda) = c \} \).

For each polynomial \( P_\lambda \) \((\lambda \in \mathcal{M})\) given in 3.7, we define its average \( \bar{P}_\lambda \) over \( S_n \) by \( \bar{P}_\lambda = \sum_{\sigma \in S_n} \sigma(P_\lambda) \). Note that \( \Delta_\lambda(\bar{P}_\mu) \) is a constant if \( \lambda, \mu \in \mathcal{M}_c \) for some \( c \). Let \( \lambda_0 = (e - 1, \cdots, e - 1) \in \mathcal{M} \). Then \( \lambda_0 \) is the longest element in \( \mathcal{M} \) with \( \ell(\lambda_0) = n(n - 1)(e - 1)/2 \). We consider the following two statements.

(3.12.1) \( \Delta_{\lambda_0}(\bar{P}_{\lambda_0}) \) is a non-zero constant.

(3.12.2) For any integer \( c \geq 1 \), the matrix \( \Delta_\lambda(\bar{P}_\mu) \) is non-singular.

We don’t know whether these two statements hold in a full generality for \( W \). It is verified that (3.12.1) holds whenever \( e \geq n \), which will be discussed in Theorem 3.14. In the case where \( n = 3 \) it is checked that (3.12.2) holds.
for small $e$. Note that (3.12.1) is a special case of (3.12.2), since the set $\mathcal{M}_c$ consists of a single element $\lambda_0$ if $c = \ell(\lambda_0)$.

**3.13.** In order to look at $P_\lambda$ more precisely, we shall extend the parameter set $\mathcal{M}$ to $\mathbb{N}^{n-1}$. For each $\lambda = (\lambda_2, \cdots, \lambda_n) \in \mathbb{N}^{n-1}$, we define a polynomial $F_n(\lambda)$ by $F_n(\lambda) = \prod_{i=2}^{n} g_{i, \lambda_i}$. Hence if $\lambda \in \mathcal{M}$, $F_n(\lambda)$ coincides with $P_\lambda$. We put $\bar{F}_n(\lambda) = \sum_{\sigma \in S_n} \sigma(F_n(\lambda))$.

For each $i$ ($1 \leq i \leq n$), let

$$
\sigma_i = \begin{pmatrix}
1 & 2 & \cdots & i & i+1 & i+2 & \cdots & n \\
1 & 2 & \cdots & n & i & i+1 & \cdots & n-1
\end{pmatrix} \in S_n.
$$

Then $\{\sigma_1, \cdots, \sigma_n\}$ is a complete set of representatives of the right cosets $S_{n-1}\setminus S_n$. For each $\mu = (\mu_2, \cdots, \mu_n) \in \mathbb{N}^{n-1}$, we define $\mu^{(i)} \in \mathbb{N}^{n-2}$, ($2 \leq i \leq n - 1$) by

$$
\mu^{(i)} = (\mu_2, \cdots, \mu_{i-1}, \mu_i + \mu_{i+1}, \mu_{i+2}, \cdots, \mu_n).
$$

Also we put $\mu^{(1)} = (\mu_3, \cdots, \mu_n) \in \mathbb{N}^{n-2}$ and $\mu^{(n)} = (\mu_2, \cdots, \mu_{n-1}) \in \mathbb{N}^{n-2}$. Then it is easy to see that

$$
(3.13.1) \quad \sigma_i(F_n(\mu)) = \begin{cases}
F_{n-1}(\mu^{(i)}) \cdot b_i(\mu) & \text{if } 1 \leq i \leq n - 1, \\
F_{n-1}(\mu^{(n)}) \cdot (x_1 \cdots x_{n-1})^{\mu_n} & \text{if } i = n,
\end{cases}
$$

where $b_i(\mu) = \mu_i + 1 + \cdots + \mu_n$ for $i = 1, \cdots, n - 1$. It follows from (3.13.1) that

$$
\sum_{\sigma \in S_{n-1}} \sigma \sigma_i F_n(\mu) = \begin{cases}
\bar{F}_{n-1}(\mu^{(i)}) \cdot b_i(\mu) & \text{if } 1 \leq i \leq n - 1, \\
\bar{F}_{n-1}(\mu^{(n)}) \cdot (x_1 \cdots x_{n-1})^{\mu_n} & \text{if } i = n.
\end{cases}
$$

