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# UPPER AND LOWER BOUNDS FOR THE NUMBER OF THE SOLITONS OF THE KdV EQUATION

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**Abstract**. The number of the solitons of the Korteweg-de Vries (KdV) equation is considered when the initial value of the solution is given. Upper and lower bounds for the number of solitons of this equation are obtained under some conditions on the initial value of the solution.

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## §1. Introduction

From the pioneering work of Gardner et al. [1], we know how to solve exactly the KdV equation

$$u_t - 6uu_x + u_{xxx} = 0 \tag{1.1}$$

by the inverse scattering method if we know the initial data $u_0(x) = u(x, t = 0)$, under the condition that

$$\int_{-\infty}^{\infty} (1 + x^2)\, |u_0(x)|\, dx \, < \, \infty. \tag{1.2}$$

This method shows rigorously that the solitons of equation (1.1) correspond to the discrete eigenvalues of the associated "Schrödinger equation"

$$L\psi(x,t) \,=\, \lambda\psi(x,t) \quad \text{for} \quad t \geq 0 \quad \text{and} \quad -\infty < x < \infty\,, \tag{1.3}$$

where the operator $L$ is $L = L(t) = -\dfrac{\partial^2}{\partial x^2} + u(x,t)$ .

In (1.3), considered as a Schrödinger equation in one dimension, $u(x,t)$ plays therefore the role of the potential in the Quantum Mechanical sense. Writing (1.1) in the Lax form [2]

$$L_t = ML - LM, \tag{1.4}$$

where $M$ is a linear operator, one can easily show that the spectrum of (1.3) is independent of $t$ : $\lambda_t = 0$ and one has

$$\psi_t = M\psi \quad \text{for} \quad t > 0 \tag{1.5}$$

which gives the time evolution of $\psi(x,t)$.

Since the discrete eigenvalues are constant in time, to count the number of solitons of (1.1) it is sufficient to count the number of discrete eigenvalues of (1.3), the so-called bound states, with the potential $u(x,0)$. The direct and inverse problem for the one dimentional Schrödinger equation has been thoroughly studied by Faddeev, and Deifet and Trubovitz [3], who showed that the condition (1.2) is sufficient for proving all what is needed for the inverse problem method to work, and also to show that the number of bound states is finite. It was then shown by Segur [4] that a bound on the number of solitons is given by

$$N \le 1 + \int_{-\infty}^{\infty} |x|\,|V(x)|\,dx \tag{1.6}$$

where

$$V(x) = u(x,0)\,\theta[-u(x,0)], \tag{1.7}$$

$\theta$ being the Heaviside function. The above bound is the extention to the whole $R$ of the well-known Bargmann bound [5] for the radial Schrödinger equation for the $S$-wave $(l=0)$ : $\varphi''(E,r) + E\varphi = V(r)\varphi$, $r \in [0,\infty)$ with Dirichlet boundary condition at $r = 0$ . Other types of bounds for the number of bound states have been found for the radial case in arbitrary number of dimensions by Setô [6]. In one dimension for the whole $R$, he shows that if the potential is negative (attractive) everywhere and symmetric, the number of discrete spectrum corresponding to even or odd wave-functions (Neumann or Dirichlet condition at $x = 0$), satisfy the bounds

$$N_1^e \le 1 + \frac{\dfrac{1}{2}\displaystyle\int_0^\infty \int_0^\infty |r - r'|\,|V(r)|\,|V(r')|\,dr dr'}{\displaystyle\int_0^\infty |V(r)|\,dr} \tag{1.8}$$

and

$$N_1^o \le \int_0^\infty r\,|V(r)|\,dr.$$

From these, we deduce, of course, a bound on $N_{total} = N_1^e + N_1^o$.
The purpose of the present paper is to show that one can improve the bound of Segur and Setô if one assumes an additional condition on the potential $u(x,0)$, besides being everywhere negative. The additional conditions turns out to be the monotonicity of the potential for $x > 0$ and $x < 0$, or its generalization.

