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POLY–MODAL LOGIC $S5_nC$

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Abstract. We investigate a poly-modal logic $S5_nC$ which has n-modalities ($n > 1$) satisfying the axioms of $S5$ and has axioms expressing permutability of modalities. We show that the logic $S5_nC$ is complete concerning Kripke semantics, has the finite model property and is decidable, however we prove $S5_nC$ is not locally finite. A main result consists of an algorithmic criterion for recognizing the admissibility of inference rules in $S5_nC$, i.e. the logic $S5_nC$ is decidable with respect to the admissibility.

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§1. Introduction

Semantics and deductive properties of many mono-modal logics are investigated very deeply. We mean the results concerning completeness and decidability of logics for the representative and rich classes, the results concerning the finite model property and the local finiteness, the results on the description of corresponding Kripke models. Mentioned results are related to the equational modal logics. But results about inference rules for modal logics belong to the quasi-equational logics, the theory of quasi-identities of modal algebras. The problems concerning inference rules for mono-modal logics are investigated rather profoundly.

These questions appeared in the study of intuitionistic logic by Moscow logical school headed by P.S.Novikov in forties, and then by Leningrad school in works of G.E.Mints and others, and in Kishinev school in works of A.V.Kuznetsov and his followers.

The main point of these studies was two related problems: (a) the Harvey Friedman problem (problem 40,[1]): about the existence of an algorithm for recognizing the admissibility of inference rules; (b) the problem of A.V.Kuznetsov about the existence of finite bases for admissible inference rules of the intuitionistic logic.

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These problems were answered by V.V.Rybakov affirmatively for Friedman problem, but negatively for Kuznetsov question, using solutions of similar questions for the modal logics $S4$ and Grz [2,3]. Also a complete description of the bases of admissible inference rules of the modal logic $S4.3$ was obtained, and it was proved that these logics are decidable with respect to the admissibility of inference rules [4].

Therefore there are a number of results concerning the admissibility of inference rules for individual logics. Recently V.V.Rybakov [6] obtained a general theorem for the description of classes of mono-modal logics concerning the existence of algorithms for recognizing the admissibility of inference rules. A complete description of the hereditarily structurally complete modal logics over $K4$ was found in [5]. Thus the problem of the admissibility of inference rules for mono-modal logics is studied in detail. As to poly-modal logics, the above mentioned problems of equational logics and especially the problems of quasi-equational logics are investigated not so deeply. It is quite natural, using the methods of mono-modal logics if possible, to investigate the corresponding and related problems for poly-modal logics.

A semantic investigation of poly-modal logics has been initiated with works of K.Segerberg (see for example [10]). A study of poly-modal logics has an independent theoretical and practical interest, and is important(see for example an interconnection discovered by M.Rennie [11]). However, in general case, poly-modal logics have as a rule more complicate structures comparing to mono-modal logics (see for instance [8]). Nevertheless it is possible sometimes to obtain rather strong positive results regarding poly-modal logics. For example, V.B.Shechtman [9] proved theorems of general character for semantic characterization of poly-modal logics, from which the completeness, the finite model property and the decidability follow for many logics.

The aim of this paper is a study of inference rules for the poly-modal logic $S5_nC$ which has commutative modalities. Namely, the logic $S5_nC$ is a logic with n modalities $\Box_1, \ldots, \Box_n$, which have all axioms of $S5$ and besides the following axioms: $\Box_i\Box_j p \equiv \Box_j\Box_i p$, $i, j = 1, \ldots, n$. The problem is interesting since on the one hand this logic promises positive results and, on the other hand, among the extensions of $S5_nC$ there are logics which are not decidable (see V.V.Rybakov [7]). In this paper we show that the logic $S5_nC$ is complete concerning Kripke semantics, has the finite model property and is decidable, however we prove that $S5_nC$ is not locally finite for $n > 1$. A main result consists of an algorithmic criterion for recognizing the admissibility in $S5_nC$, i.e. the logic $S5_nC$ is decidable with respect to the admissibility of inference rules.

We are grateful to V.V.Rybakov for offering this problem and for valuable discussions.