Hence we have a recursive formula,

$$
(3.13.2) \quad \bar{F}_n(\mu) = \sum_{i=1}^{n-1} \bar{F}_{n-1}(\mu^{(i)}) b_i(\mu) + \bar{F}_{n-1}(\mu^{(n)}) (x_1 \cdots x_{n-1})^{\mu_n}.
$$

Let $\mathcal{M}' = [0, e-1]^{n-2}$ be the set corresponding to the situation in $G(e, e, n-1)$. Then for $\lambda = (\lambda_2, \cdots, \lambda_n) \in \mathcal{M}$, the operator $\Delta_\lambda$ can be written as $\Delta_\lambda = \Delta_n[\lambda_n] \Delta_{\lambda'}$ with $\lambda' = (\lambda_2, \cdots, \lambda_{n-1}) \in \mathcal{M}'$. By applying $\Delta_\lambda$ to the formula (3.13.2), we obtain

$$
(3.13.3) \quad \Delta_\lambda(\bar{F}_n(\mu)) = \sum_{i=1}^{n-1} \Delta_n[\lambda_n](\Delta_{\lambda'}(\bar{F}_{n-1}(\mu^{(i)}))) \cdot b_i(\mu) \\
+ \Delta_n[\lambda_n](\Delta_{\lambda'}(\bar{F}_{n-1}(\mu^{(n)}))) \cdot (x_1 \cdots x_{n-1})^{\mu_n}.
$$

By making use of the formula (3.13.3), we can compute the value $\Delta_{\lambda_0}(\bar{P}_{\lambda_0})$ under a certain condition, which gives a partial answer to the conjecture (3.12.1).
**Theorem 3.14.** Assume that \( e \geq n \). Then \( \Delta_{\lambda_0}(P_{\lambda_0}) = c_{\lambda_0} \), where \( c_{\lambda_0} \) is given as in Proposition 3.8.

**Proof.** Since \( \lambda_0 = (e - 1, \ldots, e - 1) \in \mathcal{M} \), \( \Delta_{\lambda_0} \) can be written as \( \Delta_{\lambda_0} = \Delta_{n-1}[e-1] \Delta_{\lambda_0}' \), where \( \lambda_0' = (e - 1, \ldots, e - 1) \in \mathcal{M}' \). First we note the following

\[
(3.14.1) \text{Let } \mu = (\mu_2, \ldots, \mu_n) \in \mathbb{N}^{n-1}. \text{ Assume that } \mu_i \equiv 0 \pmod{e - 1} \text{ for all } i \text{ and that } e - 1 < \sum_i \mu_i < e(e - 1). \text{ Then we have } \Delta_{\lambda_0}(\bar{F}_n(\mu)) = 0.
\]

We prove (3.14.1) by induction on \( n \). We apply the formula (3.13.3) with \( \lambda = \lambda_0 \). Note that if \( \mu \) satisfies the assumption of (3.14.1), then \( \mu^{(i)} \) \((2 \leq i \leq n - 1)\) above also satisfies the same condition. Hence (3.13.3) implies, by induction hypothesis, that

\[
\Delta_{\lambda_0}(\bar{F}_n(\mu)) = \Delta_n[e - 1](\Delta_{\lambda_0'}(\bar{F}_{n-1}(\mu^{(1)}))) \cdot x_n^{b_1(\mu)} + \Delta_n[e - 1](\Delta_{\lambda_0'}(\bar{F}_{n-1}(\mu^{(n)}))) \cdot (x_1 \cdots x_{n-1})^{\mu_n}.
\]

