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VORONOI–ALGORITHM EXPANSION OF A FAMILY WITH PERIOD LENGTH GOING TO INFINITY

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Abstract. We consider a family of orders of complex cubic fields which is similar to one introduced by Levesque and Rhim. We find the Voronoi-algorithm expansions and the fundamental units.

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§ 1. Introduction

Levesque and Rhin [4] introduced two families of complex cubic fields $\mathbb{Q}(\alpha)$, each of which depends on two parameters. Adam [1] obtained the Voronoi-algorithm expansions of the order $\mathbb{Z}[\alpha]$ for these two families, for one of which Kühner [3] also found the Voronoi-algorithm expansions.

In this paper we shall consider a new family of complex cubic fields, similar but different from those families above, *i.e.* $\mathbb{Q}(\alpha)$, where α is the real root of the irreducible cubic polynomial $f(X)$ in Proposition 1.1.
We obtain the following results :
the Voronoi-algorithm expansions of the order $\mathbb{Z}[\alpha]$,
the period length of these expansions goes to infinty,
the fundamental units of the order $\mathbb{Z}[\alpha]$.
Our method, in which we use an isotropic vector of the quadratic form,
is due to Adam [1] .

Proposition 1.1. *Let* $f(X) = X^3 - c^m X^2 + (c+1)X - c^m$, *where* m, c *are intergers such that* $m \geqq 1$ *and* $c \geqq 2$. *Then* $f(X)$ *has only one real root* α *and* $f(X)$ *is irreducible except the case* $m = 1, c = 2$.
Moreover if $m \geqq 2$, *then* α *satisfies*

$$c^m - \frac{1}{c^{m-1}} - \frac{1}{c^{m+2}} < \alpha < c^m - \frac{1}{c^{m-1}} \quad . \tag{1.1}$$

Proof. Since the discriminant of $f(X)$ is

$$D_f = -\{4c^{4m} - (c^2 + 20c - 8)c^{2m} + 4(c+1)^3\} < 0 \ ,$$

$f(X)$ has only one real root α .
Since

$$f(c^m - \frac{2}{c^{m-1}}) = -c^{m+1} + \frac{6}{c^{m-2}} - \frac{2}{c^{m-1}} - \frac{8}{c^{3m-3}} < 0 \quad \text{and}$$
$$f(c^m - \frac{1}{c^{m-1}}) = \frac{c-1}{c^{m-1}} - \frac{1}{c^{3m-3}} > 0 \qquad ((m,\, c) \neq (1,\, 2)) \ ,$$
$$c^m - \frac{2}{c^{m-1}} < \alpha < c^m - \frac{1}{c^{m-1}} \qquad ((m,\, c) \neq (1,\, 2)) \ .$$

Therefore if $(m,\, c) \neq (1,\, 2)$, then $f(X)$ is irreducible.
Furthermore we have

$$f(c^m - \frac{1}{c^{m-1}} - \frac{1}{c^{m+2}}) = -\, c^{m-2} + \frac{1}{c^{m-2}} - \frac{1}{c^{m-1}} + \frac{3}{c^{m+1}} - \frac{1}{c^{m+2}} + \frac{2}{c^{m+4}} - \frac{1}{c^{3m-3}} - \frac{3}{c^{3m}} - \frac{3}{c^{3m+3}} - \frac{1}{c^{3m+6}} < 0 \qquad (m \geqq 2) \ .$$

Hence if $m \geqq 2$, then

$$c^m - \frac{1}{c^{m-1}} - \frac{1}{c^{m+2}} < \alpha < c^m - \frac{1}{c^{m-1}} \ . \qquad \square$$

§ 2. Voronoi-algorithm and preliminaries

Let K be a cubic algebraic number field of negative discriminant. Let $1, a_1, \alpha_2 \in K$ be rationally independent. We say that $\mathcal{R} = [1,\, \alpha_1,\, \alpha_2] = \mathbb{Z} + \mathbb{Z}.\alpha_1 + \mathbb{Z}.\alpha_2$ is a lattice of K with basis $\{1,\, \alpha_1,\, \alpha_2\}$. For $\omega \in \mathcal{R}$ we define $F(\omega) = \frac{N_K(\omega)}{\omega} = \omega'\omega''$, where N_K denotes the norm of K over $\mathbb{Q}$, and ω' and ω'' the conjugates of ω.

Definition 2.1. Let $\mathcal{R}$ be a lattice of K , and let $\omega(>0) \in \mathcal{R}$. We say that ω is a *minimal point* of $\mathcal{R}$ if for all φ in $\mathcal{R}$ so that $0 < \varphi < \omega$ we have $F(\varphi) > F(\omega)$. Let $\omega, \varphi \in \mathcal{R}$ such that $0 < \omega, \varphi$. We say that ω is a minimal point adjacent on the right (further on,we will not specify "right") to φ in $\mathcal{R}$ if $\omega = min\{\ \psi \in \mathcal{R}\ ;\ \varphi < \psi,\, F(\varphi) > F(\psi)\ \}$. We define the increasing chain of the minimal points of $\mathcal{R}$ by :

$$\theta_0 = 1\ ,\ \theta_{k+1} = min\{\ \psi \in \mathcal{R}\ ;\ \theta_k < \psi,\, F(\theta_k) > F(\psi)\ \} \quad \text{if} \quad k \geqq 0\ .$$

Then θ_{k+1} is the minimal point adjacent (on the right) to θ_k in $\mathcal{R}$. Let $\mathcal{O}$ be any order of K and $\mathcal{R} = \mathcal{O}$. By Voronoi we know that the previous chain is of purely periodic form :

$$\theta_0 = 1,\ \theta_1, \cdots, \theta_{l-1},\ \theta_l = \varepsilon,\ \varepsilon\theta_1, \cdots, \varepsilon\theta_{l-1}, \cdots ,$$

where l denotes the period length and ε is the fundamental unit of $\mathcal{O}$. To calculate such a sequence, it is sufficient to know how to find the minimal point adjacent to 1 in a lattice $\mathcal{R}$. Indeed, let ${\theta_g}^{(1)}$ be the minimal point adjacent to 1 in $\mathcal{R}_1 = \mathcal{O} = [1,\, \alpha_1,\, \alpha_2]$ and $\theta_1 = {\theta_g}^{(1)}$.

(i) We choose an appropriate point ${\theta_h}^{(1)}$ so that $\{1,\, {\theta_g}^{(1)},\, {\theta_h}^{(1)}\}$ is a basis of $\mathcal{R}_1$

(ii) ${\theta_g}^{(2)}$ is the minimal point adjacent to 1 in $\mathcal{R}_2 = \frac{1}{{\theta_g}^{(1)}}\mathcal{R}_1 =$ $[1,\, 1/{\theta_g}^{(1)},\, {\theta_h}^{(1)}/{\theta_g}^{(1)}]$ is equivalent to $\theta_2 = \theta_1{\theta_g}^{(2)} = {\theta_g}^{(1)}{\theta_g}^{(2)}$ being the minimal point adjacent to θ_1 in $\mathcal{R}_1$.

This process can be continued by induction.

We quote Adam[1],Lemma 2.2 from which we dropp one condition $F(0,0,1) > 1$ as Lemma 2.1 for our convenience.