§2. Definitions and preliminary results

A *normal modal n-logic* is an extension of the classical propositional calculus by adding n modalities $\Box_1, \dots, \Box_n$, which has axioms $\Box_i(p \to q) \to (\Box_i p \to \Box_i q)$ $(i = 1, \dots, n)$, and the rules $\vdash A \Rightarrow \vdash \Box_i A$ $(i = 1, \dots, n)$ and, possibly, other additional axioms. A *weak product* of the normal modal logics L_1 and L_2 with modalities $\Box_1, \dots, \Box_n$ and $\Box_{n+1}, \dots, \Box_{n+m}$ is a normal logic $L_1 \times L_2$ with modalities $\Box_1, \dots, \Box_n, \Box_{n+1}, \dots, \Box_{n+m}$, which contains all axioms of L_1 and L_2 and the axioms $\Box_i \Box_j p \equiv \Box_j \Box_i p$, $\Diamond_i \Box_j p \to \Box_j \Diamond_i p$ $(1 \le i \le n < j \le n+m)$.

The language of the propositional logic $S5_nC$ extends the language of the classical propositional calculus with modalities $\Box_1, \dots, \Box_n$. The logic $S5_nC$ has axioms

$$\Box_i(p \to q) \to (\Box_i p \to \Box_i q),\ i = 1, \dots, n,$$
$$\Box_i p \to p,\ i = 1, \dots, n,$$
$$\Diamond_i \Box_i p \to p,\ i = 1, \dots, n,$$
$$\Box_i p \to \Box_i \Box_i p,\ i = 1, \dots, n,$$
$$\Box_i \Box_j p \equiv \Box_j \Box_i p,\ i, j = 1, \dots, n,$$

and the following inference rules:

$$\vdash A, \vdash A \to B \Rightarrow \vdash B,\ \vdash A \Rightarrow \vdash \Box_i A (i = 1, \dots, n).$$

A logic λ is called *locally finite* if, for any m, there exist only a finite number of pairwise nonequivalent formulas of m propositional variables. An *n-frame* is the structure $\langle U, R_1, \dots, R_n \rangle$, where U is a nonempty set and $R_1, \dots, R_n$ are binary relations on the set U. A *Kripke n-model* is a structure $\langle U, R_1, \dots, R_n, V \rangle$, where $\langle U, R_1, \dots, R_n \rangle$ is an n-frame and V is a function from a set of propositional variables to the powerset of U (we call this function as *valuation*). When there is no risk to confuse, we will speak of <<Kripke model>> instead of <<Kripke n-model>>. The truth value of a formula can be defined in poly-modal case quite similarly as the mono-modal case. Let $\mathcal{M}$ be a Kripke model with a valuation defined on m-elements set P_m of propositional variables. We say that $\mathcal{M}$ is *m-characterizing* for the logic λ, if for any formula φ with variables from P_m, $\varphi \in \lambda \iff \mathcal{M} \Vdash \varphi$. An element v of the model $\langle U, R_1, \dots, R_n, V \rangle$ is *definable* if there is a formula α such that for any $w \in U$ $w \Vdash_V \alpha \iff w = v$. In this case we say that α *defines* v. Let $\mathcal{M} = \langle U, R_1, \dots, R_n, V \rangle$ be a Kripke model and let W be a new valuation of some propositional variables on the frame of the model $\mathcal{M}$. The valuation W is called *definable*, if for any variable p_i from the domain of W, there exists a formula α_i such that $W(p_i) = \{x \in U | x \Vdash_V \alpha_i\}$.

Suppose $\mathcal{M}_1 = \langle U', R'_1, \dots, R'_n, V' \rangle$ and $\mathcal{M}_2 = \langle U'', R''_1, \dots, R''_n, V'' \rangle$ are Kripke models. We call the model $\mathcal{M} = \langle U, R_1, \dots, R_n, V \rangle$ a *disjoint union* of models $\mathcal{M}_1$ and $\mathcal{M}_2$ ($\mathcal{M} = \mathcal{M}_1 \sqcup \mathcal{M}_2$) if the following hold: $U = U' \cup U''$, $U' \cap U'' = \emptyset$, $R_i \mid_{U'} = R'_i$, $R_i \mid_{U''} = R''_i$ and $V \mid_{U'} = V'$, $V \mid_{U''} = V''$. Let $F = \langle U, R_1, \dots, R_n \rangle$ be an n-frame. Assume U^x is the smallest subset of U containing x and satisfying the condition: if $y \in U^x$ and $(y, z) \in R_i$ for some i, $1 \leq i \leq n$, then $z \in U^x$. We use the denotation $R_i^x = R_i|_{U^x}$. We say a subframe $F^x = \langle U^x, R_1^x, \dots, R_n^x \rangle$ of a frame F is the *cone* of F with the *root* x.