Here we may assume that \( \mu^{(1)} = \lambda_0' \) or \( \mu^{(n)} = \lambda_0' \), since both of \( \Delta_{\lambda_0'}(\bar{F}_{n-1}(\mu^{(1)})) \) and \( \Delta_{\lambda_0'}(\bar{F}_{n-1}(\mu^{(n)})) \) are zero, otherwise. But if \( \mu^{(1)} = \lambda_0' \), then \( \bar{F}_1(\mu^{(n)}) = \bar{P}_{\lambda_0} \), and \( \Delta_{\lambda_0'}(\bar{P}_{\lambda_0}) \) is a constant. The same argument holds for the case \( \mu^{(n)} = \lambda_0' \). Therefore, in order to prove (3.14.1), we have only to show that

\[
(3.14.2) \Delta_n[e - 1]x_n^{b_1(\mu)} = 0, \quad (3.14.3) \Delta_n[e - 1](x_1 \cdots x_{n-1})^{\mu_n} = 0.
\]

The left hand side of (3.14.2) can be computed by making use of the formula in Lemma 3.4. In particular, it is divisible by \( c(e - 1, b_1(\mu)) \). We claim that \( c(e - 1, b_1(\mu)) = 0 \). In fact, by our assumption, \( b_1(\mu) = \mu_2 + \cdots + \mu_n \) can be written as \( b_1(\mu) = d(e - 1) \) for some \( d \) such that \( 1 < d < e \). Then there exists \( j \) \((1 \leq j \leq e - 2)\) such that \( b_1(\mu) - j \equiv 0 \pmod{e} \). This implies that \( c(e - 1, b_1(\mu)) = 0 \), and (3.14.2) holds. (3.14.3) can be proved in a similar way, by replacing \( b_1(\mu) \) by \( \mu_n \), and by using Lemma 3.3. Hence (3.14.1) is proved.

We now prove the theorem. We compute \( \Delta_{\lambda_0}(P_{\lambda_0}) \) by applying (3.13.3) with \( \lambda_0 = \mu \). Then \( \lambda_0^{(i)} \) \((2 \leq i \leq n - 1)\) satisfies the condition in (3.14.1), since \((n - 1)(e - 1) < e(e - 1)\) by our assumption. Hence, by applying (3.14.1), the terms corresponding to \( \mu^{(i)} \) \((2 \leq i \leq n - 1)\) vanish. It follows that

\[
\Delta_{\lambda_0}(P_{\lambda_0}) = \Delta_n[e - 1]x_n^{(n-1)(e-1)} \cdot \Delta_{\lambda_0'}(P_{\lambda_0'}) + \Delta_n[e - 1](x_1 \cdots x_{n-1})^{e-1} \cdot \Delta_{\lambda_0'}(\bar{P}_{\lambda_0}).
\]

But the first term of the sum goes to 0 by applying (3.14.2) with \( \mu = \lambda_0 \).

Since \((x_1 \cdots x_{n-1})^{e-1} = g_n, e-1\), the second term coincides with \( c_{\lambda_0} \), by Proposition 3.8. This proves the theorem. \( \square \)
3.15. Let \( u_0 \in S_n \) be as in 2.5, and let \( w_1 \in W \) be the element in \( W \) corresponding to \((\lambda_0, u_0) \in \mathcal{M} \times S_n\). Then \( w_1 \) is the longest element in \( W \) with \( \ell(w_1) = on(n - 1)/2 = N \), where \( N \) is the number of reflections in \( W \). Let \( Q_0 \) be as in 2.5. Then \( P_{\lambda_0}Q_0 \) is a polynomial of degree \( N \). Since \( P_\lambda \) is \( S_n \)-invariant, and \( \Delta_{u_0}(Q_0) = 1 \) by Proposition 2.6, we have

\[
\Delta_{\lambda_0}\Delta_{u_0}(P_{\lambda_0}Q_0) = \Delta_{\lambda_0}(P_{\lambda_0}) = c\lambda_0. 
\]

Before stating the next result, we prepare a simple lemma.