Lemma 2.1(Adam[1],Lemma 2.2). *Let F be a positive quadratic form in three variables with real coefficients of rank 2 such that $F(1,\, 0,\, 0) = 1$. Suppose that F has an isotropic vector $(\omega_2,\, 1,\, \omega_1)$. Then we can write*

$$F(u,v,w) = a(w-\omega_1 v)^2 + 2b(w-\omega_1 v)(u-\omega_2 v) + (u-\omega_2 v)^2 \tag{2.1}$$

and

$$\begin{aligned} F(u,v,w) =& \frac{a}{2}\{w - (\omega_1 + 2\frac{b}{a}\omega_2)v + 2\frac{b}{a}u\}^2 + \frac{a}{2}(w-\omega_1 v)^2 \\ &+ (1 - 2\frac{b^2}{a})(u-\omega_2 v)^2 \end{aligned} \tag{2.2}$$

with $b^2 < a$.

Let $\mathcal{R}$ be a lattice of K with basis $\{1,\, \alpha_1,\, \alpha_2\}$ and $F(\omega) = \omega'\omega''(\omega \in \mathcal{R})$. For $(u,\, v,\, w) \in \mathbb{Z}^3$ we define $F(u,v,w) = F(u + v\alpha_1 + w\alpha_2) =$ $(u + v\alpha_1 + w\alpha_2)'(u + v\alpha_1 + w\alpha_2)''$. Further,we can consider F as a quadratic form in three variables with real coefficients. Then, F is positive, the rank of F is 2 and $F(1,\, 0,\, 0) = 1$. Hence we can write F in the form (2.1) and (2.2) with $a = \alpha_2'\alpha_2''$, $b = \frac{1}{2}(\alpha_2' + \alpha_2'')$. We find ω_1 and ω_2 by the formulas

$$\omega_1 = -\frac{\alpha_1' - \alpha_1''}{\alpha_2' - \alpha_2''},\ \omega_2 = -\frac{1}{2}\{(\alpha_1' + \alpha_1'') + \omega_1(\alpha_2' + \alpha_2'')\}\ .$$

§ 3. Main theorem and preliminary results

Let $f(X) = X^3 - c^m X^2 + (c+1)X - c^m$, where m, c are intergers such that $m {\geqq} 2$ and $c {\geqq} 2$. By Proposition 1.1 $f(X)$ is irreducible and has only one real root.

Theorem 3.1. *Let α be the real root of the polynomial $f(X)$, $K = \mathbb{Q}(\alpha)$, and $\mathcal{O} = \mathbb{Z}[\alpha]$. Then*

(i) *The chain of the minimal points of $\mathcal{O}$ is : for $1 \leqq s \leqq m-1$*
$\theta_0 = 1$, $\theta_{3s-2} = (c^s + \alpha - c^m)(\frac{\alpha}{c^m-\alpha})^s$, $\theta_{3s-1} = (\frac{c\alpha}{c^m-\alpha})^s$,
$\theta_{3s} = \alpha(\frac{\alpha}{c^m-\alpha})^s$, $\theta_{3m-2} = \alpha(1+\alpha-c^m)(\frac{\alpha}{c^m-\alpha})^m$ *and*
$\theta_{3m-1} = \alpha(\frac{\alpha}{c^m-\alpha})^m$.

(ii) $\varepsilon = \alpha(\frac{\alpha}{c^m-\alpha})^m$ *is the fundamental unit of $\mathcal{O}$ and Voronoi-algorithm expansion period length is $l = 3m-1$.*

Remark 3.2. The following relation holds among the minimal points of $\mathcal{O}$: $\theta_2 = \alpha\theta_0 + \theta_1$, $\theta_{3s-1} = \theta_{3s-3} + \theta_{3s-2}$ *for* $2 \leqq s \leqq m-1$, $\theta_{3m-1} = \alpha\theta_{3m-3} + \theta_{3m-2}$.

For the proof of Theorem 3.1, we prepare six lemmas .
In the following lemmas we denote θ_g the minimal point adjacent to 1 in a lattice $\mathcal{R}$ of K.

Lemma 3.3. *For an integer s, $1 \leqq s \leqq m-2$,*

$$\text{if} \quad \mathcal{R} = [1, \alpha - c^m + 1, \frac{c^s}{\alpha}], \quad \text{then} \quad \theta_g = \frac{c^s(\alpha^2+1) - \alpha}{\alpha}.$$

Proof. We can write F in the form (2.1) and (2.2) with
$a = \frac{c^{2s}}{c^m}\alpha$, $2b = \frac{\alpha(c^m-\alpha)}{c^{m-s}}$, $\omega_1 = \frac{c^{m-s}}{\alpha}$, $\omega_2 = \alpha - 1$.
By (1.1) we have

$$0 < \alpha_1 < 1,\ \ 0 < \alpha_2 < 1,\ \ 0 < \omega_1 < 1,\ \ [\omega_2] = c^m - 2,$$

$$c^{2s} - \frac{c^{2s}}{c^{2m-1}} - \frac{c^{2s}}{c^{2m+2}} < a < c^{2s} - \frac{c^{2s}}{c^{2m-1}},$$

where [...] is the greatest integer function. From the last inequality we have $c^2 - 1 < a$, then $a > 3$. Since $\frac{4b^2}{a} = \frac{\alpha(c^m-\alpha)^2}{c^m} < 1$, $4b^2 < a$.

Let $\theta_g = u + v\alpha_1 + w\alpha_2$.

Claim 1. $v \neq 0$ and $uv \geqq 0$.
Suppose that $v = 0$. If $u = 0$, then $F(\theta_g) = aw^2 > 1$. If $w = 0$, then $F(\theta_g) = u^2 \geqq 1$. If $u \neq 0$ and $w \neq 0$, then $F(\theta_g) > \frac{a}{2} + (1 - 2\frac{b^2}{a}) > 1$. Therefore $v \neq 0$. Since $F(\theta_g) < 1$ and $4b^2 < a$, we have $(u - \omega_2 v)^2 < 2$; but $\omega_2 \geqq \sqrt{2} - 1$, then $uv \geqq 0$. (cf. the proof of Adam [1],Proposition 2.3)

Claim 2. $u \geqq 0$, $v > 0$, $w \geqq 0$
Since $F(\theta_g) < 1$ and $a > 3$, we have $(w - \omega_1 v)^2 < 1$, then $wv \geqq 0$.
If $v < 0$, then $u \leqq 0$ and $w \leqq 0$, which is impossible because $\theta_g > 0$, so we have $v > 0$, $u \geqq 0$ and $w \geqq 0$.

Claim 3. $w > 0$.
Since $(w-\omega_1 v)^2 < 1$, $w = [\omega_1 v]$ or $[\omega_1 v]+1$. Since $F(\theta_g) < 1$ and $4b^2 < a$, we have $(u-\omega_2 v)^2 < 2$, then $u = [\omega_2 v] + i$, where $i = -1, 0, 1$ or 2.
Suppose that $w = 0$. Since in (2.2) $\frac{a}{2}(w-\omega_1 v)^2 = \frac{c^m}{2\alpha}v^2$, $v = 1$.
Since $u = [\omega_2] + i = c^m - 2 + i$ $(i = -1, 0, 1, 2)$, we have $\theta_g = u + \alpha_1$ $= \alpha - 1 + i$ $(i = -1, 0, 1, 2)$. Since $F(\theta_g) = \frac{c^m}{\alpha} + (c^m - \alpha)(i-1) + (i-1)^2 > 1$ $(i = -1, 0, 1, 2)$, $w > 0$.