Let $\mathcal{M} = \langle U, R_1, \dots, R_n, V \rangle$ be an n-model and $x \in U$. Denote by $\mathcal{M}^x$ the submodel $\langle U^x, R_1^x, \dots, R_n^x, V' \rangle$, where the frame $\langle U^x, R_1^x, \dots, R_n^x \rangle$ is a cone of a frame $\langle U^x, R_1^x, \dots, R_n^x \rangle$ with a root x and $V'(p_i) = V(p_i) \cap U^x$. By $\mathcal{E}$ we denote a *one-element model* that is a model with one element universe and with all relations R_i ($1 \leq i \leq n$) reflexive. Note that, in case of n propositional variables, there are 2^n different one-element models. We call the frame $F_1 \times F_2 = \langle U_1 \times U_2, \hat{R}_1, \dots, \hat{R}_n, \hat{S}_1, \dots, \hat{S}_m \rangle$ as *cartesian product* of frames $F_1 = \langle U_1, R_1, \dots, R_n \rangle$ and $F_2 = \langle U_2, S_1, \dots, S_m \rangle$, where $(x_1, y_1)\hat{R}_i(x_2, y_2)$, if $x_1 R_i x_2$ and $y_1 = y_2$; $(x_1, y_1)\hat{S}_i(x_2, y_2)$, if $x_1 = x_2$ and $y_1 S_i y_2$. The *composition* of frames $F_1 = \langle U, R_1, \dots, R_n \rangle$ and $F_2 = \langle U, S_1, \dots, S_m \rangle$ is the frame

$$F_1 * F_2 = \langle U, R_1, \dots, R_n, S_1, \dots, S_m \rangle.$$

If $\mathcal{F}_1$ and $\mathcal{F}_2$ are certain classes of n-frames and m-frames, respectively, then $\mathcal{F}_1 \times \mathcal{F}_2 = \{F_1 \times F_2 | F_1 \in \mathcal{F}_1, F_2 \in \mathcal{F}_2\}$, $\mathcal{F}_1 * \mathcal{F}_2 = \{F_1 * F_2 | F_1 \in \mathcal{F}_1, F_2 \in \mathcal{F}_2\}$.

If $\mathcal{F}$ is a class of n-frames, then by $S(\mathcal{F})$ we denote an n-modal logic which contains all modal formulas which are true on the class $\mathcal{F}$.

Let V_1 be a class of all 1-frames. $V_1 \cap D$ is the class of all countable 1-frames or finite 1-frames, REF, SYM, TR, NE, E are defined by axioms $\Box p \to p$, $\Diamond\Box p \to p$, $\Box p \to \Box\Box p$, $\Diamond\top$, $\Diamond\top \to \Diamond\Box\bot$ respectively.

Theorem 2.1 ([9], theorem 3). *Let G be the semigroup generated by the classes of frames of a free semigroup of classes of frames (where the multiplication is the composition $*$) generated by classes $V_1, V_1 \cap D, REF, SYM, TR, NE, E$ and nonempty intersections of these classes. Let $\mathcal{F}$, $\mathcal{F}'$ be certain elements of G. Then the following holds*

$$S(\mathcal{F} \times \mathcal{F}') = S(\mathcal{F}) \times S(\mathcal{F}').$$

Since in the logic $S5_nC$ each inference rule of type $A_1, \dots, A_k/B$ is equivalent to the rule $A_1 \wedge \cdots \wedge A_k/B$, we presuppose that each inference rule is of the form A/B. We say that an inference rule $A(p_1, \dots, p_m)/B(p_1, \dots, p_m)$ is *admissible* in a logic λ if, for any formulas $\phi_1, \dots, \phi_m$, the condition $A(\phi_1, \dots, \phi_m) \in \lambda$ entails $B(\phi_1, \dots, \phi_m) \in \lambda$. We will also consider the logical connective *"possible"*: $\Diamond_i$, which is defined as an abbreviation for $\neg\Box_i\neg$. Note that, $\Box_i\varphi \equiv \neg\Diamond_i\neg\varphi$ and $\Diamond_i\varphi \equiv \neg\Box_i\neg\varphi$ for every formula φ in $S5_nC$ for any n.