**Lemma 3.16.** Let \( \varepsilon : S(V) \to \mathbb{C} \) denotes the evaluation at 0. Let \( I_W \) be the ideal of \( S(V) \) defined in 2.3. Then for any \( w \in W \) we have

\[
\varepsilon \Delta_w(I_W) = 0
\]

**Proof.** Let \( f \) be an element of \( I_W \). Then \( f \) can be written as

\[
f = \sum_i u_i f_i,
\]

with \( u_i \in S(V) \), \( f_i \in S(V)_W \), where \( f_i \) is homogeneous of positive degree. Then applying \( \Delta_w \) to \( f \), we obtain

\[
\Delta_w(f) = \sum_i \Delta_w(u_i)f_i,
\]

since \( f_i \) is \( W \)-invariant. Here \( \Delta_w(u_i)f_i \) is a polynomial without a constant term. This implies that \( \varepsilon \Delta_w(f) = 0 \) and the lemma follows. \( \square \)

3.17. Let \( \varepsilon_W : W \to \{\pm 1\} \) be the sign character of \( W \). Let \( Q \) be the polynomial in \( \mathbb{C}[x_1 \cdots, x_n] \) defined by \( Q = \prod_{i>j}(x_i^e - x_j^e) \). Then \( \deg Q = N \), and up to scalar, \( Q \) coincides with the product of the eigenvectors attached to all the reflections in \( W \). It is easy to see that \( Q \) generates a one-dimensional representation of \( W \) affording \( \varepsilon_W \). We define an operator \( J : S(V) \to S(V) \) by

\[
J = \sum_{w \in W} \varepsilon_W(w)w.
\]

Then \( J \) is a projection on the \( \varepsilon_W \)-isotypic subspace of \( S(V) \). We have the following remarkable result, although it is not used in the later discussion.

Note that it is an analogue of [H, IV, Prop. 1.6].

**Proposition 3.18.** Assume that \( e \geq n \). Then there exists a non-zero constant \( d \) such that \( \Delta_{w_1} = dQ^{-1}J \).
It is known that $S_W$ is a regular $W$-module, and $S_W^N$ affords the sign representation of $W$. Hence we have

$$S_W^N(V) = (I_W)^N + \mathbb{C}Q,$$

where $(I_W)^N = I_W \cap S_W^N(V)$. Now $\tilde{P}_{\lambda_0}Q_0 \in S_W^N(V)$, and (3.15.1) implies, in view of Lemma 3.16, that $\tilde{P}_{\lambda_0}Q_0 \notin I_W$. Hence there exists a non-zero constant $c' \in \mathbb{C}$ such that $Q \equiv c\tilde{P}_{\lambda_0}Q_0 \pmod{I_W}$. In particular, we have $\Delta_{w_1}(Q) = c$ with $c = c'c_{\lambda_0}$, by Theorem 3.14. Since $\Delta_{w_1}$ and $Q^{-1}J$ are $S(W)^W$-endomorphisms of $S(V)$, both of them are determined by the restriction to $S_W^N(V)$. Hence, by comparing the value at $Q$, we see that $\Delta_{w_1} = dQ^{-1}J$ with $d = c/|W|$. This proves the proposition. \hfill $\Box$

### 3.19

We now return to the condition (3.12.2). We deduce several properties of the operators $\Delta_w$ by assuming this condition. Note that for any $\lambda, \mu \in \mathcal{M}_c$, the polynomial $\Delta_\lambda \Delta_{w_0}(\tilde{P}_\mu Q_0)$ is a constant.

We denote by $A_c$ the matrix $(\Delta_\lambda \Delta_{w_0}(\tilde{P}_\mu Q_0))_{\lambda, \mu \in \mathcal{M}_c}$, under a suitable order, for a given integer $c \geq 0$. Then since $\Delta_\lambda \Delta_{w_0}(\tilde{P}_\mu Q_0) = \Delta_\lambda (\tilde{P}_\mu)$ by a similar argument as in (3.15.1), we see that

\[(3.19.1) \text{ Assume that (3.12.2) holds for } W. \text{ Then the matrix } A_c \text{ is nonsingular.}\]

We have the following lemma.