Claim 4. For an integer v if $1 \leqq v < c^{m-1}$,
then $[v\alpha] = \begin{cases} vc^m - 1 & (v < \frac{c^{m+2}}{c^3+1}) \\ vc^m - 2 & (v \geqq \frac{c^{m+2}}{c^3+1}) \end{cases}$
By (1.1) we have $vc^m - (\frac{v}{c^{m-1}} + \frac{v}{c^{m+2}}) < v\alpha < vc^m - \frac{v}{c^{m-1}}$. From this, our claim is deduced because $\frac{v}{c^{m-1}} + \frac{v}{c^{m+2}} < 1$ is equivalent to $v < \frac{c^{m+2}}{c^3+1}$.

Claim 5. $\theta_g = [c^s\omega_2] + c^s\alpha_1 + \alpha_2$.
We shall show that $v \leqq c^s - 1$ implies that $F(\theta_g) > 1$. Suppose that $v \leqq c^s - 1$. Since $[\omega_1 v] = 0$, $w = [\omega_1 v] + 1 = 1$. So $\theta_g = u + v\alpha_1 + \alpha_2 = [\omega_2 v] + i + v\alpha_1 + \alpha_2 = i - 1 + v\alpha + \frac{c^s}{\alpha}$ $(i = -1, 0, 1, 2)$. we have

$$\text{(3.1)} \qquad \begin{aligned} F(\theta_g) = &(i-1)^2 + (i-1)(c^m - \alpha)(v + \frac{\alpha}{c^{m-s}}) \\ &+ \frac{v\alpha(c^m-\alpha)^2}{c^{m-s}} + \frac{c^m}{\alpha}(v - \frac{\alpha}{c^{m-s}})^2 \quad (i = -1, 0, 1, 2). \end{aligned}$$

Clearly if $i = 2$ and $v \leqq c^s - 1$, then $F(\theta_g) > 1$. We have

$$1 - (c^m - \alpha)(c^s - 1 + \frac{\alpha}{c^{m-s}}) = \frac{J}{c^{m-s}\alpha},$$

where $J = c\alpha\{c^{m-1-s} - 1 - (c^{m-1} - c^{m-1-s})(c^m - \alpha)\} + c^m - \alpha$. By (1.1) if $(s, c) \neq (m-2, 2)$, then we have $J > 0$, so $(c^m - \alpha)(c^s - 1 + \frac{\alpha}{c^{m-s}}) < 1$. Hence if $v \leqq c^s - 1$ and $(s, c) \neq (m-2, 2)$, then we have

$$\text{(3.2a)} \quad \begin{aligned} |(i-1)(c^m-\alpha)(v + \frac{\alpha}{c^{m-s}})| &\leqq |i-1|(c^m-\alpha)(c^s - 1 + \frac{\alpha}{c^{m-s}}) \\ &< |i-1| \qquad\qquad (i = -1, 0, 2). \end{aligned}$$

If $v \leqq c^s - 1$, then we have

$$\text{(3.2b)} \quad \begin{aligned} &|(i-1)(c^m-\alpha)(v + \frac{\alpha}{c^{m-s}})| \leqq |i-1|(c^m-\alpha)(c^s - 1 + \frac{\alpha}{c^{m-s}}) \\ &< |i-1|(c^m-\alpha)(2c^s - 1) < |i-1|(c^m-\alpha)2c^s \\ &< |i-1|2c^{m-2}(\frac{1}{c^{m-1}} + \frac{1}{c^{m+2}}) \leqq |i-1|(1 + \frac{1}{2^3}) \quad (i = -1, 0, 2). \end{aligned}$$

From (3.1), (3.2b) if $i=-1$ and $v \leqq c^s-1$, then $F(\theta_g)>1$.
If $i=1$, then we have

$$F(\theta_g)=\frac{v\alpha(c^m-\alpha)^2}{c^{m-s}}+\frac{c^m}{\alpha}(v-\frac{\alpha}{c^{m-s}})^2. \tag{3.3}$$

From (1.1) $\frac{c^m-\alpha}{c^{m-s}}<\frac{1}{c^{m+1}}+\frac{1}{c^{m+4}}<\frac{1}{2^3}+\frac{1}{2^6}$, so we have

$$\frac{c^m}{\alpha}(v-\frac{\alpha}{c^{m-s}})^2 \geqq \frac{c^m}{\alpha}(c^s-\frac{\alpha}{c^{m-s}}-2)^2>(\frac{c^m-\alpha}{c^{m-s}}-2)^2>3 \quad \text{if} \quad v \leqq c^s-2. \tag{3.4}$$

Hence if $i=1$ and $v \leqq c^s-2$, then $F(\theta_g)>1$.
From (3.3) if $i=1$ and $v=c^s-1$, then we have

$$F(\theta_g)=\frac{1}{c^{m-s}\alpha}\{(c^s-1)\alpha^2(c^m-\alpha)^2-2c^m(c^m-\alpha)+c^s(c^m-\alpha)^2\}+\frac{c^m}{\alpha}.$$

From this if $i=1$ and $v=c^s-1\geqq 2$, then $F(\theta_g)>1$ because
$(c^s-1)\alpha^2(c^m-\alpha)^2-2c^m(c^m-\alpha)\geqq 2\alpha^2(c^m-\alpha)^2-2c^m(c^m-\alpha)$
$=(c^m-\alpha)\{2(c+1)\alpha-4c^m\}>0$ and $\frac{c^m}{\alpha}>1$.
If $i=1$ and $v=c^s-1=1$ (*i.e.* $s=1, c=2$), then we have $F(\theta_g)>1$, because from (3.6) in the proof of Lemma 3.3 $\alpha(\alpha-c^m)+c>0$, thus we have
$F(\theta_g)=\frac{\alpha(c^m-\alpha)^2}{c^{m-1}}+\frac{c^m}{\alpha}(1-\frac{\alpha}{c^{m-1}})^2$
$=\frac{1}{c^{m-1}}\{\alpha(\alpha-c^m)+c\}\{c^{m-1}-(c^m-\alpha)\}+1>1$.
Therefore if $i=1$ and $v\leqq c^s-1$, then we have $F(\theta_g)>1$.
If $i=0$, then we have

$$F(\theta_g)=\frac{v\alpha(c^m-\alpha)^2}{c^{m-s}}+\frac{c^m}{\alpha}(v-\frac{\alpha}{c^{m-s}})^2+1-(c^m-\alpha)(v+\frac{\alpha}{c^{m-s}}). \tag{3.5}$$

From the case $i=1$ and (3.2a) if $i=0$, $v\leqq c^s-1$ and $(s, c)\neq(m-2, 2)$, then we have $F(\theta_g)>1$. If $i=0$, $v=c^s-1$ and $(s, c)=(m-2, 2)$, then we have

$$\begin{aligned}F(\theta_g)=&\frac{(c^{m-2}-1)\alpha(c^m-\alpha)^2}{c^2}+\frac{c^m}{\alpha}(\frac{c^m-\alpha-c^2}{c^2})^2+1\\&-\frac{(c^m-\alpha)(c^m+\alpha-c^2)}{c^2}\\=&\frac{1}{c^2\alpha}\{(c^{m-2}-1)\alpha^2(c^m-\alpha)^2+c^{m-2}(c^m-\alpha)+c(c^m-\alpha)(c\alpha-c^m)\\&+c^{m+2}-\alpha(c^m-\alpha)(c^m+\alpha)\}+1>1 .\end{aligned}$$

Therefore $v\leqq c^s-1$ implies that $F(\theta_g)>1$.