§3. Main results

From our definitions above we immediately infer

Lemma 3.1. *A frame $\langle U, R_1, \dots, R_n \rangle$ is adequate to the logic $S5_nC \iff R_i$ is reflexive, symmetric, transitive and $R_iR_j = R_jR_i$ $(i, j = 1, \dots, n)$.*

Theorem 3.2. *The logic $S5_nC$ has the property of completeness for Kripke semantics and the finite model property.*

To prove this theorem we need the following denotation and lemma. Denote by $\mathcal{F}_1$ the set of all finite reflexive and transitive clusters (remind, a cluster is a frame of the kind $\langle U, R \rangle \in \mathcal{F}_1 \iff \forall x, y \in U\, [(x, y) \in R]$). It is well known that $\mathcal{F}_1$ is a characterizing class for $S5$.

Lemma 3.3. $S5_nC = S(\underbrace{\mathcal{F}_1 \times \cdots \times \mathcal{F}_1}_{n})$

Proof. By induction on n. For $n = 1$ the lemma holds. Let the lemma be proved for every $k \leq n$ and let $k = n+1$. Applying Theorem 2.1 and the inductive hypothesis we have: $S5_{n+1}C = S5_nC \times S5 = S(\underbrace{\mathcal{F}_1 \times \cdots \times \mathcal{F}_1}_{n}) \times S(\mathcal{F}_1) =$

$S((\underbrace{\mathcal{F}_1 \times \cdots \times \mathcal{F}_1}_{n}) \times \mathcal{F}_1) = S(\underbrace{\mathcal{F}_1 \times \cdots \times \mathcal{F}_1}_{n+1})$. Lemma 3.3 is proved. □

Theorem 3.2 follows immediately from Lemma 3.3.

Thus $\mathcal{F}_1^n$ is a characterizing class of frames for $S5_nC$. Let $Char_m$ be a disjoint union of all possible models of type $\langle F, R_1, \dots, R_n, V \rangle$, where $F = M_1 \times M_2 \times \cdots \times M_n$, $(x_1, \dots, x_n)R_i(y_1, \dots, x_y) \iff x_1 = y_1, \dots, x_{i-1} = y_{i-1}, x_{i+1} = y_{i+1}, \dots, x_n = y_n$, $M_1, \dots, M_n$ are any finite sets and V is any valuation of variables $p_1, \dots, p_m$ on the frame F. According to Lemma 3.3 the model $Char_m$ is an m-characterizing model for $S5_nC$.

Next theorem follows from the Theorem 3.2 and the fact that logic $S5_nC$ is finitely axiomatizable.

Theorem 3.4. *The logic $S5_nC$ is decidable.*

Theorem 3.5. *The logic $S5_nC$ for $n > 1$ is not locally finite.*

Proof. We introduce a model in the following way. We take the set $N \times N$ as a universe of our model and define the relations of accessibility $R_1, \dots, R_n$ as follows: $(r, s)R_1(q, t)$ iff $s = t$; $(r, s)R_2(q, t)$ iff $r = q$; $(r, s)R_k(q, t)$ iff $(r = q) \wedge (s = t)$ $(k = 3, 4, \dots, n)$. The valuation of the single variable p on this model is following one: $V(p) = \{(i, j) | \, i \leq j\}$. Now we introduce the set of formulas inductively :

$\alpha = p \wedge \Diamond_2 \neg p$,

$\beta_0 = \Box_2 p$,

$\gamma_0 = \Box_1 \neg\alpha,$
$\beta_1 = \Box_2(\gamma_0 \vee \alpha) \wedge \neg\beta_0,$
$\gamma_1 = \Box_1(\beta_0 \vee \beta_1 \vee \neg\alpha) \wedge \neg\gamma_0,$
$\dots \quad \dots$
$\beta_{i+1} = \Box_2(\bigvee_{j=0}^{i} \gamma_j \vee \alpha) \wedge \bigwedge_{j=0}^{i} \neg\beta_j,$
$\gamma_{i+1} = \Box_1(\bigvee_{j=0}^{i+1} \beta_j \vee \neg\alpha) \wedge \bigwedge_{j=0}^{i} \neg\gamma_j.$

It is clear that the formulas γ_i and β_j are valid only on the elements of the sets $\{(k,i)|k \in N\}$ and $\{(j,k)|k \in N\}$ respectively. This yields that there exist an infinite set of pairwise nonequivalent formulas with one propositional variable. Hence the logic $S5_nC$ is not locally finite and Theorem 3.5 is proved. □

Theorem 3.6. *An inference rule $A(p_1 \dots, p_m)/B(p_1 \dots, p_m)$ is not admissible in $S5_nC$ iff the following hold:*
1) $(\Box_1 \dots \Box_n A \to B) \notin S5_nC$,
2) there exists a valuation on one-element model $\mathcal{E}$ such that $\mathcal{E} \Vdash A$.

