SUT Journal of Mathematics
Vol. 32, No. 2 (1996), 109–120

GLOBAL STRUCTURE OF SOLUTIONS OF SOME SINGULAR OPERATORS WITH APPLICATIONS TO IMPULSIVE INTEGRODIFFERENTIAL BOUNDARY VALUE PROBLEMS*

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(Received February 16, 1996)

Abstract. For a kind of singular non-continuous operators, we prove that unbounded continua of the solution set exist. As applications, we give global structure of the solution set to some impulsive singular integrodifferential boundary value problems.

AMS 1991 *Mathematics Subject Classification.* Primary 34B15, Secondary 47H15.

Key words and phrases. Singular operators, Coninua, Fixed point index, Impulsive integrodifferential boundary value problems

1. Introduction

Boundary value problems with singular nature arise quite naturally in physics, fluid dynamics and the study of radially symmetric solutions to elliptic problems, see [1]–[4] for example, while impulsive differential equations describe processes with a sudden change of their state at certain moments, see [5]–[8] and the references therein. At present, most papers study the solvabity of such problems, where the nonlinearity is sublinear at infinity, see [1]–[4], or multiple solutions of superlinear problems with superlinear zeros at the origin, see [5]. Recently, Wong in [9] proved that for some singular boundary value problems with parameter, solutions exist when $\lambda < \lambda_0$, while no solutions exist when $\lambda > \lambda_0$. His problems involve superlinear nonlinearities at infinity, see also [14].

In this paper, we will study the global structure of the solution set of some singular nonlinear operators, which have some "approximate properties". We do not assume they are defined on the whole cone and continuous. By applying fixed point index on cones, we give the existence of unbounded continua of the solution set.

* This work is supported in part by NSF of Shandong Province and NNSF of China.

As applications, we consider the following impulsive integrodifferential boundary value problems:

$$
(1.1)\qquad
\begin{aligned}
&(Lx)(t) + p(t)f(\lambda, t, x(t), (Hx)(t), (Sx)(t)) = 0,\\
&\qquad t \in (0,1),\ t \neq t_k,\ k = 1,2,\dots,m,\\
&\Delta x\big|_{t=t_k} := x(t_k+0) - x(t_k-0) = I_k(x(t_k)), \quad k = 1,2,\dots,m,\\
&\alpha x(0) - \beta \lim_{t\to 0} p(t)x'(t) = \gamma x(1) + \delta \lim_{t\to 1} p(t)x'(t) = 0
\end{aligned}
$$

where $(Lx)(t) = \dfrac{1}{p(t)}(p(t)x'(t))'$, $f \in C[[0,\infty)\times(0,1)\times \mathbf{R}^+ \times \mathbf{R}^1 \times \mathbf{R}^1, \mathbf{R}^+]$, $\mathbf{R}^+ = (0,\infty)$, $p \in C^1[0,1]$, $p(t) > 0$ for $t \in (0,1)$, H and S are given by

$$
(1.2)\qquad (Hx)(t) = \int_0^t k(t,s)x(s)\,ds, \quad (Sx)(t) = \int_0^1 k_1(t,s)x(s)\,ds
$$

with $k, k_1 \in C[[0,1]\times[0,1],[0,\infty)]$, and $\alpha,\beta,\gamma,\delta \geq 0$, $\beta\gamma + \alpha\delta + \alpha\gamma > 0$, $I_k \in C[[0,\infty),[0,\infty)]$, $k = 1,2,\dots,m$, $0 < t_1 < t_2 < \dots < t_m < 1$. Note that the nonlinear term $f(\lambda,t,x,y,z)$ may be singular at $t = 0,1$ and $x = 0$. Using the existence principle of [11], we prove that unbounded continua of the solution set of (1.1) exist.

2. Global structure of solutions of singular operators

In order to treat global problems, we need the following auxiliary lemma. Recall that a subcontinuum is a maximal connected subspace of a topological space. In the case of metric spaces, a subcontinuum is always closed.

Lemma A. *Let X be a compact metric space, $a_n, a \in X$, $a_n \to a$, E_n is the subcontinuum of X containing a_n. Define $E = \overline{\lim}_{n\to\infty} E_n = \{x \in X :$ There exists a subsequence E_{n_k} and $x_{n_k} \in E_{n_k}$ with $x_{n_k} \to x\}$. Then E is closed and connected.*

Proof. Clearly E_n is compact. Let $x^j \in E$, $x^j \to x$. Suppose $x^j = \lim_{k\to\infty} x^j_{n_k}$, where $x^j_{n_k} \in E^j_{n_k}$. Choose $k(j)$ such that $d(x^j_{n_{k(j)}}, x^j) < 1/j$. Then $\lim_{j\to\infty} x^j_{n_{k(j)}} = x$, hence $x \in E$ by definition. Thus E is closed and compact. Suppose E has a decomposition $E = K \cup S$, where K, S are compact; nonempty and disjoint. Assume $a \in K$. Thus there exist disjoint open sets U, V such that $K \subset U$, $S \subset V$, $\operatorname{cl} U \cap \operatorname{cl} V = \emptyset$, where $\operatorname{cl} U$ denotes the closure of U. Without loss of generality we can assume that $a_n \in U$ for $n \geq 1$. Now we have two cases.