Now we shall consider the case $v = c^s$. We have

$$\begin{aligned}\theta_g &= [\omega_2 c^s] + i + c^s\alpha_1 + ([\omega_1 c^s] + j)\alpha_2 \\ &= c^{m+s} - c^s - 1 + i + c^s(\alpha - c^m + 1) + (1+j)\frac{c^s}{\alpha} \\ &= -1 + i + c^s\alpha + (1+j)\frac{c^s}{\alpha} \qquad (i = -1, 0, 1, 2), \quad (j = 0, 1) \ .\end{aligned}$$

If $i = -1$ and $j = 0$, then we have

$$F(-2 + c^s\alpha + \frac{c^s}{\alpha}) = 4 - 2c^s(c^m - \alpha)(1 + \frac{\alpha}{c^m}) - \frac{c^{2s}\alpha(c^m - \alpha)^2}{c^m}(1 + \frac{1}{\alpha^2}).$$

Since

$$2c^s(c^m - \alpha)(1 + \frac{\alpha}{c^m}) < 4c^s(c^m - \alpha) \leqq 4c^{m-2}(c^m - \alpha) < 4(\frac{1}{c} + \frac{1}{c^4}) < 3,$$

we have $F(\theta_g) > 1$. If $i = -1$ and $j = 1$, then we have

$$\begin{aligned}F(-2 + c^s\alpha + \frac{2c^s}{\alpha}) &= 4 - 4c^s(c^m - \alpha) + \frac{c^{2s}\alpha}{c^m} - \frac{2c^s\alpha(c^m - \alpha)}{c^m} \\ &\quad + \frac{2c^{2s}\alpha(c^m - \alpha)}{c^m}\{\alpha(c^m - \alpha) - 1\} + c^{2s}(c^m - \alpha)(\frac{1}{\alpha} - \frac{1}{c^m}) > 1.\end{aligned}$$

If $i = 0$ and $j = 0$, then we have $F(-1 + c^s\alpha + \frac{c^s}{\alpha}) = 1 + N$, where

$$N = -c^s(c^m - \alpha)(1 + \frac{\alpha}{c^m}) + \frac{c^{2s}\alpha(c^m - \alpha)^2}{c^m}(1 + \frac{1}{\alpha^2}).$$

Since

$$N = \frac{c^s(c^m - \alpha)}{c^m}\{-c^m - \alpha + c^s(c^m - \alpha)\alpha + \frac{c^s(c^m - \alpha)}{\alpha}\} < 0,$$

we have $F(\theta_g) < 1$. Therefore $\theta_g = [c^s\omega_2] + c^s\alpha_1 + \alpha_2$. □

Lemma 3.4. *For an integer s, $1 \leqq s \leqq m - 1$,*

$$\textit{if} \quad \mathcal{R} = [1, \frac{c^m - \alpha}{c^s + \alpha - c^m}, \frac{c^s + \alpha(\alpha - c^m)}{c^s + \alpha - c^m}], \quad \textit{then} \quad \theta_g = \frac{c^s}{c^s + \alpha - c^m}.$$

Proof. We can write F in the form (2.1) and (2.2) with
$a = \frac{1}{D}\{(c^{2m} + c^{2s} - c^{s+1} - c^s)\alpha + \frac{c^{2m}}{\alpha} - c^{m+s}\}, \quad 2b = -\frac{G}{D}$,
$\omega_1 = \frac{c^s + \alpha - c^m}{(c^m - c^s - 1)\alpha - c^{m-s+1} + c^m - c^s + c + 1}$,

$2\omega_2 = \frac{(2c^m - c^s)\alpha^2 - c^{m+s}\alpha + 2c^m}{D} + \omega_1(-2b)$,
where $D = \alpha(c^s + \alpha - c^m)'(c^s + \alpha - c^m)'' = (c^m - c^s)\alpha^2 - c^s(c^m - c^s)\alpha + c^m$,
and $G = c^s\alpha^2 + (c^{m+s} - 2c^{2s} + c^{s+1} - c^{m+1} + c^s)\alpha + c^{m+s}$.
By (1.1) we have $0 < \alpha_1 < 1$, $0 < \alpha_2 < 1$, $0 < \omega_1 < 1$, $\omega_2 > 1$, and $a > 1$.

We claim that $4b^2 < a$.
First we shall show that if $(s, c) \neq (m-1, 2), (m-1, 3)$, then $|2b| < 1$: this is equivalent to $D - G > 0$. We have

$$\begin{aligned} D - G =&(c^m\alpha - 2c^s\alpha - 2c^{m+s} - c^{s+1} - c^s + 3c^{2s} + c^{m+1})\alpha + c^m - c^{m+s} \\ =&\{c^m(c^m - 4c^s) + c^{2s} + (c^m - 2c^s)(\alpha - c^m) + c^{m+1} - c^{s+1} + 2c^{2s} \\ &- 2c^s\}\alpha + c^m + c^s(\alpha - c^m) > 0 \end{aligned}$$

if $(s, c) \neq (m-1, 2)$, $(m-1, 3)$. Since $a > 1$, if $|2b| < 1$, then $4b^2 < a$. Hence if $(s, c) \neq (m-1, 2)$, $(m-1, 3)$, then $4b^2 < a$. In each case of $(s, c) = (m-1, 2)$, $(m-1, 3)$, $4b^2 < a$ is easily checked. Hence we have $4b^2 < a$.

We claim that if $s = 1$, then $\alpha_1 > \alpha_2$ and if $s \geqq 2$, then $\alpha_1 < \alpha_2$.
we consider the defining polynomal $g(X)$ of $\alpha(c^m - \alpha)$, *i.e.*

$$g(X) = X^3 - 2(c+1)X^2 + \{c^{2m} + (c+1)^2\}X - c^{2m+1}.$$

Since $g(c - \frac{1}{c^{2m-2}}) < 0$ and $g(c) > 0$,

$$c - \frac{1}{c^{2m-2}} < \alpha(c^m - \alpha) < c \ . \tag{3.6}$$

If $s \geqq 2$, then $c^m - \alpha < c^s + \alpha(\alpha - c^m)$. Hence if $s \geqq 2$, then $\alpha_1 < \alpha_2$.
From (3.6) $c + \alpha(\alpha - c^m) < \frac{1}{c^{2m-2}}$, and from (1.1) $\frac{1}{c^{m-1}} < c^m - \alpha$.
Therefore if $s = 1$, then $\alpha_1 > \alpha_2$. We have

$$F(1 + \alpha_1) = F(\frac{c^s}{c^s + \alpha - c^m}) = \frac{c^{2s}\alpha}{(c^m - c^s)\alpha(\alpha - c^s) + c^m} = \frac{1}{H},$$

where $H = (c^{m-s} - 1)(\frac{\alpha}{c^s} - 1) + \frac{c^m}{c^{2s}\alpha}$. If $1 \leqq s \leqq m-2$, then $H > 1$.
If $s = m-1$, then

$$\begin{aligned} H &= (c-1)\frac{\alpha}{c^{m-1}} + \frac{c^m}{c^{2(m-1)}\alpha} - c + 1 \\ &\geqq \frac{\alpha}{c^{m-1}} + \frac{c}{c^{m-1}\alpha} - c + 1 = \frac{\alpha^2 + c}{c^{m-1}\alpha} - c + 1. \end{aligned}$$

From (3.6) $\alpha^2 + c > c^m\alpha$, so $\frac{\alpha^2+c}{c^{m-1}\alpha} - c + 1 > 1$. Hence if $s = m-1$, then $H > 1$. Therefore $F(1 + \alpha_1) < 1$.