**Lemma 3.20.** Assume that (3.12.2) holds for $W$. Then the operators $\{\Delta_\lambda \Delta_w | \lambda \in \mathcal{M}, w \in S_n\}$ are linearly independent over $S(V)$.

**Proof.** We consider the dependence relation

\[(3.20.1) \sum_{\lambda, w} a(\lambda, w) \Delta_\lambda \Delta_w = 0\]

on $S(V)$, where $a(\lambda, w) \in S(V)$. By induction on the length $\ell(w)$ of $w \in S_n$, we may assume that $a(\lambda, w') = 0$ for any $w' \in S_n$ such that $\ell(w') < \ell(w)$ and for $\lambda \in \mathcal{M}$. Multiplying $\Delta_{w^{-1}w_0}$ to the equation (3.20.1) from the right, and by making use of Proposition 2.7 together with induction hypothesis, we obtain

\[(3.20.2) \sum_{\lambda \in \mathcal{M}} a(\lambda, w) \Delta_\lambda \Delta_{w_0} = 0.\]

We show that $a(\lambda, w) = 0$ by induction on the length of $\mathcal{M}$. Assume that $a(\mu', w) = 0$ for any $\mu' \in \mathcal{M}$ such that $\ell(\mu') < c$. We evaluate the equation (3.20.2) at $\tilde{P}_\mu Q_0$ for $\mu \in \mathcal{M}_c$. Note that $\Delta_\lambda \Delta_{w_0}(\tilde{P}_\mu Q_0) = 0$ if $\ell(\lambda) > c$.\hfill $\Box$
Hence the non-zero contribution only comes from the terms corresponding to \( \lambda \in \mathcal{M}_c \). We consider such equations for all \( \mu \in \mathcal{M}_c \). Then it is regarded as a linear equation with variables \( a(\lambda, w) (\lambda \in \mathcal{M}_c) \), and with coefficient matrix \( A_c \). Since the matrix \( A_c \) is non-singular by (3.19.1), we see that \( a(\lambda, w) = 0 \) for any \( \lambda \in \mathcal{M}_c \). This proves the lemma.

We can now prove the following proposition, which is analogous to proposition 2.14 in [RS].

**Proposition 3.21.** Assume that (3.12.2) holds. Then the algebra \( \mathcal{D}_W \) is a free \( S(V) \)-module with basis \( \{ \Delta_w \mid w \in W \} \).

**Proof.** Let \( K \) be the quotient field of \( S(V) \). The operator \( \Delta_\alpha \) on \( S(V) \) can be extended to an operator on \( K \). We consider the subalgebra \( \mathcal{D}_W^K \) of \( \text{End}_K K \) defined by \( \mathcal{D}_W^K = K \otimes_{S(V)} \mathcal{D}_W \). Since \( \dim_K \mathcal{D}_W^K \leq |W| \), Lemma 3.20 implies that

\[(3.21.1) \text{ The set } \{ \Delta_w \mid w \in W \} \text{ gives a basis of } \mathcal{D}_W^K \text{ as a } K\text{-vector space.}\]

By a similar argument as in the proof of Lemma 2.14 in [RS], the proof of the proposition is reduced to showing the following lemma.

**Lemma 3.22.** Let \( \Delta \) be a \( d \)-product of \( \Delta_s \) \((s \in S)\). Then \( \Delta \) can be written as

\[
\Delta = \sum_{w \in W} a_w \Delta_w,
\]

where \( a(w) \) are elements in \( S(V) \) satisfying the following conditions.

\[(3.22.1) \begin{cases} 
a_w = 0 & \text{if } \ell(w) < d, \\
a_w \in S^{(\ell(w)-d)}(V) & \text{if } \ell(w) \geq d.
\end{cases}\]