First if there exists N such that $E_n \subset U$ for $n > N$, then by definition $E \subset \operatorname{cl} U$, which contradicts S is nonempty. Next if there exists a subsequence E_{n_k} with $E_{n_k} \not\subset U$. Since E_n is connected, we can find $x_{n_k} \in E_{n_k} \cap \mathrm{b}U$, where b$U$ denotes the boundary of U. By the compactness of X we get $E \cap \mathrm{b}U \neq \emptyset$. This is a contradiction. Q.E.D.

Let $\mathfrak{X}$ be a Banach space, P a cone of $\mathfrak{X}$, $\mathfrak{X}^*$ be a linear vector space. Consider an operator $A\colon \mathbf{R}^* \times \mathcal{D}(A) \to \mathfrak{X}^*$, where $\mathcal{D}(A)$ is a subset of P, $\mathbf{R}^* = [0, \infty)$. Note that $\mathcal{D}(A)$ need not be open or closed. We will study the following operator equation

$$A(\lambda, x) = 0, \quad (\lambda, x) \in \mathbf{R}^* \times \mathcal{D}(A). \tag{2.1}$$

Define $\Sigma \subset \mathbf{R}^* \times \mathcal{D}(A)$ to be the set of all solutions of (2.1). For $\lambda = 0$, we write

$$\Omega^0 = \{x \in \mathcal{D}(A)\colon (0, x) \in \Sigma\}.$$

We always understand Σ to be a metric space with its induced topology from $\mathbf{R}^* \times P$. Let $x^0 \in \Omega^0$, and denote by $E(x^0)$ the subcontinuum of Σ containing $(0, x^0)$. Define

$$E = \mathrm{cl}\Big(\bigcup\{E(x^0)\colon x^0 \in \Omega^0\}\Big)$$

where the closure is taken in the space $\mathbf{R}^* \times P$. Associated with the operator A, we will consider an approximate operator A_n, where $A_n\colon \mathbf{R}^* \times P \to P$ is continuous. Denote the solution set of the following equation

$$x = A_n(\lambda, x) \tag{2.2}$$

by Σ_n, i.e., $\Sigma_n = \{(\lambda, x)\colon (\lambda, x) \in \mathbf{R}^* \times P, (\lambda, x) \text{ is a solution of (2.2)}\}$. Again, we define

$$\Omega_n^0 = \{x \in P\colon (0, x) \in \Sigma_n\}.$$

For $x^0 \in \Omega_n^0$, denote by $E_n(x^0)$ the subcontinuum of Σ_n containing $(0, x^0)$. Write

$$E_n = \mathrm{cl}\Big(\bigcup\{E_n(x^0)\colon x^0 \in \Omega_n^0\}\Big).$$

We will assume the following conditions to be satisfied:

- (N_0) Σ is closed and locally compact in $\mathbf{R}^* \times P$.
- (N_1) A_n are completely continuous on $\mathbf{R}^* \times P$, for any integer $n \in \mathbf{N}$.
- (N_2) Ω_n^0 are nonempty, for any integer $n \in \mathbf{N}$.
- (N_3) If $(\lambda_n, x_n) \in \Sigma_n$ and is a bounded sequence, then there exists a subsequence (λ_{n_k}, x_{n_k}) satisfying $(\lambda_{n_k}, x_{n_k}) \to (\lambda, x)$ and $(\lambda, x) \in \Sigma$.
- (N_4) $\displaystyle\lim_{\|x\| \to \infty} \frac{\|A_n(0, x)\|}{\|x\|} = 0$ for any integer $n \in \mathbf{N}$.

Throughout this section, we use bD to denote the boundary of the set D in the metric space $\mathbf{R}^* \times P$.

REMARK. Condition (N_3) is an approximate hypothesis, which relates the operator A_n with A.