Let $\theta_g = u + v\alpha_1 + w\alpha_2$.
(1) By Claim 1 in the proof of lemma 3.3 we have $v \neq 0$ and $uv \geqq 0$.
(2) We claim that $u \geqq 0$, $v > 0$, $w \geqq 0$.
Since $F(\theta_g) < 1$ and $a > 1$, we have $(w - \omega_1 v)^2 < 2$, then $wv \geqq 0$ or $|w| \leqq 1$. If $wv \geqq 0$, then $v < 0$ implies that $u \leqq 0$ and $w \leqq 0$, which is impossible because $\theta_g > 0$, so we have $v > 0$, $u \geqq 0$ and $w \geqq 0$. If $wv < 0$, then $|w| = 1$. If $w = 1$, then $v < 0$ and $u \leqq 0$. If $u = 0$, we have $F(\theta_g) > \frac{a}{2} + (1 - 2\frac{b^2}{a}) > 1$, and if $u < 0$, we have $\theta_g < 0$, which is impossible.
If $w = -1$, then $v > 0$ and $u \geqq 0$. We assume now that $w = -1$. Since $a(w - \omega_1 v)^2 > 1$, $2b < 0$, and $w - \omega_1 v < 0$, by (2.1) $u - \omega_2 v < 0$. Hence $u = [\omega_2 v] - 1$, or $[\omega_2 v]$. If $u = [\omega_2 v] - 1$, then by (2.2) we have $F(\theta_g) > 1$. Hence $u = [\omega_2 v]$, so $\theta_g = [\omega_2 v] + v\alpha_1 - \alpha_2$. If $v \geqq 2$, then we have $\theta_g \geqq [2\omega_2] + 2\alpha_1 - \alpha_2 > 1 + \alpha_1$. If $v = 1$ and $[\omega_2] \geqq 2$, then $\theta_g > 1 + \alpha_1$. If $v = 1$, $[\omega_2] = 1$ and $s \geqq 2$, then $\theta_g < 1$ because $\alpha_1 < \alpha_2$. If $v = 1$, $[\omega_2] = 1$ and $s = 1$, then $F(\theta_g) > 1$ because

$$\begin{aligned} F(1 + \alpha_1 - \alpha_2) &= F\Big(\frac{\alpha(c^m - \alpha)}{c^s + \alpha - c^m}\Big) \\ &= \frac{c^{2m}(\alpha^2 + 1)}{(c^m - c)^2\alpha^2 + (c^2 + c - c^{m+1})\alpha + c^m(c^m - c)} > 1. \end{aligned}$$

Hence the case $w = -1$ is impossible. Therefore $u \geqq 0$, $v > 0$, $w \geqq 0$.
(3) We claim that $v = 1$.
Since $u = [\omega_2 v] + i$ $(i = -1, 0, 1, 2)$, if $v \geqq 2$, then $\theta_g \geqq 1 + 2\alpha_1 + w\alpha_2 > 1 + \alpha_1$.
(4) We claim that $\theta_g = 1 + \alpha_1$.
Since $(w - \omega_1 v)^2 = (w - \omega_1)^2 < 2$, we have $0 \leqq w \leqq 2$. If $u = 0$, then $w = 1$ or 2. If $w = 1$ or 2, then $F(\theta_g) > 1$ because in (2.1) $2b < 0$, $w - \omega_1 v > 0$, $u - \omega_2 v < 0$ and $(u - \omega_2 v)^2 > 1$. Hence $u \neq 0$. If $u \geqq 2$, then $\theta_g > 1 + \alpha_1$. Therefore $\theta_g = 1 + \alpha_1$. □

Lemma 3.5. *For an integer* s, $1 \leqq s \leqq m - 1$,

$$\textit{if} \quad \mathcal{R} = [1, \frac{c^s + \alpha - c^m}{c^s}, \frac{c^s + \alpha(\alpha - c^m)}{c^s}], \quad \textit{then} \quad \theta_g = \frac{\alpha}{c^s}.$$

Proof. we can write F in the form (2.1) and (2.2) with
$a = \frac{1}{c^{2s}\alpha^2}(c^{2s}\alpha^2 - c^{m+s}\alpha + c^{2m}) + c^{2m-2s} - \frac{c+1}{c^s}$,
$2b = \frac{1}{c^s\alpha}\{(2c^s - c - 1)\alpha - c^m\}$, $\omega_1 = \frac{1}{\alpha}$, $\omega_2 = \frac{c^{m-s}(\alpha^2+1) - \alpha^2 - \alpha}{\alpha^2}$.
By (1.1) we have $0 < \alpha_1 < 1$, $0 < \alpha_2 < 1$, $0 < \omega_1 < \frac{1}{3}$, $[\omega_2] = c^{m-s} - 2$, $a > 2$ and $4b^2 < a$. If $(s, c) \neq (m - 1, 2)$, then $\omega_2 > 1$. If $(s, c) = (m - 1, 2)$, then $\frac{\sqrt{2}}{2} < \omega_2 < 1$. $F([\omega_2] + 1 + \alpha_1 - \alpha_2) = F(\frac{-\alpha^2 + (c^m+1)\alpha - c^s}{c^s})$
$= \frac{\alpha}{c^s} + \frac{1}{c^{2s}\alpha}\{(c^{2m} - c^{m+s} + c^m - c^{s+1} + c^{2s} - 2c^2)\alpha + c^s(\alpha - c^m) + \frac{c^{2m}}{\alpha}\} > 1$.
$F([\omega_2] + 1 + \alpha_1) = F(\frac{\alpha}{c^s}) = \frac{c^m}{c^{2s}\alpha} < 1$.