We prove Lemma 3.22. Here we recall that any \( \Delta_{w'} \) \((w' \in W)\) can be written as \( \Delta_{w'} = \Delta_\lambda \Delta_w \) with \( \lambda \in \mathcal{M}, w \in S_n \). Hence by (3.21.1) \( \Delta \) can be expressed as

\[(3.22.2) \Delta = \sum_{\lambda \in \mathcal{M}, w \in S_n} a(\lambda, w) \Delta_\lambda \Delta_w,
\]

with \( a(\lambda, w) \in K \). We write \( a(\lambda, w) = a_{w'} \) if \( w' \in W \) corresponds to \((\lambda, w)\). We shall prove that \( a(\lambda, w) \) satisfies the condition (3.22.1) by induction on the length \( \ell(\lambda) \) of \( \mathcal{M} \), and on the length \( \ell(w) \) of \( S_n \). We fix \( w \in S_n \) and assume that (3.22.1) is verified for any \( a(\lambda', w') \) such that \( \lambda' \in \mathcal{M} \) and that \( w' \in S_n \) with \( \ell(w') < \ell(w) \). Also we assume that it is verified for any \( a(\mu', w) \) such
that $\ell(\mu') < c$ for an integer $c \geq 0$. We show that $a(\lambda, w)$ satisfies (3.22.1) for any $\lambda \in \mathcal{M}_c$. By multiplying $\Delta_{w^{-1}w_0}$ on both sides of (3.22.2) from the right, we have

$$\Delta_{w^{-1}w_0} = \sum_{\lambda \in \mathcal{M}} a(\lambda, w)\Delta_\lambda \Delta_{w_0} + \sum_{\lambda', w'} a(\lambda', w')\Delta_{\lambda'} \Delta_{w''},$$

where in the second sum, $\lambda'$ runs over all the elements in $\mathcal{M}$, and $w'$ in $S_n$ such that $\ell(w') < \ell(w)$. Here $w'' \in S_n$ is given by $w'' = w'w^{-1}w_0$ with $\ell(w'') = \ell(w') - \ell(w) + \ell(w_0)$. We evaluate the equation (3.22.3) at $P_\mu Q_0$, with $\mu \in \mathcal{M}_c$, which is a polynomial of degree $c + \ell(w_0)$. Then the non-zero contribution in the first sum comes from the terms corresponding to $\lambda \in \mathcal{M}_1$, where

$$\mathcal{M}_1 = \{\lambda \in \mathcal{M} | \ell(\lambda) \leq c\}.$$

First assume that $c + \ell(w) < d$. Then for any $\lambda \in \mathcal{M}_1$, we have $\ell(\lambda) < d$. Hence by induction hypothesis, we have $a(\lambda, w) = 0$ for $\lambda \in \mathcal{M}_1$ such that $\ell(\lambda) < c$. On the other hand, again by induction hypothesis, $a(\lambda', w')\Delta_{\lambda'} \Delta_{w''}(P_\mu Q_0)$ is a homogeneous polynomial of degree $c + \ell(w) - d < 0$. This means that there are no contributions from the terms in the second sum, and we have

$$\Delta_{w^{-1}w_0}(P_\mu Q_0) = \sum_{\lambda \in \mathcal{M}_c} a(\lambda, w)\Delta_\lambda \Delta_{w_0}(P_\mu Q_0).$$