Proof. **Necessity.** Suppose an inference rule $A(p_1, \dots, p_m)/B(p_1, \dots, p_m)$ is not admissible in $S5_nC$. Hence there exist formulas $\phi_1, \dots, \phi_m$ such that $A(\phi_1, \dots, \phi_m) \in S5_nC$, $B(\phi_1, \dots, \phi_m) \notin S5_nC$. Therefore

$$\Box_1 \dots \Box_n A(\phi_1, \dots, \phi_m) \in S5_nC.$$

According to Lemma 3.3 there is a certain frame F in $\mathcal{F}_1^n$ such that the formula $B(\phi_1, \dots, \phi_m)$ is false on F. Let V be a valuation which disproves the formula $B(\phi_1, \dots, \phi_m)$ on the frame F. Since $\Box_1 \dots \Box_n A(\phi_1, \dots, \phi_m) \in S5_nC$ we have $F \Vdash_V \Box_1 \dots \Box_n A(\phi_1, \dots, \phi_m)$. Now we introduce a new valuation V' on the frame F in the following way: $V'(p_i) = \{x| \, x \Vdash \phi_i\}$. By induction on the length of formula we can prove $F \Vdash_{V'} \Box_1 \dots \Box_n A(p_1, \dots, p_m)$ and $F \nVdash_{V'} B(p_1, \dots, p_m)$. Therefore $(\Box_1 \dots \Box_n A(p_1, \dots, p_m) \to B(p_1, \dots, p_m)) \notin S5_nC$ and 1) is proved.

Since the one-element frame E with reflexive relations $R_1, \dots, R_n$ is adequate to $S5_nC$ we have $E \Vdash_V A(\phi_1, \dots, \phi_m)$ for any valuation V. We fix the valuation V and introduce on E the new valuation $V'(p_i) = \{x|x \Vdash_V \phi_i\}$, then $E \Vdash_{V'} A(p_1 \dots, p_m)$ and 2) hold.

Sufficiency. Suppose the conditions 1) and 2) of our theorem hold. Then there exists a model $\mathcal{M}$, which is a cone of $Char_m$, such that $\mathcal{M} \nVdash (\Box_1 \dots \Box_n A \to B)$. Let the power of the model $\mathcal{M}$ is k. We define an extension of the valuation V of variables $p_1, \dots, p_m$ on the model $\mathcal{M}$ to the valuation of $p_1, \dots, p_{m+k}$ such that, for $i = m+1, \dots, m+k$, the set $V(p_i)$ contains exactly one element of the model $\mathcal{M}$ and if $i, j \in \{m+1, \dots, m+k\}$, $i \neq j$ then $V(p_i) \cap V(p_j) = \emptyset$. Under the valuation V, extended in such a manner, all elements of the model $\mathcal{M}$ are definable, $\mathcal{M}$ is a submodel of the model $Char_{m+k}$ and $\mathcal{M} \nVdash (\Box_1 \dots \Box_n A \to B)$ holds. We define $V(p_i)$ on $\mathcal{E}$ for $i = m+1, \dots, m+k$ in such a way that, $\mathcal{E} \nVdash p_i$. According to the definition of V, all elements of the model $\mathcal{N} = \mathcal{M} \sqcup \mathcal{E}$ are definable. Now we obtain that the model $\mathcal{N} = \mathcal{M} \sqcup \mathcal{E}$ is a submodel of the model $Char_{m+k}$.