Let $\theta_g = u + v\alpha_1 + w\alpha_2$.
(1) By Claim 1 in the proof of Lemma 3.4 we have $v \neq 0$ and $uv \geqq 0$.
(2) We claim that $u \geqq 0$, $v > 0$, $w \geqq 0$.
By (2) in the proof of Lemma 3.4 if $wv \geqq 0$, then $u \geqq 0$, $v > 0$, $w \geqq 0$, and if $wv < 0$, then $w = -1$. Suppose that $w = -1$. Then $\theta_g = u + v\alpha_1 - \alpha_2$, $u \geqq 0$, $v > 0$ and $(w - \omega_1 v)^2 > 1$. If $u \leqq [\omega_2 v] - 1$, then $(u - \omega_2 v)^2 > 1$ and $F(\theta_g) > 1$. If $u = [\omega_2 v]$, then $\theta_g = [\omega_2 v] + v\alpha_1 - \alpha_2$. If $v \geqq 3$, then $\theta_g \geqq [\omega_2 + 2\omega_2] + 3\alpha_1 - \alpha_2 \geqq [\omega_2] + 1 + \alpha_1 + 2\alpha_1 - \alpha_2 > [\omega_2] + 1 + \alpha_1$. If $v = 2$ and $\omega_2 > 1$, then $\theta_g \geqq [\omega_2] + 1 + \alpha_1 + \alpha_1 - \alpha_2 > [\omega_2] + 1 + \alpha_1$. If $v = 2$ and $\frac{\sqrt{2}}{2} < \omega_2 < 1$, then $\theta_g = [2\omega_2] + 2\alpha_1 - \alpha_2 = 1 + \alpha_1 + \alpha_1 - \alpha_2 > [\omega_2] + 1 + \alpha_1$. If $v = 1$, then $F([\omega_2] + \alpha_1 - \alpha_2) = \frac{1}{c^{2s}}\{2c^{2s} + \frac{c^{2m}}{\alpha^2} + \frac{c^m}{\alpha}(c^m + 1 - 2c^s) + \frac{2c^s}{\alpha}(c^s\alpha - c^m) + 2c^s\alpha(\alpha - c^m + 1) + c^m(c^m + 1 - 2c^s)\} > 1$. If $u = [\omega_2 v] + 1$ and $v \geqq 2$, then $\theta_g \geqq [2\omega_2] + 1 + 2\alpha_1 - \alpha_2 > [\omega_2] + 1 + \alpha_1$. If $u = [\omega_2 v] + 1$ and $v = 1$, then $\theta_g = [\omega_2] + 1 + \alpha_1 - \alpha_2$. If $u \geqq [\omega_2 v] + 2$, then $\theta_g > [\omega_2] + 1 + \alpha_1$.
Therefore the case $wv < 0$ is impossible.
Therefore we have $u \geqq 0$, $v > 0$, $w \geqq 0$.
(3) We claim that $v = 1$ or 2 and $w = 0$ or 1.
We have $\theta_g = [\omega_2 v] + i + v\alpha_1 + w\alpha_2$ $(i = -1, 0, 1, 2)$.
If $v \geqq 3$, then $\theta_g \geqq [3\omega_2] + i + 3\alpha_1 + w\alpha_2 \geqq [\omega_2] + 2\alpha_1 + \alpha_1 + w\alpha_2 > [\omega_2] + 1 + \alpha_1$. Hence $v = 1$ or 2. Since $a > 2$ and $\omega_1 < \frac{1}{3}$, $(w - \omega_1 v)^2 < 1$, so $w = 0$ or 1.
(4) We claim that $v = 1$.
Suppose that $v = 2$. Then $\theta_g = [2\omega_2] + i + 2\alpha_1 + w\alpha_2$, so $i = -1$ or 0.
If $w = 1$, then $\theta_g = i + \frac{\alpha^2 + (2 - c^m)\alpha}{c^s}$ and $F(\theta_g) = i^2 + \frac{1}{c^s}\{(\alpha - 2)(\alpha - c^m) - \frac{2c^m}{\alpha}\}i + \frac{c^m}{c^{2s}\alpha}\{(c^m - 4)\alpha + 2(\alpha - c^m + 2) + \frac{c^m}{\alpha}\} > 1$ $(i = -1, 0)$.
If $w = 0$, then $\theta_g = -1 + i + \frac{2\alpha}{c^s}$ and

$$F(\theta_g) = \frac{1}{c^{2s}\alpha}\{4c^m + 2(i-1)c^s\alpha(c^m - \alpha)\} + (i-1)^2 > 1 \quad (i = -1, 0).$$

Therefore we have $v = 1$.
(5) We claim that $\theta_g = [\omega_2] + 1 + \alpha_1$.
We have $\theta_g = [\omega_2] + i + \alpha_1 + w\alpha_2$ $(i = -1, 0, 1)$. If $w = 1$, then $i = -1$, or 0 and

$$\begin{aligned} F(\theta_g) =& i^2 + \frac{1}{c^s}\{(\alpha - 1)(\alpha - c^m) - \frac{2c^m}{\alpha}\}i \\ &+ \frac{c^m}{c^{2s}\alpha}\{(c^m - 2)\alpha + \alpha - c^m + 1 + \frac{c^m}{\alpha}\} > 1 \quad (i = -1, 0). \end{aligned}$$

If $w = 0$, then

$$F(\theta_g) = \frac{1}{c^{2s}\alpha}\{c^m + (i-1)c^s\alpha(c^m - \alpha)\} + (i-1)^2 > 1 \quad (i = -1, 0).$$

Therefore we conclude that $\theta_g = [\omega_2] + 1 + \alpha_1$. □

Lemma 3.6. *If* $\mathcal{R} = [1, \alpha - c^m + 1, \frac{c^m}{\alpha} - 1]$, *then* $\theta_g = 1 - \alpha + \alpha^2$.

Proof. We can write F in the form (2.1) and (2.2) with
$a = \alpha^2 + 1$, $2b = \alpha(c^m - \alpha) - 2$, $\omega_1 = \frac{1}{\alpha}$, $\omega_2 = \alpha + \frac{1}{\alpha} - 1$.
By (1.1) we have $0 < \alpha_1 < 1$, $0 < \alpha_2 < 1$, $0 < \omega_1 < 1$, $\omega_2 > 1$, $a > 4$ and $4b^2 < a$.

Let $\theta_g = u + v\alpha_1 + w\alpha_2$.
(1) By Claim 1 in the proof of Lemma 3.3 we have $v \neq 0$ and $uv \geqq 0$.
(2) By Claim 2 in the proof of Lemma 3.3 we have $u \geqq 0$, $u > 0$ and $w \geqq 0$.
(3) We shall show that if $v \leqq c^m - 1$, then $w = 1$.
Suppose that $v \leqq c^m - 1$. Since $4b^2 < a$, we have $(u - \omega_2 v)^2 < 2$, so $u = [\omega_2 v] + i$ $(i = -1, 0, 1, 2)$. Since $a > 4$, we have $(w - \omega_1 v)^2 < 1$, so $w = [\omega_1 v] + j = j$ $(j = 0, 1)$. Since $[v\alpha] \leqq [v\alpha + \frac{v}{\alpha}] \leqq [v\alpha + 1] = [v\alpha] + 1$,
we have

$$[\omega_2 v] = [v(\alpha + \frac{1}{\alpha} - 1)] = [v\alpha] + k - v \quad (k = 0, 1).$$

Hence we have

$$\theta_g = [\omega_2 v] + i + v\alpha_1 + j\alpha_2 = [v\alpha] - c^m v + i + k + v\alpha + j(\frac{c^m}{\alpha} - 1).$$

If we put $x = [v\alpha] - c^m v + i + k$, since $c^m - 1 < \alpha < c^m$, we have
$-v + i + k \leqq x \leqq -1 + i + k$. Hence

$$\begin{aligned} x(c^m - \alpha) + v\frac{c^m}{\alpha} &\geqq (-v + i + k)(c^m - \alpha) + v\frac{c^m}{\alpha} \\ &= (i + k)(c^m - \alpha) + v\{\frac{c^m}{\alpha} - (c^m - \alpha)\} > 0. \end{aligned}$$

