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ON THE CUBIC FIELDS $\mathbf{Q}(\theta)$ DEFINED BY $\theta^3 - 3\theta + b^3 = 0$

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Abstract. We consider families of complex cubic fields introduced by Ishida. Using the Voronoi continued fraction expansion, we find all the reduced principal ideals.

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1. Introduction

Let $\mathbf{Z}$ be the set of rational integers, and let θ be the real root of the irreducible cubic polynomial

$$x^3 - 3x + b^3, \quad b\ (\neq 0) \in \mathbf{Z}. \tag{1.1}$$

The discriminant of (1.1) is equal to $-27(b^6-4)$ and negative provided $b \neq \pm 1$. Let $K = \mathbf{Q}(\theta)$ be the cubic field formed by adjoining θ to the rationals $\mathbf{Q}$, and let $\mathbf{Q}[\theta]$ be the ring of algebraic integers in K. These families of complex cubic fields were introduced by Ishida[1]. Ishida constructed an unramified cyclic exrension, of degree 3^2, of K provided $b \equiv 1 \pmod{3^2}$.

In this paper we shall consider the case that $\{1, \theta, \theta^2\}$ *is a basis of* $\mathbf{Q}[\theta]$ and $|b| \geq 2$. We obtain all the reduced principal ideals and a few facts about the ideal class group Cl_K of K. Our method is mainly the algorithm of Voronoi as descrived in Williams, Cormack and Seah[3]. As most of the proofs are elementary or routine, we often omit cumbersome calculations.

Remark 1. If $b \not\equiv 0 \pmod 3$, then K is of Eisenstein type with respect to 3 (cf. [1]).

Remark 2. Since $(\theta^2/3)^3 - 2(\theta^2/3)^2 + (\theta^2/3) - (b^6/27) = 0$, if $b \equiv 0 \pmod 3$, then $\theta^2/3 \in \mathbf{Q}[\theta]$. Hence, if $b \equiv 0 \pmod 3$, then $\{1, \theta, \theta^2\}$ cannot be a basis of $\mathbf{Q}[\theta]$.

Remark 3. Let $\delta_1 = \left(-\frac{b^3}{2} + \sqrt{\frac{b^6}{4} - 1}\right)^{1/3}$, $\delta_2 = \left(-\frac{b^3}{2} - \sqrt{\frac{b^6}{4} - 1}\right)^{1/3}$, $\varepsilon = \delta_1 - \delta_2$. Then the roots of (1.1) are $\theta = \delta_1 + \delta_2$, $\theta' = -\frac{\theta}{2} + i\sqrt{3}\frac{\varepsilon}{2}$, $\theta'' = -\frac{\theta}{2} - i\sqrt{3}\frac{\varepsilon}{2}$.

2. Lattices and Ideals

Let G be an additive abelian group, and let $\alpha_1, \alpha_2, \alpha_3 \in G$. We denote by $[\alpha_1, \alpha_2, \alpha_3]$ the set $\{x_1\alpha_1 + x_2\alpha_2 + x_3\alpha_3; x_i \in \mathbf{Z}\}$. If $\alpha \in K$, we denote its conjugates by α' and α''. Let $\sigma\colon K \to \mathbf{R}^3$ be the monomorphism of $\mathbf{Q}$-vector spaces defined by $\alpha^\sigma = (\alpha, \mathrm{Im}(\alpha'), \mathrm{Re}(\alpha'))$, where $\mathrm{Re}(z)$ and $\mathrm{Im}(z)$ are the real and imaginary parts of the complex number z. Let $\alpha_1, \alpha_2, \alpha_3 \in K$ are rationally independent. We say that $\mathcal{R} = [\alpha_1, \alpha_2, \alpha_3]$ is a lattice of K with basis $\{\alpha_1, \alpha_2, \alpha_3\}$. If $\mathcal{R}$ has a basis of the form $\{1, \alpha_2, \alpha_3\}$, we call $\mathcal{R}$ a 1-lattice. If $\mathcal{R} = [\alpha_1, \alpha_2, \alpha_3]$ and $\gamma\,(\neq 0) \in K$, we define $\gamma\mathcal{R}$ to be the lattice $[\gamma\alpha_1, \gamma\alpha_2, \gamma\alpha_3]$. If $\mathcal{R}_1$ and $\mathcal{R}_2$ are both 1-lattices and $\mathcal{R}_2 = \gamma\mathcal{R}_1$, we say that $\mathcal{R}_1$ and $\mathcal{R}_2$ are *similar* and write this $\mathcal{R}_1 \sim \mathcal{R}_2$. This relation is clearly an equivalence relation. Let $\mathcal{R}$ be a lattice of K, and let $\omega \in \mathcal{R}$. We define $C(\omega)$ to be

$$C(\omega) = \{(x, y, z) \in \mathbf{R}^3; |x| \le |\omega|, y^2 + z^2 \le \omega'\omega''\}.$$