Since $d + \ell(w^{-1}w_0) > \ell(\mu) + \ell(w_0)$, we have $\Delta_{w^{-1}w_0}(P_\mu Q_0) = 0$. This implies that $a(\lambda, w) = 0$ for any $\lambda \in \mathcal{M}_c$, since the matrix $A_c$ is non-singular by (3.19.1). Next assume that $c + \ell(w) \geq d$. Take $\lambda \in \mathcal{M}$ such that $\ell(\lambda) < c$. Then by induction hypothesis, $a(\lambda, w)$ is a homogeneous polynomial of degree $\ell(\lambda) + \ell(w) - d$ for such $\lambda$, if it is positive, and $a(\lambda, w) = 0$ if $\ell(\lambda) + \ell(w) - d < 0$. Hence $a(\lambda, w)\Delta_\lambda \Delta_{w_0}(P_\mu Q_0)$ is a homogeneous polynomial of degree $c + \ell(w) - d$, if it is non-zero. On the other hand, by a similar argument as before we see that the term in the second sum $a(\lambda', w')\Delta_{\lambda'} \Delta_{w''}(P_\mu Q_0)$ is also a homogeneous polynomial of degree $c + \ell(w) - d$, if it is non-zero. Moreover, $\Delta_{w^{-1}w_0}(P_\mu Q_0)$ is a homogeneous polynomial of the same degree. Since the matrix $A_c$ is a non-singular $\mathbb{C}$-matrix, we see that $a(\lambda, w)$ is a homogeneous polynomial of degree $c + \ell(w) - d$ for any $\lambda \in \mathcal{M}_c$. This shows that $a(\lambda, w)$ satisfies the condition in (3.22.1). The lemma is now proved and the proposition follows.

The following lemma can be proved in a similar way as Lemma 2.16 in [RS], in view of [RS, Remark 2.10].

**Lemma 3.23.** Let $P$ be a homogeneous polynomial of degree $N$. Let $I$ be a graded ideal of $S(V)$ containing $I_W$, but not containing $P$. Then $I = I_W$. 

3.24 Let $S(V)^* \equiv \oplus_{i \geq 0} S^i(V)^*$, where $S^i(V)^*$ denotes the dual space of $S^i(V)$ over $\mathbb{C}$. We have a natural pairing $\langle \cdot, \cdot \rangle : S(V) \times S(V)^* \to \mathbb{C}$, $\langle u, f \rangle = f(u)$. Let $\varepsilon : S(V) \to \mathbb{C}$ denote the evaluation at 0. Then for each $\Delta \in D_W$ we can regard $\varepsilon \Delta$ as an element in $S(V)^*$. Let $\tilde{D}_W$ be the subspace of $S(V)^*$ generated by $\varepsilon \Delta$ with $\Delta \in D_W$. Let $H_W$ be the dual space of $D_W$. Then we have a natural map $c : S(V) \to H_W$, which sends $u \in S(V)$ to the restriction to $\tilde{D}_W$ of the map $\langle u, \cdot \rangle : S(V) \to \mathbb{C}$. We can now state the main theorem, which is an analogue of [RS, Th. 2.18].

**Theorem 3.25.** Assume that the conjectures (3.12.1) and (3.12.2) hold for $W$. Then there exists a unique graded $\mathbb{C}$-algebra structure on $H_W$ such that $c$ induces an isomorphism $S_W \cong H_W$. The set $\{\varepsilon \Delta_w | w \in W\}$ gives a basis of the $\mathbb{C}$-vector space $\tilde{D}_W$. In particular, if we denote by $\{X_w | w \in W\}$ the dual basis of $\{\varepsilon \Delta_w | w \in W\}$, the map $c$ can be described, for $u \in S(V)$, as

$$c(u) = \sum_{w \in W} \varepsilon \Delta_w(u) X_w.$$ 

**Proof.** It follows from proposition 3.21 that $\{\varepsilon \Delta_w | w \in W\}$ gives rise to a basis of $D_W$. Since $\dim S_W = |W|$, in order to prove the theorem it is enough to prove that $\ker c = I_W$. Since $D_W$ has a structure of a right $S(V)$-module, we see that $\ker c$ is a graded ideal of $S(V)$. It also follows from Lemma 3.16 that $I_W \subset \ker c$. Now (3.12.1) asserts that $\Delta_{\lambda_0} \Delta_{\omega_0}(P_{\lambda_0} Q_0) \neq 0$ (see (3.15.1)). Hence $P_{\lambda_0} Q_0$ is a polynomial with $\deg P_{\lambda_0} Q_0 = N$, which is not contained in $I$. Then one can apply Lemma 3.23 with $P = P_{\lambda_0} Q_0$ and we conclude that $I = I_W$. This proves the theorem. 

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**References**


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