Therefore if $j = 0$ and $x \neq 0$, then we have $F(\theta_g) = F(x + v\alpha) = x^2 + v\{x(c^m - \alpha) + v\frac{c^m}{\alpha}\} > 1$. If $j = 0$ and $x = 0$, then $F(\theta_g) = v^2\frac{c^m}{\alpha} > 1$. Therefore we conclude $j (= w) = 1$.
(4) We claim that if $v \leqq c^m - 2$, then $F(\theta_g) > 1$.
If $v \leqq c^m - 2$, then we have

$$\begin{aligned} &\frac{a}{2}(w - \omega_1 v)^2 = \frac{a}{2}(1 - \omega_1 v)^2 \geqq \frac{a}{2}(1 - \frac{c^m - 2}{\alpha})^2 \\ &= \frac{1}{2}(c^m - \alpha)^2 + \frac{1}{2}(1 - \frac{c^m - 2}{\alpha})^2 + 2(\alpha - c^m) + 2 > 2 - \frac{2}{c^{m-1}} - \frac{2}{c^{m+2}}. \end{aligned}$$

Hence if $(m, c) \neq (2, 2)$, then $\frac{a}{2}(w - \omega_1 v)^2 > 1$, so $F(\theta_g) > 1$.
In the case $(m, c) = (2, 2)$, $\frac{a}{2}(w - \omega_1 v)^2 > 1$ is easily checked. Therefore if $v \leqq c^m - 2$, then $F(\theta_g) > 1$.
(5) We claim that $\theta_g = [(c^m - 1)\omega_2] + (c^m - 1)\alpha_1 + \alpha_2$.
Now we shall consider the case $v = c^m - 1$. First we shall show that

$$(3.7) \quad c^{2m} - c^m - c + 1 < A < c^{2m} - c^m - c + 2, \quad \text{where} \quad A = (c^m - 1)(\alpha + \frac{1}{\alpha}).$$

We observe that $c^{2m}-c^m-c+1<A$ is equivalent to

$$c\alpha+c^m\alpha(\alpha-c^m)+\alpha(c^m-\alpha)-1+c^m-\alpha>0. \tag{3.8}$$

From (3.6) we have $c(\alpha-c^m)<c\alpha+c^m\alpha(\alpha-c^m)$, further from (1.1) $-\frac{1}{c^{m-2}}-\frac{1}{c^{m+1}}<c\alpha+c^m\alpha(\alpha-c^m)$. From this and (3.6) we have (3.8). Hence we have $c^{2m}-c^m-c+1<A$. In the same way, $A<c^{2m}-c^m-c+2$ is more easily proved. Therefore by (3.7) we have $[A]=[(c^m-1)(\alpha+\frac{1}{\alpha})]=c^{2m}-c^m-c+1$. So if $v=c^m-1$, then $\theta_g=[\omega_2 v]+i+v\alpha_1+\alpha_2=[(c^m-1)(\alpha+\frac{1}{\alpha})-(c^m-1)]+i+(c^m-1)(\alpha-c^m+1)+\frac{c^m}{\alpha}-1=-c+i+c^m\alpha+\frac{c^m}{\alpha}-\alpha=\alpha^2-\alpha+1+i$ $(i=-1,0,1,2)$. If $i=-1$ and $v=c^m-1$, then $F(\theta_g)=F(\alpha^2-\alpha)=\frac{c^m}{\alpha}(\alpha-c^m+1+\frac{c^m}{\alpha})>1$. If $i=0$ and $v=c^m-1$, then $F(\theta_g)=F(\alpha^2-\alpha+1)=\frac{c^m}{\alpha^2}(\alpha-c^m)(\alpha-1)+(c^m-\alpha)^2-(c^m-\alpha)+1<1$. Therefore we conclude $\theta_g=[(c^m-1)\omega_2]+(c^m-1)\alpha_1+\alpha_2=\alpha^2-\alpha+1$. □

Lemma 3.7. *If* $\mathcal{R}=[1,\alpha-c^m+1,\frac{c^m-1}{\alpha}]$, *then* $\theta_g=\frac{1-c\alpha+\alpha^2}{c}$.

Proof. We can write F in the form (2.1) and (2.2) with $a=c^{m-2}\alpha$, $2b=\frac{\alpha(c^m-\alpha)}{c}$, $\omega_1=\frac{c}{\alpha}$, $\omega_2=\alpha-1$.
By (1.1) we have $0<\alpha_1<1$, $0<\alpha_2<1$, $0<\omega_1<1$, $\omega_2>1$ and $a>3$.
By (3.6) $2b<1$, so $4b^2<a$.

Let $\theta_g=u+v\alpha_1+w\alpha_2$.
(1) By Claim 1 in the proof of Lemma 3.3 we have $v\neq 0$ and $uv\geqq 0$.
(2) By Claim 2 in the proof of Lemma 3.3 we have $v\geqq 0$, $v>0$, and $w\geqq 0$.
(3) By Claim 2 in the proof of Lemma 3.3 we have $w>0$.
(4) we claim that $\theta_g=-1+(c^{m-1}-1)\alpha_1+\alpha_2$.
We shall show that $v\leqq c^{m-1}-2$ implies that $F(\theta_g)>1$. Suppose that $v\leqq c^{m-1}-2$. Since $[\omega_1 v]=0$, $w=[\omega_1 v]+1=1$. By Claim 4 in the proof of Lemma 3.3 we have $[v\alpha]=vc^m+k$ ($k=-2$ *or* -1). So we have $\theta_g=u+v\alpha_1+\alpha_2=[\omega_2 v]+i+v\alpha_1+\alpha_2=x+v\alpha+\frac{c^{m-1}}{\alpha}$, where $x=k+i$ $(i=-1,0,1,2)$. We have

$$\begin{aligned}F(\theta_g)=&x^2+x(c^m-\alpha)(v+\frac{\alpha}{c})\\&+\frac{v\alpha(c^m-\alpha)^2}{c}+\frac{c^m}{\alpha}(v-\frac{\alpha}{c})^2\quad(-3\leqq x\leqq 1).\end{aligned} \tag{3.9}$$

By (1.1) if $v\leqq c^{m-1}$ and $x\neq 0$, then we have

$$\begin{aligned}|x(c^m-\alpha)(v+\frac{\alpha}{c})|&\leqq|x|(c^m-\alpha)(c^{m-1}+\frac{\alpha}{c})\\&<|x|(c^m-\alpha)2c^{m-1}<|x|2c^{m-1}(\frac{1}{c^{m-1}}+\frac{1}{c^{m+2}})\\&=|x|(2+\frac{2}{c^3})\leqq|x|(2+\frac{1}{2^2})\quad(x\neq 0).\end{aligned} \tag{3.10}$$

Also by (1.1), $\frac{c^m-\alpha}{c} < \frac{1}{c^m} + \frac{1}{c^{m+3}} < \frac{1}{3}$, so if $v \leqq c^{m-1} - 2$, then we have

$$\frac{c^m}{\alpha}\left(v - \frac{\alpha}{c}\right)^2 \geqq \frac{c^m}{\alpha}\left(c^{m-1} - \frac{\alpha}{c} - 2\right)^2 \geqq \left(\frac{c^m - \alpha}{c} - 2\right)^2 \geqq 2 + \frac{2}{3}. \tag{3.11}$$

from (3.9), (3.10), (3.11) if $-3 \leqq x \leqq 1$, then $F(\theta_g) > 1$.
Therefore if $v \leqq c^{m-1} - 2$, then $F(\theta_g) > 1$.