We say that ω is a *relative minimum* of $\mathcal{R}$ if

$$C(\omega) \cap \mathcal{R}^\sigma = \{0^\sigma, \omega^\sigma, -\omega^\sigma\}.$$

If ω and φ are relative minima of $\mathcal{R}$ such that

$$0 < \varphi < \omega, \quad \varphi'\varphi'' > \omega'\omega'',$$

and there does not exist a $\psi \in \mathcal{R}$ such that

$$\varphi < \psi < \omega, \quad \varphi'\varphi'' > \psi'\psi'',$$

we call ω the relative minimum *adjacent* to φ. If $\mathcal{R}$ is a 1-lattice in which 1 is a relative minimum, we call $\mathcal{R}$ a *reduced* lattice.

If $\{1, \omega_2, \omega_3\}$ is a basis of $\mathbf{Q}[\theta]$, we know that any ideal $\mathcal{I}$ of $\mathbf{Q}[\theta]$ has a basis $\{\alpha_1, \alpha_2, \alpha_3\}$, *where*

$$\alpha_2 = a_1 + a_2\omega_2, \quad \alpha_3 = a_3 + a_4\omega_2 + a_5\omega_3.$$

Here $\alpha_1, a_i \in \mathbf{Z}$, $\alpha_1, a_2, a_5 > 0$, and α_1, a_2, a_5 are uniquely determined by $\mathcal{I}$. We let $L(\mathcal{I})$ denote α_1. If we let $N(\mathcal{I})$ denote the norm of $\mathcal{I}$, then $N(\mathcal{I}) = \alpha_1 a_2 a_5$. If we put $\mathcal{R}(\mathcal{I}) = [1, \alpha_2/\alpha_1, \alpha_3/\alpha_1]$, we say that $\mathcal{R}(\mathcal{I})$ is the 1-lattice which *corresponds* to the ideal $\mathcal{I}$. Let $\mathcal{I}$ be a primitive ideal. We say that $\mathcal{I}$ is a *reduced* ideal if $\mathcal{R}(\mathcal{I})$ is a reduced lattice. We say that two ideals $\mathcal{I}$ and $\mathcal{J}$ are equivalent, written $\mathcal{I} \sim \mathcal{J}$, when there exist $\gamma\ (\neq 0) \in K$ such that $\mathcal{J} = \gamma\mathcal{I}$. From the definitions, it is clear that $\mathcal{I} \sim \mathcal{J}$ if and only if $\mathcal{R}(\mathcal{I}) \sim \mathcal{R}(\mathcal{J})$ (cf. [4], Lemma 2.1). Notice that if $\mathcal{I}$ and $\mathcal{J}$ are both primitive ideals and $\mathcal{R}(\mathcal{I}) = \mathcal{R}(\mathcal{J})$, then $\mathcal{I} = \mathcal{J}$.

3. Preliminaries

Definition 3.1. Let $\mathcal{R}$ be a lattice of K, and let $\omega \in \mathcal{R}$. We define

$$X_\omega = (2\omega - \omega' - \omega'')/2 \ (= \omega - \mathrm{Re}(\omega')), \quad Y_\omega = (\omega' - \omega'')/2i \ \ (= \mathrm{Im}(\omega')),$$
$$Z_\omega = (\omega' + \omega'')/2 \ \ (= \mathrm{Re}(\omega')), \quad P(\omega) = (X_\omega, Y_\omega) \in \mathbf{R}^2,$$
$$C = \{(x, y, z) \in \mathbf{R}^3; y^2 + z^2 \le 1\}.$$

Let ω^* be that one of elements of $\mathcal{R}$ such that $P(\omega^*) = P(\omega)$, $(\omega^*)^\sigma \in C$ and $|\omega^*|$ is minimal. Note that ω^* does not necessarily exist.