Lemma 2.1. *Let (N_1) (N_2) (N_4) be satisfied. Then E_n is unbounded for every $n \in \mathbf{N}$.*

Proof. Clearly Σ_n is locally compact from condition (N_1). Suppose that E_n is bounded for some n. Let $B_R = \{x \in P \colon \|x\| \le R\}$, $Q_R = [0, R] \times B_R$. Then we can choose $R > 0$ such that $E_n \subset Q_R$ and $E_n \cap \mathrm{b}Q_R = \emptyset$, where

$$\mathrm{b}Q_R = ([0, R] \times \mathrm{b}B_R) \cup (\{R\} \times B_R), \quad \mathrm{b}B_R = \{x \in P \colon \|x\| = R\}.$$

Let $X_n = \Sigma_n \cap Q_R$, then X_n is a compact metric space and $E_n \subset X_n$. Define $Y_n = \Sigma_n \cap \mathrm{b}Q_R$, hence E_n, Y_n are disjoint compact subset of X_n.

Next we will prove that there does not exist a subconinuum of X_n meeting both E_n and Y_n. Suppose the contrary, and Z be a subcontinuum of X_n with $Z \cap E_n \neq \emptyset$, $Z \cap Y_n \neq \emptyset$. Choose $(\lambda, x) \in Z \cap E_n$. First assume $(\lambda, x) \in E_n(x^0)$ where $x^0 \in \Omega_n^0$. Then $Z \cup E_n(x^0)$ is connected. But $E_n(x^0)$ is maximal, hence $Z \cup E_n(x^0) = E_n(x^0)$, in contradiction with $Z \cap Y_n \neq \emptyset$. Thus there exist $x_j^0 \in \Omega_n^0$ and $(\lambda_j, y_j) \in E_n(x_j^0)$ such that $(\lambda_j, y_j) \to (\lambda, x)$. By Lemma A, $E^* = \overline{\lim}_{j \to \infty} E_n(x_j^0)$ is closed and connected. Also since X_n is compact we find a subsequence $x_j^{0\prime}$ of x_j^0 such that $x_j^{0\prime} \to x^0 \in \Omega_n^0$. Clearly $(0, x^0) \in E^*$ by definition, hence $E^* \subset E_n(x^0)$ and $(\lambda, x) \in E_n(x^0)$. By the above step we know this is also a contradiction.

From Lemma 1.1 of [12] we know that there exist disjoint compact subsets K_1, K_2 such that $X_n = K_1 \cup K_2$, $K_1 \supset E_n$, $K_2 \supset Y_n$, hence $K_1 \cap \mathrm{b}Q_R = \emptyset$. Since X_n is a metric space, we get an open set $U \subset Q_R$ with $K_1 \subset U$, $U \cap \mathrm{b}Q_R = \emptyset$, $U \cap K_2 = \emptyset$, $\mathrm{b}U \cap K_2 = \emptyset$, $\mathrm{b}U \cap K_1 = \emptyset$. Thus $\mathrm{b}U \cap \Sigma_n = \emptyset$. By the general homotopy invariance of the fixed point index on cones (see [13] Theorem 11.3) we have

$$i(A_n(\lambda, \cdot), U(\lambda), P) = \mu = \text{const}.$$

where $U(\lambda) = \{x \colon (\lambda, x) \in U\}$. Evidently $U(R) = \emptyset$, hence $\mu = 0$ for $\lambda \in [0, R]$. But when $\lambda = 0$, we have $\Omega_n^0 \subset U(0)$ since $E_n \subset K_1 \subset U$. As a result, $A_n(0, \cdot)$ has no fixed points outside $U(0)$. Thus

$$\mu = i(A_n(0, \cdot), U(0), P) = i(A_n(0, \cdot), B_T, P)$$

where T is large enough. From condition (N_4) and the index computation formula of cone compresion (see [14]) we get $\mu = 1$. Thus the proof is complete.

Lemma 2.2. *Suppose that for every bounded open set G of $\mathbf{R}^* \times P$ which contains $\{0\} \times \Omega^0$, $\mathrm{b}G \cap \Sigma$ is nonempty, then E is unbounded.*

Proof. Suppose the contrary. Then we can choose $R > 0$ such that $E \subset Q_R$, $E \cap \mathrm{b}Q_R = \emptyset$. Let $Y = \Sigma \cap \mathrm{b}Q_R$, $X = \Sigma \cap Q_R$. Since Y and E are disjoint, similar to the proof of Lemma 2.1, we get disjoint compact subsets K_1, K_2 of

X such that $E \subset K_1$, $Y \subset K_2$, $K_1 \cap \mathrm{b}Q_R = \emptyset$, $X = K_1 \cup K_2$. Because $\mathbf{R}^* \times P$ is a regular space, there exists a bounded open set $U \subset \mathbf{R}^* \times P$ such that $K_1 \subset U \subset Q_R$, $U \cap K_2 = \emptyset$, $U \cap \mathrm{b}Q_R = \emptyset$. Furthermore, choose oepn set G satisfying $K_1 \subset G \subset \operatorname{cl} G \subset U$. Consequently $\Sigma \cap \mathrm{b}G = \emptyset$, which contradicts our hypothesis. The proof is complete.