Now we shall consider the case $v = c^{m-1} - 1$. We have $\theta_g = x + (c^{m-1} - 1)\alpha + \frac{c^{m-1}}{\alpha}$ $(-3 \leqq x \leqq 1)$. From (3.9), (3.10) if $x = -3$, then $F(\theta_g) > 1$. By (3.9) if $x = -2$, then we have $F(\theta_g) = 4 - 2(c^m - \alpha)(c^{m-1} - 1 + \frac{\alpha}{c}) + \frac{(c^{m-1}-1)\alpha(c^m-\alpha)^2}{c} + \frac{c^m}{\alpha}(c^{m-1} - 1 - \frac{\alpha}{c})^2$ $= \frac{1}{c\alpha}\{c^m(c - \alpha(c^m - \alpha)) + c\alpha - c^m(c^m - \alpha)c^{m-1} + (c^{m-1} - c)\alpha(c^m - \alpha) + (c^m - \alpha)^2\} + 1 > 1$. By (3.9) if $x = -1$, then we have

$$\begin{aligned} F(\theta_g) =& 1 - (c^m - \alpha)(c^{m-1} - 1 + \frac{\alpha}{c}) \\ &+ \frac{(c^{m-1} - 1)\alpha(c^m - \alpha)^2}{c} + \frac{c^m}{\alpha}\left(c^{m-1} - 1 - \frac{\alpha}{c}\right)^2 \\ =& 1 + c^{m-2}(c^m - \alpha)\{\alpha(c^m - \alpha) - c\} + \frac{1}{c\alpha}(c^m - \alpha)^2(c^{m-1} - \alpha) \\ &+ \frac{1}{c\alpha}(c^m - \alpha)\{c + 1 - c^m + \alpha(c^m - \alpha - 1)\} < 1. \end{aligned}$$

Therefore we conclude $\theta_g = -1 + (c^{m-1} - 1)\alpha + \frac{c^{m-1}}{\alpha}$. □

Lemma 3.8. *If* $\mathcal{R} = [1, \frac{c^m-\alpha}{\alpha-c^m+1}, \frac{c^m-\alpha}{\alpha(\alpha-c^m+1)}]$, *then* $\theta_g = \frac{1}{\alpha-c^m+1}$.

Proof. We can write F in the form (2.1) and (2.2) with
$a = \frac{c^m\alpha^2-c\alpha+c^m}{(c^m-1)\alpha^2-(c^m-1)\alpha+c^m}$, $2b = -\frac{c\alpha^2-(c-1)\alpha+c^m}{(c^m-1)\alpha^2-(c^m-1)\alpha+c^m}$,
$\omega_1 = \frac{1}{\alpha^2-\alpha+1}$, $2\omega_2 = 2 + \frac{\alpha^2+(c^m-2)\alpha}{(c^m-1)\alpha^2-(c^m-1)\alpha+c^m} - \omega_1(2b)$.
By (1.1) we have $0 < \alpha_1 < 1$, $0 < \alpha_2 < 1$, $0 < \omega_1 < 1$, $a > 1$ and $|2b| < 1$, so $4b^2 < a$. Since $2 < 2\omega_2 < 4$, $1 < \omega_2 < 2$. We have

$$\begin{aligned} F([\omega_2] + \alpha_1) = F(1 + \alpha_1) &= F\left(\frac{1}{\alpha - c^m + 1}\right) \\ &= \frac{\alpha}{(c^m - 1)\alpha^2 - (c^m - 1)\alpha + c^m} < 1. \end{aligned}$$

So by Adam [1], Proposition 2.3 $\theta_g = [\omega_2] + \alpha_1$, $[\omega_2] + \alpha_1 - \alpha_2$ or $[\omega_2] - 1 + \alpha_1$. Since $F([\omega_2] + \alpha_1 - \alpha_2) = F(\frac{2\alpha - c^m}{\alpha(\alpha-c^m+1)}) = \frac{c^m\alpha^2-2(c-2)\alpha+2c^m}{(c^m-1)\alpha^2-(c^m-1)\alpha+c^m} > 1$ and $[\omega_2] - 1 + \alpha_1 = \alpha_1 < 1$, $\theta_g = [\omega_2] + \alpha_1 = \frac{1}{\alpha-c^m+1}$. □

§ 4. Proof of the main theorem

Proof of Theorem 3.1. First we define
$\theta_g{}^{(1)} = 1 - \alpha + \alpha^2 = \alpha\frac{c+\alpha-c^m}{c^m-\alpha}$ and $\theta_h{}^{(1)} = \alpha$,
$\theta_g{}^{(3s-2)} = \frac{c^{s-1}(\alpha^2+1)-\alpha}{\alpha} = \frac{c^s+\alpha-c^m}{c^m-\alpha}$ and $\theta_h{}^{(3s-2)} = \alpha$ for $2 \leqq s \leqq m-1$,
$\theta_g{}^{(3s-1)} = \frac{c^s}{c^s+\alpha-c^m}$ and $\theta_h{}^{(3s-1)} = \frac{c^s+\alpha(\alpha-c^m)}{c^s+\alpha-c^m}$ for $1 \leqq s \leqq m-1$,
$\theta_g{}^{(3s)} = \frac{\alpha}{c^s}$ and $\theta_h{}^{(3s)} = \frac{\alpha(\alpha-c^m)+\alpha}{c^s}$ for $1 \leqq s \leqq m-1$,
$\theta_g{}^{(3m-2)} = \frac{1-c\alpha+\alpha^2}{c} = \alpha\frac{1+\alpha-c^m}{c^m-\alpha}$ and $\theta_h{}^{(3m-2)} = \alpha$,
$\theta_g{}^{(3m-1)} = \frac{1}{1+\alpha-c^m}$.
Next we define
$\mathcal{R}_1 = [1, \alpha, \alpha^2]$, $\mathcal{R}_n = [1, 1/\theta_g{}^{(n-1)}, \theta_h{}^{(n-1)}/\theta_g{}^{(n-1)}]$ for $2 \leqq n \leqq 3m-1$.
It is easily seen that $\mathcal{R}_n = [1, \theta_g{}^{(n)}, \theta_h{}^{(n)}]$ for $1 \leqq n \leqq 3m-2$.
By Lemma 3.6 $\theta_g{}^{(1)}$ is the minimal point adjacent to 1 in $\mathcal{R}_1$.
By Lemma 3.3 $\theta_g{}^{(3s-2)}$ is the minimal point adjacent to 1 in $\mathcal{R}_{3s-2}$.
By Lemma 3.4 $\theta_g{}^{(3s-1)}$ is the minimal point adjacent to 1 in $\mathcal{R}_{3s-1}$.
By Lemma 3.5 $\theta_g{}^{(3s)}$ is the minimal point adjacent to 1 in $\mathcal{R}_{3s}$.
By Lemma 3.7 $\theta_g{}^{(3m-2)}$ is the minimal point adjacent to 1 in $\mathcal{R}_{3m-2}$.
By Lemma 3.8 $\theta_g{}^{(3m-1)}$ is the minimal point adjacent to 1 in $\mathcal{R}_{3m-1}$.
We define $\theta_n = \prod_{i=1}^{n} \theta_g{}^{(i)}$. Then we have $N_K(\theta_{3m-1}) = 1$ and $N_K(\theta_i) \neq 1$ if $1 \leqq i \leqq 3m-2$. Therefore, θ_{3m-1} is the fundamental unit ε of $\mathcal{O}$, and the Voronoi-algorithm expansion period length is $l = 3m-1$. □

Remark 4.1. In fact (ii) in Theorem 3.1 is valid for $m = 1$ provided $c \geqq 4$.

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