Definition 3.2. Let $\{1, M, N\}$ be a basis of $\mathcal{R}$. We say that $\{1, M, N\}$ is *normalized* provided that

$$\text{(a)} \quad 0 < X_M < X_N, \qquad \text{(b)} \quad Y_M Y_N < 0,$$
$$\text{(c)} \quad |Y_N| < 1/2, \quad 1/2 < |Y_M|.$$

Definition 3.3. Let $V_1, V_2, V_3 \in \mathbf{Q}$. We define $F(V_1, V_2, V_3) = N_K(V_1 + V_2\theta + V_3\theta^2) = 9V_1V_3^2 + 3b^3V_2V_3^2 + b^6V_3^3 + 6V_1^2V_3 - 3V_1V_2^2 + 3b^3V_1V_2V_3 - b^3V_2^3 + V_1^3$, where N_K denotes the norm of K over $\mathbf{Q}$.

Lemma 3.4. *Let $V = V_1 + V_2\theta + V_3\theta^2$ ($V_i \in \mathbf{Q}$) be any element of $\mathcal{R}$.*

1. $X_V = \dfrac{3}{2}(-2V_3 + V_2\theta + V_3\theta^2)$.
2. $Y_V = \dfrac{\sqrt{3}}{2}\varepsilon(V_2 - V_3\theta)$.
3. $Z_V = \dfrac{1}{2}(2V_1 + 6V_3 - V_2\theta - V_3\theta^2)$.

Proof. These are all easy calculations from definitions. $\square$

Lemma 3.5.

1. $V > 0 \iff N_K(V) > 0, \quad V < 0 \iff N_K(V) < 0.$
2. $|Y_V| > \sqrt{m}/2 \iff U(V) = U_1 + U_2\theta + U_3\theta^2 > 0,$
   $|Y_V| < \sqrt{m}/2 \iff U(V) = U_1 + U_2\theta + U_3\theta^2 < 0,$

where $m\,(> 0) \in \mathbf{Z}$, $U_1 = -12V_2^2 + 6b^3V_2V_3 - m$, $U_2 = 6V_2V_3 - 3b^3V_3^2$, $U_3 = -3V_3^2 + 3V_2^2$.

3. $V^\sigma \in C \iff W(V) = W_1 + W_2\theta + W_3\theta^2 \le 0,$

where $W_1 = -1 - 3V_2^2 + b^3V_2V_3 + (V_1 + 3V_3)^2$, $W_2 = -b^3V_3^2 - V_1V_2$, $W_3 = -3V_3^2 + V_2^2 - V_1V_3$.

4. $F(aV_1, aV_2, aV_3) = a^3F(V_1, V_2, V_3)$, *where $a \in \mathbf{Q}$.*

Proof. 3. $V^\sigma \in C \iff Y_V^2 + Z_V^2 \le 1$. Otheres are easy to verify. $\square$

4. All the reduced principal ideals

From now on, we shall consider the case that $\{1,\theta,\theta^2\}$ is a basis of $\mathbf{Q}[\theta]$. For a detailed description of our method in the following Theorems we refer the reader to Williams, Cormack and Seah[3], Williams, Dueck and Schmid[4] and Williams[5].

Theorem 4.1. *If $\mathbf{Z}[\theta]=\mathbf{Q}[\theta]$ and $b\geq 2$, then all the reduced principal ideals of $\mathbf{Q}[\theta]$ are $\mathcal{I}_1=[1,\theta,\theta^2]$, $\mathcal{I}_2=[3b^2-2,(3b^2-2)\theta,\theta^2+2b\theta+b^2-1]$, $\mathcal{I}_5=[3b,3b\theta,\theta^2+2b\theta+b^2-3]$,*

(1) b is even: $\mathcal{I}_3=[\frac{3}{2}b^2-2,(\frac{3}{4}b^2-1)\theta,\theta^2+\frac{b}{2}\theta+b^2-4]$, $\mathcal{I}_4=[\frac{3}{2}b^3-3b^2+3b-2,(\frac{3}{4}b^3-\frac{3}{2}b^2+\frac{3}{2}b-1)\theta+\frac{3}{4}b^3-\frac{3}{2}b^2+\frac{3}{2}b-1,\theta^2+(\frac{3}{2}b^2-b)\theta+\frac{3}{2}b^3-\frac{7}{2}b^2+3b-3]$,

(2) b is odd: $\mathcal{I}_3=[3b^2-4,(3b^2-4)\theta,\theta^2+(\frac{3}{2}b^2+\frac{1}{2}b-2)\theta+b^2-4]$, $\mathcal{I}_4=[3b^3-6b^2+6b-4,(3b^3-6b^2+6b-4)\theta,\theta^2+(\frac{3}{2}b^3-\frac{3}{2}b^2+2b-2)\theta+\frac{3}{2}b^3-\frac{7}{2}b^2+3b-3]$.