Theorem 2.3. *Suppose* (N_0)–(N_4) *hold. Then E is unbounded.*

Proof. We need only to verify the hypotheses of Lemma 2.2. Suppose G is open and bounded which contains $\{0\} \times \Omega^0$. First we prove that $\Omega_n^0 \subset G$ for n large enough. In fact, suppose there exist $(0, x_n) \in \Omega_n^0 \backslash G$. From ($\mathrm{N}_4$) we know x_n is bounded. Thus from (N_3) we can write $(0, x_n) \to (0, x) \in \Sigma$ (without loss of generality). Obviously $(0, x) \in (\mathbf{R}^* \times P) \backslash G$ and $(0, x) \in \Omega^0$. This contradicts $\Omega^0 \subset G$. Hence there exists N such that $\Omega_n^0 \subset G$ for $n > N$. Since E_n are unbounded we can find $x_n^0 \in \Omega_n^0$ such that $E_n(x_n^0) \cap \mathrm{b}G \neq \emptyset$. Consequently $\Sigma_n \cap \mathrm{b}G \neq \emptyset$. Then condition ($\mathrm{N}_3$) yields $\Sigma \cap \mathrm{b}G \neq \emptyset$. Thus the proof is complete by Lemma 2.2.

Theorem 2.4. *Suppose Ω^0 is bounded, and* (N_0)–(N_4) *hold. Then there exists $x^0 \in \Omega^0$ such that the subcontinuum $E(x^0)$ emanating from $(0, x^0)$ is unbounded.*

Proof. Suppose that $E(x^0)$ is bounded for any $x^0 \in \Omega^0$. Then from Theorem 2.3 there exist $x^n \in \Omega^0$ such that the bound of $E(x^n)$ tends to infinity. Without loss of generality we can assume that $x^n \to x^0 \in \Omega^0$. Denote by $E(x^0)$ the subcontinuum containing $(0, x^0)$. Then $E(x^0)$ is bounded. Choose $R > 0$ such that $E(x^0) \subset Q_R$, $E(x^0) \cap \mathrm{b}Q_R = \emptyset$, where $Q_R = [0, R] \times B_R$. Take $X = \Sigma \cap Q_R$ which is compact and closed. Then $E(x^0)$ is a compact closed subset of X. Define $Y = \Sigma \cap \mathrm{b}Q_R$, hence $E(x^0)$ and Y are disjoint and compact. Consequently, there exist compact disjoint subsets K_1, K_2 of X such that $X = K_1 \cup K_2$, $E(x^0) \subset K_1$, $Y \subset K_2$, $K_1 \cap \mathrm{b}Q_R = \emptyset$. Thus there exists a bounded open set $U \subset \mathbf{R}^* \times P$ satisfying $K_1 \subset U \subset Q_R$, $U \cap (K_2 \cup \mathrm{b}Q_R) = \emptyset$. Again we get a bounded open set G with $K_1 \subset G \subset \operatorname{cl} G \subset U \subset Q_R$, hence $\Sigma \cap \mathrm{b}G = \emptyset$. Since $x^n \to x^0$ while $(0, x^0) \in E(x^0) \subset K_1$. Therefore $(0, x^n) \in G$ for n large enough. So the unboundedness of $E(x^n)$ yields $E(x^n) \cap \mathrm{b}G \neq \emptyset$. Take $(\lambda_n, y_n) \in E(x^n) \cap \mathrm{b}G$. Because Σ is locally compact there exists subsequence $(\lambda_n', y_n') \to (\lambda, x) \in \Sigma \cap \mathrm{b}G$ which is a contradiction. The proof is complete.

3. Applications to impulsive integrodifferential boundary value problems

In this section, we will apply the abstract results of the previous section to impulsive integrodifferential boundary value problems. Specifically we will show that the solution set of problem (1.1) has unbounded continua. For simplicity we will assume $\beta\delta = 0$ in this section. Now we list the main assumptions below. Recall that $\mathbf{R}^* = [0, \infty)$, $\mathbf{R}^+ = (0, \infty)$.