**Remark 1.** *It is a well-known theorem that, for the Schrödinger equation in the radial case, $r \in [0,\infty)$, the number of bound states is equal to the number of nodes (zeros) of the solution at zero energy ($E = 0$). This nodal theorem, which was at the basis of the original proof of Bargmann bound, will be systematically used in what follows. It applies in all cases, whatever the b.c. may be, whether Dirichlet, Neumann or mixed. In the Dirichlet case, one should not count the zero at $r = 0$.*

## §2. Calogero-type bounds

For the radial case $r \in [0,\infty)$, with Dirichlet boundary condition at $r = 0$, it has been shown by Calogero [7], that if one assumes that the potential, being everywhere negative, is also an increasing function (nondecreasing)

$$V(r) \le 0, \quad V'(r) \ge 0, \tag{2.1}$$

one has the bound

$$N \;\le\; \frac{2}{\pi}\int_0^\infty \sqrt{|V(r)|}\,dr. \tag{2.2}$$

In order to apply this bound to our case: $x \in (-\infty,\infty)$, we also assume that

$$V(-x) = V(x), \quad x > 0. \tag{2.3}$$

This means, since $V(\pm\infty) = 0$ , that the potential is decreasing for $x < 0$, and increasing for $x > 0$, and its absolute minimum is reached at $x = 0$. Since the potential is symmetrical, we study separately the odd and even solutions. For the odd solutions, we have $\psi_{odd}(0) = 0$, and we have therefore the Calogero bound (2.2) on each side of $x$. By symmetry, since $\psi_{odd}(-x) = -\psi_{odd}(x)$ for $x > 0$, it follows that if $\psi_{odd}$ is $L^2(0,\infty)$, it is also $L^2(-\infty,0)$. The number of odd bound states admits therefore the bound

$$N_{odd} \;\le\; \frac{2}{\pi}\int_0^\infty \sqrt{|V(x)|}\,dx. \tag{2.4}$$

As for the even bound states, they correspond to $\psi'_{even}(0) = 0$. Now, there is a well-known theorem [8] according to which the zeros (nodes) of $\psi_{odd}$ and $\psi_{even}$ are interlacing. Therefore, the number of the nodes of $\psi_{even}$ exceeds the number of the nodes of $\psi_{odd}$ at most by one, and so we have

$$N_{even} \;\leq\; 1 + \frac{2}{\pi}\int_0^\infty \sqrt{|V(x)|}\,dx. \qquad (2.5)$$

Adding up these, we get the following theorem.

**Theorem 1.** *If the initial data $u_0(x) = u(x, t = 0)$ is negative, is symmetric with respect to the origin, and is an increasing function of $x$ for $x > 0$, then the total number of solitons admits the bound*

$$N \;\leq\; 1 + \frac{4}{\pi}\int_0^\infty \sqrt{|V(x)|}\,dx. \qquad (2.6)$$

In examples 1 and 2, we show the comparison between our bound, and those of Segur and Setô.

*Example* 1.

$$u(x,0) = -n\,(n+1)\,\mathrm{sech}^2 x.$$

| $n$ | 1 | 2 | 3 | 4 | 5 | upper bound |
|---|---|---|---|---|---|---|
| exact value | 1 | 2 | 3 | 4 | 5 | $n$ |
| Segur's bound | 3.8 | 9.3 | 17.6 | 28.7 | 42.6 | $1 + 2n(n+1)\log 2$ |
| Setô's bound | 3.0 | 7.0 | 13.0 | 21.0 | 31.0 | $1 + n(n+1)$ |
| Our bound | 3.8 | 5.9 | 7.9 | 9.9 | 12.0 | $1 + 2\sqrt{n(n+1)}$ |

*Example* 2.