Proof. Let $\theta_g^{(i)}$ denote the relative minimum adjacent to 1 in a lattice $\mathcal{R}_i$. Let $\mathcal{A}_i=\{N_i,M_i,N_i-M_i\}$ and $\mathcal{B}_i=\{[-Z_\beta]+j+\beta; j\in\{0,1\},\beta\in\mathcal{A}_i\}$, where $[\dots]$ is the greatest integer function and $\mathcal{R}_i=[1,M_i,N_i]$ (cf. [5], p.646 and [3], Corollary 5.1.3).

(1) Let $\mathcal{R}_1=[1,\theta,\theta^2]$, $M_1=-b\theta+\theta^2$, $N_1=-(b^2+1)\theta+b\theta^2$. Clearly $\mathcal{R}_1=[1,M_1,N_1]$. First we shall show that $\{1,M_1,N_1\}$ is normalized and $|Y_{M_1}|<\sqrt{3}/2$.

(a) We have $X_{M_1}=-3-\frac{3}{2}b\theta+\frac{3}{2}\theta^2$, $X_{N_1}=-3b-\frac{3}{2}(b^2+1)\theta+\frac{3}{2}b\theta^2$ and $X_{N_1}-X_{M_1}=-3b+3-\frac{3}{2}(b^2-b+1)\theta+\frac{3}{2}(b-1)\theta^2$. Further, $F(-3,-\frac{3}{2}b,\frac{3}{2})=\frac{27}{8}(2b^6+3b^4+6b^2-2)>0$, $F(-3b,-\frac{3}{2}(b^2+1),\frac{3}{2}b)=\frac{27}{8}b(2b^8+6b^6+12b^4+11b^2+6)>0$ and $F(-3b+3,-\frac{3}{2}(b^2-b+1),\frac{3}{2}(b-1))=\frac{27}{8}(2b^9-6b^8+12b^7-17b^6+24b^5-30b^4+32b^3-24b^2+12b-4)>0$; hence, $0<X_{M_1}<X_{N_1}$.

(b) We have $Y_{M_1}=\frac{\sqrt{3}}{2}\varepsilon(-b-\theta)$, $Y_{N_1}=\frac{\sqrt{3}}{2}\varepsilon\{-(b^2+1)-b\theta\}$ and $Y_{M_1}Y_{N_1}=\frac{3}{4}\varepsilon^2\{b(b^2+1)+(2b^2+1)\theta+b\theta^2\}$. Further, $F(b(b^2+1),2b^2+1,b)=-3b<0$; hence $Y_{M_1}Y_{N_1}<0$.

(c) Since $N_K(U(N_1))=-(81b^{12}+324b^{10}+810b^8+1125b^6+1089b^4540b^2+208)<0$, we have $|Y_{N_1}|<1/2$. Also, since $N_K(U(M_1))=162b^8-162b^6-261b^4-1152b^2-100>0$, we have $1/2<|Y_{M_1}|$.

(d) Since $N_K(U(M_1))=-(486b^6+891b^4+1620b^2+432)<0$, we have $|Y_{M_1}|<\sqrt{3}/2$:

Next we shall show that $\theta_g^{(1)}=[-Z_{M_1}]+M_1$. We have $Z_{M_1}=3+\frac{1}{2}b\theta-\frac{1}{2}\theta^2$, $N_K(b^2-2+Z_{M_1})=-\frac{9}{8}b^4-\frac{3}{2}b^2+\frac{1}{4}<0$ and $N_K(b^2-1+Z_{M_1})=\frac{9}{8}b^4+\frac{3}{4}b^2+\frac{1}{2}>0$. Therefore $[-Z_{M_1}]=b^2-2$. We have $Z_{N_1}=3b+\frac{1}{2}(b^2+1)\theta-\frac{1}{2}b\theta^2$, $N_K(b^3-b+Z_{N_1})=-\frac{9}{4}b^5-\frac{27}{8}b^3-\frac{3}{2}b<0$ and $N_K(b^3-b+1+Z_{N_1})=\frac{9}{4}b^6-\frac{9}{4}b^5+\frac{9}{2}b^4-\frac{3}{8}b^3+\frac{3}{4}b^2+\frac{3}{2}b+\frac{1}{4}>0$. Therefore $[-Z_{N_1}]=b^3-b$. We also have $Z_{N_1-M_1}=3(b-1)+\frac{1}{2}(b^2-b+1)\theta-\frac{1}{2}(b-1)\theta^2$, $N_K(b^3-b^2-b+1+Z_{N_1-M_1})=-\frac{9}{8}b^6+\frac{9}{4}b^4-\frac{15}{4}b^3+\frac{9}{2}b^2-3b+1<0$ and $N_K(b^3-b^2-b+2+Z_{N_1-M_1})=\frac{9}{8}b^6-\frac{9}{2}b^5+9b^4-\frac{21}{2}b^3+\frac{15}{2}b^2-3b+\frac{1}{2}>0$. Therefore

$[-Z_{N_1-M_1}] = b^3-b^2-b+1$. Since $\theta < 0$, it is easily seen that the least positive element of $\mathcal{B}_1$ is $[-Z_{M_1}] + M_1$. Since $N_K(W([-Z_{M_1}] + M_1)) = -9b^2 < 0$, $([-Z_{M_1}]+M_1)^\sigma \in C$. Therefore $\theta_g^{(1)} = [-Z_{M_1}]+M_1$. $N_K(\theta_g^{(1)}) = 3b^2-2 \neq 1$. Let $\theta_h^{(1)} = [-Z_{N_1}] + N_1 = b^3 - b - (b^2+1)\theta + b\theta^2$.

(2) Since following procedures ((2) to (5)) are the same as (1), we only state obtained results. Let $\mathcal{R}_2 = [1, 1/\theta_g^{(1)}, \theta_h^{(1)}/\theta_g^{(1)}]$. Let $M_2 = 1/\theta_g^{(1)} = \frac{1}{3b^2-2}(-b^2+1-2b\theta-\theta^2)$, $N_2 = \theta_h^{(1)}/\theta_g^{(1)} = \frac{1}{3b^2-2}\{b^3-b+(-b^2+2)\theta+b\theta^2\}$. Then $\{1, M_2, N_2\}$ is normalized, $|Y_{M_2}| < \sqrt{3}/2$. $[-Z_{N_2}] = -1$, and then $[-Z_{M_2}] = 0$.

(i) If $b \geq 3$, then $[-Z_{N_2-M_2}] = -1$.

(ii) If $b = 2$, then $[-Z_{N_2-M_2}] = -2$.

Since $N_K(W([-Z_{M_2}]+1+M_2)) = -\frac{9b^2}{(3b^2-2)^2} < 0$, $([-Z_{M_2}]+1+M_2)^\sigma \in C$. $\text{Min}\{\omega \in \mathcal{B}_2; \omega > 0, \omega^\sigma \in C\} = [-Z_{M_2}] + 1 + M_2$; therefore $\theta_g^{(2)} = [-Z_{M_2}] + 1 + M_2$. $N_K(\theta_g^{(1)}\theta_g^{(2)}) = 3b^2-4 \neq 1$. Let $\theta_h^{(2)} = [-Z_{N_2}] + N_2$.

(3) Let $\mathcal{R}_3 = [1, 1/\theta_g^{(2)}, \theta_h^{(2)}/\theta_g^{(2)}]$. We have $1/\theta_g^{(2)} = \frac{1}{3b^2-4}(2b^2-8+b\theta+2\theta^2)$ and $\theta_h^{(2)}/\theta_g^{(2)} = \frac{1}{3b^2-4}\{b^3-2b^2-4b+8+(-b^2-b+2)\theta+(b-2)\theta^2\}$. Let $M_3 = \theta_h^{(2)}/\theta_g^{(2)}$, $N_3 = \frac{1}{3b^2-4}\{b^3-4b+(-b^2+2)\theta+b\theta^2\}$. Then $\mathcal{R}_3 = [1, M_3, N_3]$, $\{1, M_3, N_3\}$ *is normalized.* $|Y_{M_3}| < \sqrt{3}/2$. $[-Z_{N_3}] = 0$, $[-Z_{M_3}] = 0$, and then $[-Z_{N_3-M_3}] = -1$. Since $N_K(W([-Z_{M_3}] + M_3)) = -\frac{1}{(3b^2-4)^2}(9b^5 - 36b^4 + 90b^3 - 162b^2 + 168b - 72) < 0$, $([-Z_{M_3}] + M_3)^\sigma \in C$. $\text{Min}\{\omega \in \mathcal{B}_3; \omega > 0, \omega^\sigma \in C\} = [-Z_{M_3}] + M_3$; therefore $\theta_g^{(3)} = [-Z_{M_3}] + M_3$. $N(\theta_g^{(1)}\theta_g^{(2)}\theta_g^{(3)}) = 3b^3-6b^2+6b-4 \neq 1$. Let $\theta_h^{(3)} = [-Z_{N_3}] + N_3$.