Define $M = \max\{k(t,s)\colon t, s \in [0,1]\}$, $M_1 = \max\{k_1(t,s)\colon t, s \in [0,1]\}$. Let $J = [0,1]$, $\mathfrak{X} = PC(J) = \{x\colon x$ is a function from J to $\mathbf{R}^1$, continuous at $t \neq t_k$, left continuous at $t = t_k$, and right hand limit at $t = t_k$ exist for $k = 1, 2, \dots m\}$. Recall that $PC(J)$ is a Banach space with norm $\|x\| = \sup_{t\in J} |x(t)|$. Denote the normal cone of $PC(J)$ by $P = \{x\colon x \in PC(J), x(t) \geq 0, t \in [0,1]\}$. A function $x \in PC(J)$ is called a positive solution of (1.1) if $x(t) > 0$, $t \in (0,1)$, $x \in PC(J)$ and satisfies (1.1). Throughout this paper, we use C to denote a constant, and $C(\varepsilon)$ a constant dependent of ε, even if they may be different at different places. Write

$$\Delta(px')\big|_{t_k} = \lim_{\varepsilon\to 0}[p(t_k+\varepsilon)x'(t_k+\varepsilon) - p(t_k-\varepsilon)x'(t_k-\varepsilon)],$$

and introduce the following condition (see [11]):

$$\Delta(px')\big|_{t_k} = -\frac{\gamma I_k(x(t_k))}{\delta + \gamma\tau_1(t_k)}, \quad k = 0, 1, \dots, m. \tag{3.1}$$

Define $\mathcal{D}(A) = \{x\colon x \in \mathfrak{X}, x(t) > 0, t \in (0,1), x'(t)$ and $p(t)x'(t)$ are continuous at $t \in (0,1)$, $t \neq t_k$, $k = 1, 2, \dots, m$, and x satisfies (3.1)$\}$. Let $\mathfrak{X}^* = \{x\colon x$ in a real function on $J\backslash\{t_1, t_2, \dots, t_m\}\}$, and

$$A(\lambda, x) = Lx + f(\lambda, t, x, Hx, Sx,), \quad t \in (0,1), t \neq t_k, k = 1, 2, \dots, m.$$

Suppose $\int_0^1 1/p(t)\,dt < \infty$. Then $A\colon \mathbf{R}^* \times \mathcal{D}(A) \to \mathfrak{X}^*$. Note that $\mathcal{D}(A)$ need not be open or closed. Denote:

$$\tau_1(t) = \int_t^1 \frac{1}{p(t)}dt, \quad \tau_0(t) = \int_0^t \frac{1}{p(t)}dt,$$

then we have $\tau_1, \tau_0 \in C[0,1]$. Let $\rho^2 = \beta\gamma + \alpha\delta + \alpha\gamma\int_0^1 1/p(t)\,dt$, and write

$$u(t) = (1/\rho)[\delta + \gamma\tau_1(t)], \quad v(t) = (1/\rho)[\beta + \alpha\tau_0(t)].$$

Note that $\gamma v + \alpha u \equiv \rho$. Define

$$G(t,s) = \begin{cases} u(t)v(s)p(s), & 0 \leq s \leq t \leq 1, \\ v(t)u(s)p(s), & 0 \leq t \leq s \leq 1, \end{cases}$$

$$\begin{aligned} \theta(s) &= \tau_1(s), \quad \text{when } \beta > 0,\ \delta = 0,\ s \in (0,1), \\ \theta(s) &= \tau_0(s), \quad \text{when } \beta = 0,\ \delta > 0,\ s \in (0,1), \end{aligned}$$

$\theta(s) = \tau_0(s)$ for $s \in [0, 1/2]$, and $\theta(s) = \tau_1(s)$ for $s \in (1/2, 1]$ when $\beta = 0$, $\delta = 0$. Write $f_n(\lambda, t, x, y, z) = f_n(\lambda, t, \max\{1/n, x\}, y, z)$. Define

(3.2)
$$\begin{aligned}(A_n(\lambda, x))(t) = \int_0^1 & G(t,s) f_n(\lambda, s, x(s), (Hx)(s), (Sx)(s))\, ds \\ & + (\delta + \gamma\tau_1(t)) \sum_{0<t_k<t} \frac{I_k(x(t_k))}{\delta + \gamma\tau_1(t_k)}.\end{aligned}$$

We will make the following assumptions

- (H_0) $\displaystyle\int_0^1 1/p < \infty.$
- (H_1) $f(\lambda, t, x, y, z) \le \psi(t)\varphi(\lambda, x, y, z)$, $t \in (0, 1)$, $\lambda, y, z \in \mathbf{R}^*$, $x \in \mathbf{R}^+$, where $\psi \in C[(0, 1), \mathbf{R}^+]$, $\varphi \in C[\mathbf{R}^* \times \mathbf{R}^+ \times \mathbf{R}^1 \times \mathbf{R}^1, \mathbf{R}^+]$ and $\displaystyle\int_0^1 \theta p \psi < \infty$.
- (H_2) $\theta(s)p(s)$ is bounded for $s \in (0, 1)$.
- (H_3) $\displaystyle\lim_{x\to\infty} I_k(x)/x = 0$, $k = 1, 2, \dots, m$.
- (H_4) For any $R > 0$, there exist $\zeta \in C[0, 1]$ with $\zeta(t) \ge 0$ for $t \in [0, 1]$ and $\zeta(t) \not\equiv 0$ such that $f(\lambda, t, x, y, z) \ge \zeta(t)$ for $t \in (0, 1)$, $\lambda, x, y, z \in (0, R]$.
- (H_5) $\displaystyle\lim_{|x|+|y|+|z|\to\infty} \frac{\varphi(0, x, y, z)}{|x| + |y| + |z|} = 0.$