$$u(x,0) = \begin{cases} -V_0 & (|x| \leq a) \\ 0 & (|x| > a) \end{cases} \qquad (\, V_0 > 0, a > 0\,).$$

| $\sqrt{V_0\,a^2}$ | $\frac{1}{4}\pi$ | $\frac{3}{4}\pi$ | $\frac{5}{4}\pi$ | $\frac{7}{4}\pi$ | $\frac{9}{4}\pi$ | upper bound |
|---|---|---|---|---|---|---|
| exact value | 1 | 2 | 3 | 4 | 5 | $N$ (*) |
| Segur's bound | 1.6 | 6.6 | 16.4 | 31.2 | 51.0 | $1 + V_0\,a^2$ |
| Setô's bound | 1.4 | 4.2 | 10.0 | 18.6 | 30.1 | $1 + \frac{7}{12}V_0\,a^2$ |
| new bound | 2.0 | 4.0 | 6.0 | 8.0 | 10.0 | $1 + \frac{4}{\pi}\sqrt{V_0\,a^2}$ |

(*) Number of discrete eigenvalues is $N$ if

$$\left[\ \frac{2}{\pi}\sqrt{V_0\, a^2}\ \right] \ =\ N-1$$

where the symbol [ ] denotes the integral part.

As is obvious from (2.6) compared to (1.6) and (1.8), the larger $u_0(x)$ is (negative!), the more the r.h.s. of the latter become too large. In fact, it is known that when we consider a $u_0$ of the form $gV(x), V(x)$ having any sign, and $g \to\ \infty$, we have the asymptotic bound [9]

$$N_{total}\ \sim\ \frac{2}{\pi}g^{\frac{1}{2}}\int_0^\infty \sqrt{|V_-(x)|}\,dx, \tag{2.7}$$

where $V_-$ is the negative part of $V(x)$. This is more in agreement with our bound (2.6). Notice also that in this bound the extra 1 cannot be removed. Indeed, it is well-known that in full one dimension, a negative potential, no matter how weak it is, has always a bound state.

## §3. More General Upper Bound

So far, we have assumed that the potential is symmetrical with respect to $x = 0$ , is negative everywhere, and is nondecreasing for $x \geq 0$. We still keep the symmetry now, but relax the monotonicity of the potential for $x \geq 0$, and replace it by the more general condition

$$\frac{d}{dx}[x^{2p-1}(-V(x))^{p-1}] \geq 0,\ \ x \geq 0, \tag{3.1}$$

where $p \in [1/2, 1]$. For $p = 1$, this condition imposes nothing on $V$, and we are back to Segur and Seto. For $p = 1/2$, we get $V'(x) \geq 0$ , and we are back to (2.2). For $p$ in between, it is easily seen that the potential, altough everywhere negative, may have oscillations (see below). But these oscillations are not too sharp, i.e. the derivative of $V$ cannot be too negative. It has been shown recently [10] that (3.1), in the radial case $r \in [0, \infty)$, with Dirichlet b.c. at $r = 0$, leads to the bound :

$$n(V)\ \leq\ p(1-p)^{p-1}\int_0^\infty (-r^2V)^p\frac{dr}{r} \tag{3.2}$$

which is intermediate between the Bargmann bound and the Calogero bound. Arguing as in the previous section by using the nodal theorem, we end up, for the entire line $x \in (-\infty, \infty)$, with

**Theorem 2.** *For an initial value $u_0(x) = u(x, t = 0)$ of the KdV equation, which is negative everywhere, is symmetrical with respect to the origin $x = 0$, and satisfies the condition* (3.1)*, the number of solitons has the upper bound*

$$N \ \leq\ 1 + 2p(1-p)^{p-1} \int_0^\infty [-x^2 u_0(x)]^p \frac{dx}{x}. \tag{3.3}$$

As an example of potentials that satisfy (3.1), we have

$$V(x) \ =\ -x^{(2p-1)/(1-p)}[\int_x^\infty q(t)dt]^{1/(1-p)}, \ \ x \geq 0, \tag{3.4}$$

where $q(x)$ is any positive function which is $L^1$, goes to zero fast enough at infinity, and is such that $V(x)$ is less singular than $x^{-1}$ at $x = 0$ in order to satisfy (1.2). Taking for instance $q(x) = \exp(-x)$, one sees that $V(x)$ vanishes at $x = 0$, and has a minimum at some $x_0$ ($> 0$) before going to zero at infinity. Other forms of $q(x)$ can lead to more oscillations.