(4) Let $\mathcal{R}_4 = [1, 1/\theta_g^{(3)}, \theta_h^{(3)}/\theta_g^{(3)}]$. Let $M_4 = 1/\theta_g^{(3)} = \frac{1}{3b^3-6b^2+6b-4}\{-b^3 + b^2 - 2b + 2 + (b^2-4b+4)\theta + (2b-2)\theta^2\}$, $N_4 = \theta_h^{(3)}/\theta_g^{(3)} = \frac{1}{3b^3-6b^2+6b-4}\{b^3 - 2b^2+2b+(-b^2-b+2)\theta+(b-2)\theta^2\}$. Then $\mathcal{R}_4 = [1, M_4, N_4]$ and $\{1, M_4, N_4\}$ *is nomalized.* $|Y_{M_4}| < \sqrt{3}/2$. $[-Z_{N_4}] = 0$, $[-Z_{M_4}] = 0$, and then $[-Z_{N_4-M_4}] = -1$. Since $N_K(W([-Z_{N_4}]+N_4)) = -\frac{1}{(3b^3-6b^2+6b-4)^2}(9b^6-27b^5+36b^4-24b^3+9b^2-12b-8) < 0$, $([-Z_{N_4}] + N_4)^\sigma \in C$. $\text{Min}\{\omega \in \mathcal{B}_4; \omega > 0, \omega^\sigma \in C\} = [-Z_{N_4}] + N_4$; therefore $\theta_g^{(4)} = [-Z_{N_4}] + N_4$. $N(\theta_g^{(1)}\theta_g^{(2)}\theta_g^{(3)}\theta_g^{(4)}) = 3b \neq 1$. Let $\theta_h^{(4)} = [-Z_{M_4}] + M_4$.

(5) Let $\mathcal{R}_5 = [1, 1/\theta_g^{(4)}, \theta_h^{(4)}/\theta_g^{(4)}]$. We have $1/\theta_g^{(4)} = \frac{1}{3b}(-b^2+3b-6+b\theta+2\theta^2)$ and $\theta_h^{(4)}/\theta_g^{(4)} = \frac{1}{3b}(-b^2+3-2b\theta-\theta^2)$. Let $M_5 = \theta_h^{(4)}/\theta_g^{(4)}$, $N_5 = \frac{1}{3b}(-2b^2 + 3b - 3 - b\theta + \theta^2)$. Then $\mathcal{R}_5 = [1, M_5, N_5]$ and $[1, M_5, N_5]$ is normalized and then $[-Z_{N_5}] = b-1$,

(i) If b is even, then $[-Z_{M_5}] = \frac{b}{2} - 1$, $[-Z_{N_5-M_5}] = \frac{b}{2} - 1$, $[-Z_{N_5+M_5}] = \frac{3}{2}b - 1$, $[-Z_{2N_5+M_5}] = \frac{5}{2}b - 2$,

(ii) If b is odd, then $[-Z_{M_5}] = \frac{b-1}{2}$, $[-Z_{N_5-M_5}] = \frac{b-3}{2}$, $[-Z_{N_5+M_5}] = \frac{3b-3}{2}$, $[-Z_{2N_5+M_5}] = \frac{5b-5}{2}$.