Lemma 3.1. *Suppose (H_0) (H_1) (H_2) hold. Then $A_n(\lambda, x)$ maps $\mathbf{R}^* \times P$ into P and is completely continuous, i.e., condition (N_1) is satisfied.*

Proof. It is straightforward. See [11] Lemma 2.3. Q.E.D.

Next we will make the convention that all our symbols associated with the solution set have the same meaning as in section 2. Then from an existence principle obtained in [11] (see [11] Theorem 3.5) we know that Ω_n^0 are nonempty, i.e., condition (N_2) is valid for (1.1), provided that (H_0)–(H_5) hold.

Remark 3.2. If (H_0)–(H_5) are satisfied, $(\lambda, x) \in \Sigma_n$, then $x \in \mathcal{D}(A)$. Furthermore x verifies (3.1), see [11].

Lemma 3.3. *Let (H_0)–(H_5) be satisfied, then Ω^0 is bounded.*

Proof. It is essentially the same as Lemma 3.1 of [11]. Q.E.D.

Lemma 3.4. *Suppose (H_0)–(H_5) hold. Then the solution set Σ of* (1.1) *is locally compact in $\mathbf{R}^* \times P$.*

Proof. Let $(\lambda, x) \in \Sigma$ with $0 \le \lambda \le R$, $\|x\| \le R$, where $R > 0$ is a constant. We will prove our lemma in three steps.

(i) There exists $x^* \in C[0, 1]$ such that $x^*(t) > 0$ for $t \in (0, 1)$ and $x(t) \ge x^*(t)$, $t \in (0, 1)$, where x^* is independent only on R.

In fact, let ζ be determined by (H_4). Then

$$-Lx \geq \zeta(t), \quad t \in (0,1), t \neq t_k.$$

Define:

$$y(t) = \int_0^1 G(t,s)\zeta(s)\,ds + \big(\delta + \gamma\tau_1(t)\big) \sum_{0<t_k<t} \frac{I_k(x(t_k))}{\delta + \gamma\tau_1(t_k)},$$
$$x^*(t) = \int_0^1 G(t,s)\zeta(s)\,ds.$$

Then y satisfies the boundary condition and

$$\begin{cases} -(Ly)(t) = \zeta(t), \quad t \in (0,1), t \neq t_k, \\ \Delta y\big|_{t=t_k} = I_k(x(t_k)), \quad k = 1, 2, \ldots, m, \end{cases}$$

$$\Delta(py')\big|_{t_k} = -\frac{\gamma I_k(x(t_k))}{\delta + \gamma\tau_1(t_k)}, \quad k = 0, 1, \ldots, m.$$

Let $z = x - y$, then $-Lz \geq 0$, $t \neq t_k$, $\Delta z\big|_{t_k} = 0$, $\Delta(pz')\big|_{t_k} = 0$. Hence $z \in C^1(0,1)$, and z satisfies the boundary conditions. Thus it is easy to show that $z(t) \geq 0$, $t \in (0,1)$, by using elementary comparison technique.

In fact, suppose $\beta > 0$, $\delta = 0$ for example. Then $z(1) = 0$ from the boundary conditions. Since $Lx \leq 0$, $t \neq t_k$, $p(t)z'(t)$ decreases in $(0,1)$. First if $z(0) < 0$, then the boundary conditions yield $\beta \lim_{t\to 0} p(t)z'(t) = \alpha z(0) \leq 0$. Thus $p(t)z'(t) \leq 0$ in $(0,1)$ and $z'(t) \leq 0$ in $(0,1)$. This contradicts with $z(1) = 0$. So we have $z(0) \geq 0$. Suppose $z(t)$ assumes its negative minimum $z(c)$ with $c \in (0,1)$. Then $z'(c) = 0$ and $p(t)z'(t) \leq 0$ in $(c,1)$, hence $z'(t) \leq 0$ in $(c,1)$. This again contradicts with $z(1) = 0$. Therefore $z(t) \geq 0$ in $(0,1)$.