Further we construct a definable valuation W on the model $Char_{m+k}$. Denote the elements of $\mathcal{N} = \mathcal{M} \sqcup \mathcal{E}$ as $a_1, \dots, a_{k+1}$, where a_{k+1} is R_i maximal element for any R_i. The model $\mathcal{M}$ contains the elements $a_1, \dots, a_k$ and a_{k+1} is the element of the model $\mathcal{E}$. Suppose $h_1, \dots, h_{k+1}$ are the formulas defining the elements $a_1, \dots, a_{k+1}$ in the model $\mathcal{N}$,respectively. Put $f_i = h_i \wedge \bigwedge_{j, j \neq i, k+1} \neg h_j$, $i = 1, \dots, k$, $f_{k+1} = \bigwedge_{j=1}^{k} \neg h_j$. Now the formulas $f_1, \dots, f_{k+1}$ define the elements $a_1, \dots, a_{k+1}$ in the model $\mathcal{N}$, respectively. Consider following formulas :

$$g_a = f_a \wedge \bigwedge_i (\bigwedge_{aR_iy} \Diamond_i f_y) \wedge \bigwedge_i (\Box_i \bigvee_{aR_iy} f_y), \tag{1}$$

$$\varphi_a = g_a \wedge (\bigwedge_{x \in \mathcal{M}} \Box_1 \dots \Box_n (f_x \to g_x)), \tag{2}$$

where $a \in \{a_1, \dots, a_k\}$.

$$\varphi_{a_{k+1}} = \neg(\bigvee_{a \in \mathcal{M}} \varphi_a). \tag{3}$$

Note that the elements $a_1, \dots, a_{k+1}$ are defined by the formulas $\varphi_{a_1}, \dots \varphi_{a_{k+1}}$ on $\mathcal{M} \sqcup \mathcal{E}$, respectively. We will prove that, for any $x \in Char_m$, if $x \Vdash_V \varphi_a$ and $x \Vdash_V \varphi_b$ then $a = b$. If $a = a_{k+1}$ or $b = a_{k+1}$ then the statement follows from the formula (3). Let $a, b \in \{a_1, \dots, a_k\}$ and $a \neq b$. Then according to (2) and (1) we have $x \Vdash_V f_a$, $x \Vdash_V f_b$. By construction of f_a and f_b we have $x \Vdash_V h_a \wedge \neg h_b$ and $x \Vdash_V h_b \wedge \neg h_a$, a contradiction.

We introduce the following formulas :

$$\Phi_i = \bigvee_{a \in \mathcal{N}, a \Vdash p_i} \varphi_a \qquad (i = 1, \dots, m). \tag{4}$$

If, for some i, $\forall a \in \mathcal{N} a \nVdash_V p_i$ then we put by definition $\Phi_i = p_i \wedge \neg p_i$.

Now we define a new valuation W on the frame of $Char_{m+s}$ as follows:

$$W(p_i) = \{ \, x \mid x \Vdash_V \Phi_i \, \}.$$

This valuation is defined by the formulas Φ_i $(1 \leq i \leq n)$ and $W(p_i) = V(p_i)$ on the model $\mathcal{N}$. It means $Char_{m+s} \nVdash_W B(p_1, \dots, p_m)$. We need to show that $Char_{m+s} \Vdash_W A(p_1, \dots, p_m)$. For this it is necessary to prove

Lemma 3.7. *Let $\alpha(p_1,\dots,p_m)$ be an n-modal formula. Suppose an element x from the characterizing model $Char_{m+s}$ is such that $x \Vdash_V \varphi_a$, where a is an element of the model $\mathcal{N} = \mathcal{M} \sqcup \mathcal{E}$. Then*

$$x \Vdash_W \alpha \iff a \Vdash_V \alpha.$$

Proof. Induction on the length of the formula α. Let $\alpha = p_j$. If $a \Vdash_V p_j$, then by the formula (4) Φ_j contains φ_a as a disjunctive part, hence according to the condition of lemma: $x \Vdash_V \Phi_j$ and by the definition of W $x \Vdash_W p_j$. Suppose $x \Vdash_W p_j$, then according to the definition of the valuation W there exists some disjunctive term φ_b in Φ_j where $b \in \mathcal{N}$ for which $x \Vdash_V \varphi_b$. As stated above this implies $b = a$. Therefore $a \Vdash_V p_j$ by the definition of Φ_j. We accomplished the first step of our inductive proof.

The proof for logical connectives $\vee, \wedge, \neg, \to$ is trivial.