**Remark 2.** *The bound* (3.2) *(and therefore Theorem 2) is not quite optimal. For $p = 1$, we get indeed the Bargmann bound, but for $p = 1/2$, we do not recover the Calogero bound since the coefficient in front of the integral of* (2.2) *is* 1.111 $(2/\pi)$ *instead of* $(2/\pi)$*. However,* (3.2) *is the only bound actually known for $p \in [1/2, 1]$. Nevertheless, it leads again to a substantial improvement of previous bounds for large initial data if we make the mild assumption* (3.1) *on $u_0(x)$.*

## §4. Lower Bounds

We assume again the same conditions on the potential $u_0(x)$: it is negative everywhere, is symmetrical with respect to $x = 0$, and is nondecreasing for $x \geq 0$ . Of course, it satisfies (1.2).

To obtain the number of the discrete eigenvalues, we count the zeros of each solution for $x > 0$ as in the preceding sections, separately for $\psi_{odd}(x)$ and $\psi_{even}(x)$.
As for $\psi_{odd}(x)$ for $x \geq 0$ with $\psi_{odd}(0) = 0$ and $\psi'_{odd}(0) = 1$ , the nondecreasing of $u_0$ for $x \geq 0$ allows us to apply the lower bound for the number of the bound states of the Schrödinger equation, which was obtained by Calogero [7]. As for the $\psi_{even}(x)$, the initial condition at the origin is chosen as $\psi_{even}(0) = 1$ and $\psi'_{even}(0) = 0$. Interlacing theorem [8] tells that the zeros of $\psi_{odd}(x)$ and $\psi_{even}(x)$ with the above boundary condition are interlacing each other. Thus, we obtain the lower bound for the total number $N_{total}$ of the discrete spectra as

$$N_{tot} \geq \mathrm{Max}(2N_{odd}, 1), \tag{4.1}$$

where $N_{odd}$ is the number of the nodes of $\psi_{odd}(x)$ for $x > 0$. Calogero's lower bound [7] for $N_{odd}$ leads now

$$N_{odd} \geq \ [[1/2 + (1/\pi) \int_0^\infty dx \ \mathrm{Min}(q, -V(x)/q)]] \tag{4.2}$$

where $q$ is an arbitrary positive constant. It can be rewritten as

$$N_{odd} \geq [[1/2 + (qx^*)/\pi + (1/\pi) \int_{x^*}^\infty dx |V(x)|/q]] \tag{4.3}$$

where $q^2 + V(x^*) = 0$. The double bracket [[ ]] notes the integral part. The lower bounds for the number of the solitons in the exactly solvable cases are calculated as follows.

*Example* 3.

$$u_0(x) = -n\,(n+1)\ \mathrm{sech}^2 x$$

In this case, the exact number of the discrete spectra is $n$. The lower bound is calculated as

$$N \geq \mathrm{Max}(2[[1/2 + (2/\pi)\sqrt{n(n+1)}\sqrt{2x^* - 1}]], 1) \tag{4.4}$$

where

$$(1 - 2x^*)\exp(2x^*) + 1 = 0. \tag{4.5}$$

We can easily show that $5/8 < x^* < 3/4$ . Using the approximate value $x^* = 5/8$, we have

$$N \geq \mathrm{Max}(2[[1/2 + (1/\pi)\sqrt{n(n+1)}]], 1). \tag{4.6}$$

The easier way to get this non-optimal evaluation is to put $q = \sqrt{n(n+1)}$ as $V(x) \geq -n(n+1)$. Substituting this value for $q$ into eq.(4.3) gives,

$$N_{odd} \geq [[1/2 + \sqrt{n(n+1)}/\pi]]. \tag{4.7}$$

*Example* 4.