(ii) Denote by t_x^0 the zeros of $x'(t)$, including limit zeros of px'. Then there exists η independent of n such that

(1)	$t_x^0 \leq 1 - \eta,$	when $\beta > 0$, $\delta = 0$,
(2)	$t_x^0 \geq \eta,$	when $\delta > 0$, $\beta = 0$,
(3)	$\eta \leq t_x^0 \leq 1 - \eta,$	when $\beta = \delta = 0$.

In fact, let $\beta > 0$, $\delta = 0$ for brevity. Then the boundary conditions become $x(1) = 0$, $\alpha x(0) - \beta \lim_{t\to 0} p(t)x'(t) = 0$. In this case $t_x^0 < 1$. Otherwise $-Lx \geq 0$ in $(t_m, 1)$, then $x' \geq 0$, hence $x(t) = 0$ in $(t_m, 1)$, which is a contradiction. If the required η does not exist. Then we get a sequence of solutions x with $t^0 = t_x^0 \to 1$, $t^0 \in (t_m, 1)$. Evidently $|Hx| \leq MR \leq C$, $|Sx| \leq M_1 R \leq C$. Define

$$\Phi(u) = \max\{\varphi(\lambda, x, y, z)\colon 0 \leq \lambda \leq R,\ u \leq x \leq R,\ 0 \leq y, z \leq C\} + 1. \tag{3.3}$$

Then Φ decreases and for $t \in (t_0, 1)$ we get

$$0 \le -(px')' \le p(t)\psi(t)\varphi(\lambda, x, Hx, Sx) \le p(t)\psi(t)\Phi(x).$$

Evidently $px' \le 0$ and $x' \le 0$ on $(t^0, 1)$. Thus integration yields:

$$0 \le -(px')(t) \le \int_{t^0}^{t} p(s)\psi(s)\Phi(x(s))\,ds \le \int_{t^0}^{t} p(s)\psi(s)\,ds.$$

Let $T(u) = \int_0^u dv/\Phi(v)$, $z = T(x)$, then

$$0 \le -z'(t) \le \frac{1}{p}\int_{t^0}^{t} p(s)\psi(s)\,ds.$$

$$z(t^0) \le \int_{t^0}^{1} \frac{1}{p}\int_{t^0}^{t} p(s)\psi(s)\,ds = \int_{t^0}^{1} p(s)\psi(s)\tau_1(s)\,ds \to 0.$$

Hence $x(t^0) \to 0$. But (3.1) gives $\Delta(px')\big|_{t_k} < 0$. Thus x increases in $(0, t^0)$. So $x(t^0) = \|x\| \to 0$. This contradicts with (i).

(iii) Now we assume $\beta > 0$, $\delta = 0$, then $\theta = \tau_1$. Other cases are similar. Let $\|x\| = x(t^0 + 0)$. If $t^0 = t_k$, $1 \le k \le m$, then $x'(t_k + 0) \le 0$. First suppose $x'(t_k - 0) \ge 0$. From (3.1) we know

$$0 \le -\Delta(px')\big|_{t_k} \le C, \quad \text{where } C \text{ is independent on } x. \tag{3.4}$$

Thus $0 \le -px'\big|_{t_{k+0}} \le -\Delta(px')\big|_{t_k} \le C$, and $|x'(t_k + 0)| \le C$. From step (i) and the continuity of f we know

$$|x'(t)| \le C, \quad t \in [t_1, t_m]. \tag{3.5}$$

When $t \in (0, t_1)$, from (3.1), (3.5) and integration we get

$$0 \le -(Lx)(t) \le \psi(t)\varphi(\lambda, x, Hx, Sx) \le \psi(t)\Phi(x(t)),$$

$$\begin{aligned} 0 \le p(t)x'(t) &\le p(t_1)x'(t_1) + \int_t^{t_1} p(s)\psi(s)\Phi(x(s))\,ds \\ &\le C + \Phi(x(t))\int_t^{t_1} p(s)\psi(s)\,ds. \end{aligned}$$

Let $z = T(x)$, then

$$\begin{aligned} 0 \le z'(t) &\le \frac{C}{p(t)} + \frac{1}{p(t)}\int_t^{t_1} p(s)\psi(s)\,ds \\ &\le \frac{C}{p(t)} + \frac{1}{p(t)}\int_0^{t_1} p(s)\psi(s)\,ds \in L_1[0, t_1]. \end{aligned}$$

For $t \in [t_m, 1]$, similarly we obtain

$$0 \le -z'(t) \le \frac{C}{p(t)} + \frac{1}{p(t)} \int_{t_m}^{t} p(s)\psi(s)\,ds \in L_1[t_m, 1].$$