Since $N_K(W([-Z_{N_5}] + N_5)) = \frac{-21b^2+4}{9b^2} < 0$. $([-Z_{N_5}] + N_5)^\sigma \in C$. Let $\mathcal{B}'_5 = \{[-Z_{N_5+M_5}] + N_5 + M_5, [-Z_{N_5+M_5}] + 1 + N_5 + M_5, [-Z_{2N_5+M_5}] + 2N_5 + M_5, [-Z_{2N_5+M_5}] + 1 + 2N_5 + M_5\}$. $\mathrm{Min}\{\omega \in \mathcal{B}_5 \cup \mathcal{B}'_5; \omega > 0, \omega^\sigma \in C\} = [-Z_{N_5}] + N_5$; therefore $\theta_g^{(5)} = [-Z_{N_5}] + N_5$. $N_K(\theta_g^{(1)}\theta_g^{(2)}\theta_g^{(3)}\theta_g^{(4)}\theta_g^{(5)}) = 1$.

(6) From the results above ((1) to (5)), it follows that $\{\mathcal{R}; \mathcal{R}$ is a reduced lattice, $\mathcal{R}_1 \sim \mathcal{R}\} = \{\mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3, \mathcal{R}_4, \mathcal{R}_5\}$ (cf. [4], p.243).

(7) Let $\mathcal{I}_1 = [1, \theta, \theta^2]$, $\mathcal{I}_2 = [3b^2 - 2, -b^2 + 1 - 2b\theta - \theta^2, b^3 - b + (-b^2 + 2)\theta + b\theta^2] = [3b^2 - 2, (3b^2 - 2)\theta, \theta^2 + 2b\theta + b^2 - 1]$, $\mathcal{I}_5 = [3b, -b^2 + 3 - 2b\theta - \theta^2, -2b^2 + 3b - 3 - b\theta + \theta^2] = [3b, 3b\theta, \theta^2 + 2b\theta + b^2 - 3]$, $\mathcal{J}_3 = [3b^2 - 4, b^3 - 2b^2 - 4b + 8 + (-b^2 - b + 2)\theta + (b - 2)\theta^2, b^3 - 4b + (-b^2 + 2)\theta + b\theta^2]$, $\mathcal{J}_4 = [3b^3 - 6b^2 + 6b - 4, -b^3 + b^2 - 2b + 2 + (b^2 - 4b + 4)\theta + (2b - 2)\theta^2, b^3 - 2b^2 + 2b + (-b^2 - b + 2)\theta + (b - 2)\theta^2]$. Then $\mathcal{R}(\mathcal{I}_1) = \mathcal{R}_1$, $\mathcal{R}(\mathcal{I}_2) = \mathcal{R}_2$, $\mathcal{R}(\mathcal{I}_5) = \mathcal{R}_5$, $\mathcal{R}(\mathcal{J}_3) = \mathcal{R}_3$, $\mathcal{R}(\mathcal{J}_4) = \mathcal{R}_4$. Clearly $\mathcal{I}_1, \mathcal{I}_2, \mathcal{I}_5$ are reduced.

(i) b is even: Let $b = 2m$. Then $\mathcal{J}_3 = 2[6m^2 - 2, (3m^2 - 1)\theta, \theta^2 + m\theta + 4m^2 - 4]$, $\mathcal{J}_4 = 2[12m^3 - 12m^2 + 6m - 2, (6m^3 - 6m^2 + 3m - 1)\theta + 6m^3 - 6m^2 + 3m - 1, \theta^2 + (6m^2 - 2m)\theta + 12m^3 - 14m^2 + 6m - 3]$.

(ii) b is odd: Let $b = 2m + 1$. Then $\mathcal{J}_3 = [12m^2 + 12m - 1, (12m^2 + 12m - 1)\theta, \theta^2 + (6m^2 + 7m)\theta + 4m^2 + 4m - 3]$, $\mathcal{J}_4 = [24m^3 + 12m^2 + 6m - 1, (24m^3 + 12m^2 + 6m - 1)\theta, \theta^2 + (12m^3 + 12m^2 + 7m)\theta + 12m^3 + 4m^2 + m - 2]$.