$$u_0(x) = \begin{cases} -V_0 & (|x| \leq a) \\ \ \ 0 & (|x| > a) \end{cases} \qquad (\ V_0 > 0, \quad a > 0\ ).$$

In this case, the integral is easily carried out and $q = \sqrt{V_0}$ . The result does not depend on $x^*$ and we get,

$$N \geq \mathrm{Max}(2[[1/2 + \sqrt{V_0 a^2}/\pi]], 1). \tag{4.8}$$

These lower bounds (though non-optimal values for the case (i)) for the number of the solitons of the KdV equation are given in the tables together with the upper

bound obtained in the section 2.
Finally, we note that the asymptotic behaviour of the lower bound is $g^{1/2}$ as $g \to \infty$ when we consider a $u_0$ of the form $gV(x)$.

$$V(x) = u(x,0) = -n\,(n+1)\ \operatorname{sech}^2 x.$$

| n | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| exact value | 1 | 2 | 3 | 4 | 5 |
| upper bound | 3.8 | 5.9 | 7.9 | 9.9 | 12.0 |
| lower bound (non-optimal) | 1 | 2 | 2 | 2 | 4 |

**Table 1**

$$V(x) = u(x,0) = \begin{cases} -V_0 & (|x| \le a) \\ 0 & (|x| > a) \end{cases} \qquad (\, V_0 > 0, \quad a > 0\,).$$

| $\sqrt{V_0\, a^2}$ | $\frac{1}{4}\pi$ | $\frac{3}{4}\pi$ | $\frac{5}{4}\pi$ | $\frac{7}{4}\pi$ | $\frac{9}{4}\pi$ |
|---|---|---|---|---|---|
| exact value | 1 | 2 | 3 | 4 | 5 |
| upper bound | 2.0 | 4.0 | 6.0 | 8.0 | 10.0 |
| lower bound | 1 | 2 | 2 | 4 | 4 |

**Table 2**

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## References

1. C.S.Gardner, J.M.Greene, M.D.Kruscal and R.M.Miura, *Method for solving the Kortweg-de Vries equation*, Phys. Rev. Letters **19** (1967), 1095-1097.
2. P.Lax, *Integrals of nonlinear equations of evolution and solitary waves*, Comm. Pure Appl. Math. **21** (1968), 467-490.

3. P.Deift and E.Trubovitz, *Inverse scattering on the line*, Comm. Pure Appl. Math. **32** (1979), 121-251.
4. H.Segur, *The Kortweg-de Vries equation and water waves. Solutions of the equation. Part 1.*, J. Fluid Mech. **59** (1973), 721-736.
5. V.Bargmann, *On the number of bound states in a central field of force*, Proc. Nat. Acad. Sci. **38** (1952), 961-966.
6. N.Setô, *Bargmann's inequalities in spaces of arbitrary dimension*, Publ. RIMS **9** (1974), 429-461.
7. F.Calogero, *Upper and lower limits for the number of bound states in a given central potential*, Comm. Math. Phys. **1** (1965), 80-88.
8. Interlacing theorem : see, for example, E.Hille, *Lectures on Ordinary Differential Equations* . ( Addison-Wesley, Reading, MA 1969)
9. K.Chadan, *The asymptotic behaviour of the number of bound states of a given potential in the limit of large coupling*, Nuovo Cimento **58A** (1968), 191-204. In the radial case, the coefficient in front of the integral is $(1/\pi)$. Here, we have to add odd and even eigenvalues, and therefore we get $(2/\pi)$.
10. K.Chadan, A.Martin and J.Stubbe, *The Calogero bound for nonzero angular momentum*, J. Math. Phys. **36** (1995), 1616-1624.

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