Now suppose $x'(t_k - 0) \le 0$. By induction we can assume without loss of generality that $x'(t_1 + 0) < 0$, or otherwise (3.5) holds. In the former case, $\alpha x(0) = \beta \lim_{t\to 0} p(t)x'(t) \ge 0$. Therefore we can find a zero t^* of x' (including limit zeros of px'). For $t \in (0, t^*)$, we have

$$\begin{aligned} 0 \le p(t)x'(t) &\le \int_t^{t^*} p(s)\psi(s)\varphi(\lambda, x, Hx, Sx)\,ds \\ &\le C\int_t^{t^*} \Phi(x(s))p(s)\psi(s)\,ds \le C\Phi(x(t))\int_t^{t^*} p\psi. \end{aligned}$$

Let $z = T(x)$, then (Note that $\theta = \tau_1$)

$$0 \le z'(t) \le \{C/p\}\int_t^{t^*} p\psi \le \{C/p\}\int_0^{t_1} p\psi \in L_1[0, t_1]. \tag{3.6}$$

For $t \in (t^*, t_1)$, we have

$$\begin{aligned} 0 \le -p(t)x'(t) &\le \int_{t^*}^{t} p(s)\psi(s)\varphi(\lambda, x, Hx, Sx)\,ds \\ &\le C\int_{t^*}^{t} p(s)\psi(s)\Phi(x(s))\,ds \le C\Phi(x(t))\int_{t^*}^{t} p(s)\psi(s)\,ds, \end{aligned}$$

$$\begin{aligned} 0 \le -z'(t) &\le \{1/p\}\int_{t^*}^{t} p(s)\psi(s)\,ds \\ &\le \{1/p\}\int_0^{t_1} p(s)\psi(s)\,ds \in L_1[0, t_1]. \end{aligned}$$

Also we have

$$|x'(t_1)| = |z'(t_1)|\Phi(x(t_1)) \le \Phi(x^*(t_1))|z'(t_1)| \le C.$$

Hence (3.5) holds again. For $t \in [t_m, 1]$, similar reasoning yields

$$0 \le -z'(t) \le \frac{C}{p(t)} + \frac{1}{p(t)}\int_{t_m}^{t} p\psi \in L_1[t_m, 1].$$

By the standard Arzela's technique we know that $\{z(t)\}$ is compact. Hence $\{x(t)\}$ is compact. If $\|x\| = x(t^0)$, $t^0 \in (0, 1)$, $t \ne t_k$, $k = 1, 2, \dots, m$, or $\|x\| = x(0)$. The proof is similar. Q.E.D.

Lemma 3.5. *Suppose* (H_0)–(H_5) *hold. Then* Σ *is closed.*

Proof. Let $(\lambda_n, x_n) \in \Sigma$ and $(\lambda_n, x_n) \to (\lambda, x)$ in $\mathbf{R}^* \times P$. Evidently (λ_n, x_n) is bounded, hence from step (i) of the proof of Lemma 3.4 we know $x_n(t) \geq x^*(t)$ and $x(t) \geq x^*(t)$, where $x^* \in C[0,1]$, and $x^*(t) > 0$ for $t \in (0,1)$. Again we assume $\beta > 0$, $\delta = 0$ for simplicity. Since f is continuous, then $p(t)f(\lambda, t, Hx_n, Sx_n)$ converges in $PC[\varepsilon, 1-\varepsilon]$, where $\varepsilon > 0$. As a result, x_n', px_n' converges in $PC[\varepsilon, 1-\varepsilon]$. Thus $x \in \mathcal{D}(A)$ and satisfies the impulsive conditions. It is easy to show that (λ, x) is a solution of (1.1), using technique similar to Theorem 5.1 of [15]. The proof is complete.

Now we come to our main theorem of this section.

Theorem 3.6. *Suppose* (H_0)–(H_5) *hold. Then there exists* $x^0 \in \Omega^0$ *such that the subcontinuum* $E(x^0)$ *emanating from* $(0, x^0)$ *of the solution set* Σ *is unbounded.*

Proof. This follows from Theorem 2.4 and the previous lemmas. Note that condition (N_3) is valid by Theorem 3.5 of [11]. Q.E.D.

Corollary 3.7. *Let the hypotheses of Theorem 3.6 be satisfied. Then one of the following assertions holds:*

(i) *Problem* (1.1) *is solvable for any* $\lambda \geq 0$.

(ii) *The solution set* Σ *of* (1.1) *has an asymptotical bifurcation point in* $\mathbf{R}^*$.

Proof. The projection of $E(x^0)$ in Theorem 3.6 onto $\mathbf{R}^*$ is connected, hence is an interval. If this interval is unbounded, then assertion (i) hold. If this interval is bounded, then Σ has an asymptotical bifurcation point in this interval, see Guo and Lakshimikantham [10]. The proof is complete.

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