Therefore if we put $\mathcal{I}_3 = \mathcal{J}_3/2$, $\mathcal{I}_4 = \mathcal{J}_4/2$ (when b is even) and $\mathcal{I}_3 = \mathcal{J}_3$, $\mathcal{I}_4 = \mathcal{J}_4$ (when b is odd), then $\mathcal{I}_3, \mathcal{I}_4$ are reduced and $\mathcal{R}(I_3) = \mathcal{R}_3$, $\mathcal{R}(I_4) = \mathcal{R}_4$. □

Corollary 4.2. *Only under the asumption $b \geq 2$ (without the asumption $\mathbf{Q}[\theta] = \mathbf{Z}[\theta]$), the Voronoi continued fraction expansion for the order $\mathbf{Z}[\theta]$ has period length '5'and the fundamental unit of the order $\mathbf{Z}[\theta]$ is $b^4 - b^2 + 1 - (b^3 + b)\theta + b^2\theta^2$.*

Proof. The parts (1) to (5) in the proof of Theorem 4.1 and no other than the Voronoi continued fraction for the order $\mathbf{Z}[\theta]$ (cf. [6], p. 248). So $\theta_g^{(1)}\theta_g^{(2)}\theta_g^{(3)}\theta_g^{(4)}\theta_g^{(5)} = b^4 - b^2 + 1 - (b^3 + b)\theta + b^2\theta^2$ is the fundamental unit of the order $\mathbf{Z}[\theta]$. □

5. About Cl_K

Definition 5.1. If $\mathcal{I}$ is an ideal of K, we define $Cl(\mathcal{I})$ to be the ideal class of $\mathcal{I}$ in the ideal class group Cl_K.

Theorem 5.2. *If $\mathbf{Z}[\theta] = \mathbf{Q}[\theta]$, $b \not\equiv 0 \pmod 3$ and $b \geq 2$, then Cl_K contains a cyclic subgroup generated by $Cl(\mathcal{I})$ of order 3, where $\mathcal{I} = [b, b\theta, \theta^2 - 3]$.*

Proof. We shall consider the case $b \not\equiv 0 \pmod 3$ *because of* Remark 2. Let $\mathcal{I} = [b, b\theta, \theta^2 - 3]$. It is easily seen that $\mathcal{I}$ is a ideal of K. Since $L(\mathcal{I}) = b$, $N(\mathcal{I}) = b^2$, by [5,Theorem 9.1] $\mathcal{I}$ is a reduced ideal.

We shall show that $\mathcal{I}^2 = [b^2, b^2\theta, \theta^2 - 3]$ *is a reduced ideal.*

We consider $\mathcal{R}(\mathcal{I}^2) = [1, \theta, -\frac{3}{b^2} + \frac{1}{b^2}\theta^2]$.

(1) The case $b \geq 4$.

Let $M = \frac{1}{b^2}\{-3b + 3 - b^2\theta + (b-1)\theta^2\}$, $N = \frac{1}{b}(-3 - b\theta + \theta^2)$. Clearly $\mathcal{R}(\mathcal{I}^2) = [1, M, N]$. By the same argument as in Theorem 4.1 we obtain following results. $\{1, M, N\}$ is normalized, $|Y_M| < \sqrt{3}/2$, $[-Z_N] = b$, $[-Z_M] = b - 1$ and $[-Z_{N-M}] = 0$. Let $\mathcal{B} = \{N^*, M^*, (N-M)^*\}$ (cf. [4], p.266). Then $\mathcal{B}^\sigma \cap C(1) \neq \emptyset$. Therefore $\mathcal{R}(\mathcal{I}^2)$ is reduced.

(2) The case $b = 2$.

Let $M = -\frac{3}{4} + \frac{1}{4}\theta^2$, $N = -\frac{3}{4} - \theta + \frac{1}{4}\theta^2$. Then $\mathcal{R}(\mathcal{I}^2) = [1, \theta, -\frac{3}{4} + \frac{1}{4}\theta^2] = [1, M, N]$, $\{1, M, N\}$ is normalized, $|Y_M| < \sqrt{3}/2$, $[-Z_N] = 2$, $[-Z_M] = 0$ and $[-Z_{N-M}] = 1$. Let $\mathcal{B} = \{N^*, M^*, (N-M)^*\}$. Then $\mathcal{B}^\sigma \cap C(1) = \emptyset$. Therefore $\mathcal{R}(\mathcal{I}^2)$ is reduced.

From (1) and (2), $\mathcal{I}^2$ is a reduced ideal. Therefore by Theorem 4.1 $Cl(\mathcal{I})$, $Cl(\mathcal{I}^2) \neq Cl(1)$. Since $\theta\mathcal{I}^3 = \theta[b^3, b^3\theta, -3 + \theta^2] = b^3[1, \theta, \theta^2]$, $\mathrm{ord}Cl(\mathcal{I}) = 3$. □

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