Now consider the case $\alpha = \Box_i\beta$.

a) Put $a = a_{k+1}$, i.e. $x \Vdash \neg(\bigvee_{a\in\mathcal{M}} \varphi_a)$. Let $K = \{z | xR_iz\}$. Then $\forall z \in K$ $z \Vdash_V \neg(\bigvee_{a\in\mathcal{M}} \varphi_a)$. For, if not, there exists an element $b \in \mathcal{M}$, for which $z \Vdash_V \varphi_b$. According to the definition of K and the properties of the relation R_i we have $(z, x) \in R_i$. Since $\Box_i(\bigvee_{bR_iy} f_y)$ is a conjunctive member of the formula φ_b there is an element $c \in \mathcal{M}$ for which $x \Vdash_V f_c$. From (2) it follows $x \Vdash_V \varphi_c$, but this contradicts with the condition $x \Vdash_V \varphi_{a_{k+1}}$. Therefore in our case if $V|_{\mathcal{E}} = \{a_{k+1}\}$ then $W(p_i) \supset K$ otherwise $W(p_i) \cap K = \emptyset$ holds, i.e. the valuation W does not distinguish the elements of the set K. It means $\forall\gamma\forall y \in K[y \Vdash_W \gamma \iff a_{k+1} \Vdash_V \gamma]$. Thus in this case our lemma is proved.

b) Let $a \neq a_{k+1}$. Suppose $x \Vdash_W \Box_i\beta$. Let b be some element for which the relation aR_ib holds. Since the relation aR_ib holds the formula φ_a contains $\Diamond_i f_b$ as a conjunctive term. Now the condition $x \Vdash_V \varphi_a$ yields $x \Vdash_V \Diamond_i f_b$. Hence $\exists t[(xR_it) \wedge (t \Vdash_V f_b)]$. Since the formula $\Box_1 \dots \Box_n(f_b \to g_b)$ is a conjunctive term of the formula φ_a and the relation xR_it holds, the assertion $t \Vdash_V g_b$ follows from the facts $x \Vdash_V \varphi_a$ and $t \Vdash_V f_b$. Now from this assertion and the construction of the formula φ_b we have $t \Vdash_V \varphi_b$. The relation xR_it and the condition $x \Vdash_W \Box_i\beta$ yield $t \Vdash_W \beta$. Therefore $b \Vdash_V \beta$ holds by the inductive hypotheses. Since this holds for any element b, for which the relation aR_ib holds we have $a \Vdash_V \Box_i\beta$.

Vice versa, let $a \Vdash_V \Box_i\beta$. That is $\forall b \in \mathcal{M} : (aR_ib) \to (b \Vdash_V \beta)$. Note $x \Vdash_V \varphi_a$ by hypotheses, φ_a contains $\Box_i \bigvee_{aR_iy} f_y$, hence $\forall z : (xR_iz) \to (z \Vdash_V \bigvee_{aR_iy} f_y)$. Let xR_iz. This yields $\exists b[(aR_ib) \wedge (z \Vdash_V f_b)]$. Because the formula φ_a contains the formula $\Box_1 \dots \Box_n(f_b \to g_b)$ as a conjunctive term, we have $z \Vdash_V g_b$, that yields $z \Vdash_V \varphi_b$. So we get $(b \Vdash_V \beta) \wedge (z \Vdash_V \varphi_b)$, therefore by inductive hypothesis $z \Vdash_W \beta$. And since z is any element for which xR_iz we have $x \Vdash_W \Box_i\beta$. The lemma is proved. □

Proof. To complete the proof of Theorem 3.6 we put $A(p_1,\dots,p_m)$ instead of the formula α in Lemma 3.7. Note that $\forall x \in Char_{m+k}$ if $\forall a \in \mathcal{M}\, x \nVdash_V \varphi_a$

then according to (3) $x \Vdash_V \varphi_{a_{k+1}}$, that is $\forall x \in Char_{m+s} \exists a \in \mathcal{N}(x \Vdash_V \varphi_a)$. Since according to the hypothesis $\mathcal{N} \Vdash_V A(p_1, \ldots, p_m)$ applying Lemma 3.7 we arrive at: $Char_{m+k} \Vdash_W A(p_1, \ldots, p_m)$. Remind that $Char_{m+k} \nVdash_W B(p_1, \ldots, p_m)$. The model $Char_{m+k}$ is $(m+k)$-characterizing model of the logic $S5_nC$ and the valuation W is definable. Therefore the inference rule $A(p_1, \ldots, p_m)/B(p_1, \ldots, p_m)$ is not admissible in logic $S5_nC$. The Theorem 3.6 is proved. □

Using this theorem we can construct an algorithm for verifying the admissibility of the inference rules in the logic $S5_nC$. Hence we have

Corollary 3.8. *The logic $S5_nC$ is decidable with respect to admissibility of the inference rules.*

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